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Technological devices in the development of pupils' expertise in the use of selected heuristic strategies *

Abstract. This article shows how technological devices can be integrated into the use of four selected heuristic strategies by pupils in solving mathematical problem solving: Systematic experimentation, Introduction of an auxiliary element, Analogy, Solution drawing. Our research examined the change in the pupils' success rate and their attitude towards problem solving as a consequence of being taught the four selected heuristic strategies while actively using technological devices. A statistically increased success rate of problem solving and a statistically significant decrease in the frequency of the number of "no response" cases (the solver does not start to solve the problem) were expected.

The study was carried out by about 342 pupils in the 13–17 age group in the Czech Republic. The results of the study show the pupils' success rate in the solution of problems effectively solvable using the strategies systematic experimentation and solution drawing improved significantly if the use of these strategies went hand in hand with the active use of technological devices. The number of "no response" decreased significantly.

1. Introduction

In a technology-based society, where mathematics provides essential knowledge tools, several studies indicate declining interest in key mathematics topics (e.g. OECD 2006; 2014). The decline in interest is worrying, given that mathematical literacy serves as one of the foundational areas of knowledge that drives scientific and technological advancement in knowledge-based economies (European Commission 2010–2013). Research suggests that learning mathematics is critical

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for young children since the early years of schooling are especially important for children's development (Sarama, Clements, 2009).

Methods of teaching mathematics mainly in a traditional manner in schools have been identified as contributing to the falling interest in mathematics learning. (Yeh et al., 2019; Samuelsson, 2008) "Empirical classroom research over several decades shows that, with some notable exceptions, mathematics instruction has been characterized by a traditional, abstract formulation which seems to be readily understood by only a small fraction of students." (Mavrou, Meletiou-Mavrotheris, 2014) In some cases, the teaching of mathematics can be viewed as outdated and unconnected with pupils' interests and experience. Our experience shows that ideas are often presented in a theoretical and abstract manner without sufficient opportunities for pupils to engage in problem solving and experimentation. Now, as before (e.g. NCTM, 2000), it is true that educational leaders and professional organizations in mathematics education (NCTM, 2014) have been advocating the adoption of more active learning environments that would motivate learners and encourage them through authentic inquiry to establish the relevance and meaning of mathematical concepts. They have been stressing the fact that the core of school mathematics should no longer be the teaching of techniques and calculations that computers can do much faster and more reliably, but the development of problem solving skills that pupils will need to effectively live and function in a highly complex society (NCTM, 2014).

The article is a contribution offering one of the possibilities that could increase the understanding of problem solving by pupils and strengthen their skills in creative thinking. We believe that this possibility lies in a connection of the following areas – solving mathematical problems using heuristic strategies with the help of technology.

Without any doubt, problem solving is the unsubstitutable essence of mathematics as such and is the basic initiator of mathematical development (Pólya, 1973). This fact is reflected in educational systems: problem solving is an integral part of teaching mathematics in primary and secondary schools. The effort to integrate suitable problems into mathematics education is reflected in the basic curricular documents in the Czech Republic (Jeřábek et al., 2013) as well as in other countries (ACARA, 2015; NCTM, 2000).

School mathematics is based on learning the basic principles and procedures in mathematics that enable the solution of usual (standard) and expected problems (Brousseau, Novotná, 2008). However, one can also come across problems that fall out of this category. These may be real-life problems and purely mathematical problems, both providing space for the development of pupils in other areas.

The idea of using heuristic approaches in problem solving is far from new. Specific principles of this approach can already be found in the problems included in the Rhind papyrus (Přibyl et al., 2018). In more recent history, this approach can become visible in the works of philosophers and mathematicians such as R. Descartes, G. W. Leibniz and his followers or B. Bolzano (Hertwig, Pachur, 2015). In our opinion, the boom in the perception of heuristic strategies as a suitable tool for problem solving in mathematics was initiated by the work of G. Pólya, especially by the first publication of the book How to Solve It? in

1945. "Pólya's (1945) procedures consisted of simple rules, such as dividing the process toward a solution into simple steps by, for instance, finding an analogy to a problem, finding a more specialized problem, or decomposing and recombining the problem." (Hertwig, Pachur, 2015, p. 830)

It was the mathematically gifted students that Polya had in mind, when he created the idea of modern heuristics. However, in our longitudinal research, we focused on the use of heuristic strategies by pupils in normal classes, not on mathematically gifted pupils. It turned out that some of these strategies are also suitable for working with pupils who are below average in mathematics. (Eisenmann et al., 2015).

In a long-term perspective, we find all positive effects are essential, because these effects help to change pupils' attitudes toward mathematics as such (Eysenck, 2001; Eysenck, Keane, 2010). We believe that improving pupils' ability to solve problems is one of the most efficient ways to achieve these positive changes in pupils' attitudes.

2. Theoretical framework

2.1. Objectives

Similarly, to other areas of our lives, mathematics education is currently flooded with new technologies, for example, smartphones, tablets, apps, streaming, and cloud storage. It takes a short time for new technologies to enter schools and mathematics education as well (Jančařík, Novotná, 2011).

New technologies have fast become integral to everyday pupils' lives, which means they should be handled in the classroom. As Uzunboylu and Tugun (2016, p. 83) state:

> During the recent years from number of surveys assumed that the use of technology in the classroom has left positive impact on students, interesting in learning the desire course, help them learn knowledge and skills, increase the students' motivation, and focused on teachers and students a rich educational environment.

However, we must also draw attention to the fact that their use in teaching mathematics is not always effective and may not support pupils' comprehension of the subject and its concepts (e.g. Jančařík, Novotná, 2011).

The article can be considered as a contribution to this perspective. It focuses on the use of technologies (computers, laptops, tablets and software) in mathematics education to improve the pupil's ability to solve problems. The research was inspired by the finding that pupils who had access to computers while solving problems with the use of heuristic strategies used them for their work spontaneously (Eisenmann et al., 2015). Their use of computers was purposeful, certainly not "for show" in the sense of (Jančařík, Novotná, 2011).

In the design of our research, we came out primarily of Pólya (1973) and his definition of heuristic strategies and the model of a solving procedure.

Regarding the model, we decided to use the modified Pólya's model for the solving procedure (Pólya, 1973). The solving procedure is considered as divided into three basic operations: *grasping the problem assignment* (grasping all data and relationships necessary for creating a suitable mathematical model and the model creation); *solution of the mathematical model and mathematical verification*; *return to the context and contextual verification*. In practice, the procedure may not happen in the presented order, the solver may return to some of the parts or even skip some of the parts. In the model of pupils' solving procedures, we proceed from the assumption that information processing starts with a reading assignment (Novotná, 2000a; 2000b). Let us state here that in this article, we focus exclusively on the operation solution of a mathematical model.

The heuristic strategies were originally conceived principally for pupils solving problems in mathematical Olympiads and other mathematical competitions, where usually non-routine problems were solved. We had to adapt the concept of heuristic strategies to our needs (Eisenmann et al., 2015).

We perceive problem solving as a cognitive process that can be carried out in three ways depending on the solver's engagement, abilities and skills (see Fig. 1).

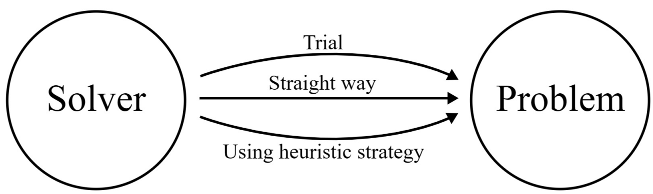

Figure 1: Approaches to problem solving

The first approach is referred to as a *trial*. It is the simplest way of dealing with the problem, and it is based only on the external motivation of the solver. The solver does not ask whether he has solved the problem correctly. His only goal is to "solve the problem", usually once, without having to check whether the solution is correct.

The second approach is referred to as a *straight way*. This way of solving is based on the application of already learned knowledge. The solver knows the desired solution procedure, is able to recognize that he should use it in solving the problem, and applies it.

The third approach is referred to as *using a heuristic strategy*. The solver either does not have the required knowledge or does not recognize that the learned knowledge is relevant to the problem and therefore cannot solve the problem with straight way. However, the solver is intrinsically motivated to solve the problem, and it is the heuristic strategy that enables him to solve the problem. (Eisenmann et al., 2015)

Let us note, in the sense of mathematical education, an expected situation is to master the mathematical content so that pupils can use it successfully in solving school tasks and practical problems. Solving problems in a straight way is usually an efficient approach to problem solving. However, in cases where pupils do not know, or are unable to use, the desired solution method, a heuristic strategy may become an efficient solution method. In some cases, it becomes appropriate to use the heuristic strategy with a technological device.

We perceive the heuristic strategy as a certain approach to solving problems, and this strategy can be perceived from different angles (e.g. Fan, Zhu, 2007; Pólya, 1973; Stacey, 1991). Unlike Pólya (1973), in some cases we do not see heuristic strategies as certain incentives – for example: Go back to definitions, but as a specific way of solving a problem based on specific steps. Our definition of a heuristic strategy is based on mathematical content and does not work with instructions that guide some of the solver's actions (such as: go back to the definition (read it again), read carefully, write down the important information from the task, etc.). Thus, we perceive the heuristic strategy as a problem-solving procedure that does not rely on a learned algorithm, but requires the student's own inventiveness or some insight based on mathematical knowledge. The heuristic strategy is only part of the whole process of solving the problem. The usual phases as defined by Pólya (1973) – (a) understanding the problem; (b) devising a plan; (c) carrying out the plan; and (d) looking back – are still present in the process of solving the problem.

The article discusses four selected heuristic strategies: systematic experimentation, introduction of an auxiliary element, analogy, and solution drawing because we expected these strategies to have the potential for the use of technological devices within the solving process. Specific reasons for the selection of these are explained in section 3.3.

The use of technology within problem solving has been described in the literature (e.g. ACARA, 2015; NCTM, 2000). However, we did not come across any research that would focus on the use of technology in solving mathematical problems using heuristic strategies in lower secondary schools. Bu et al. (2011) have attempted to work on linking heuristic strategies to technology in teaching mathematics, but their work remained more or less theoretical.

The success of a pupil while solving a problem is undoubtedly affected by their experience as it has an influence on their attitude to problem solving as such. It is a generally accepted fact that pupils' performance in problem solving improves during repeated encounters with problem-types proportionally to the use of their previous experience (Eysenck, 2001).

Several researchers are known to be concerned with the attitudes of pupils towards problem solving (Huang et al., 2015). The impact of positive attitudes on the process of education is discussed for example by (Villavicencio, Bernardo, 2015), who also take into account the solving of problems and tasks.

We believe that one of the possible ways to improve a pupil's attitude to problem solving is to teach them to actively use selected heuristic strategies. Our belief is supported by the Self-Determination Theory (SDT). There are two key assumptions of SDT: "The first assumption of self-determination theory is that people are actively directed toward growth. Gaining mastery over challenges and taking in new experiences are essential for developing a cohesive sense of self. ...self-determination theory focuses primarily on internal sources of motivation such as a need to gain knowledge or independence." (Cherry, 2021) The improvement described by SDT is not automatic but requires a permanent subsistence. It explains why pupils benefit when teachers support their autonomy (Reeve, 2002). Reeve claims that pupils benefit from the use of autonomy-supportive teachers' behaviours. A pupil will not refuse to solve a problem at the beginning even if they do not know the relevant solving algorithm. Their positive attitude combined with the above-mentioned heuristic strategies enables a pupil to start experimentation and gives them a chance not only to start solving the problem but also to solve it successfully.

It clearly follows from the text above that our study deals with the variables given in figure 2.

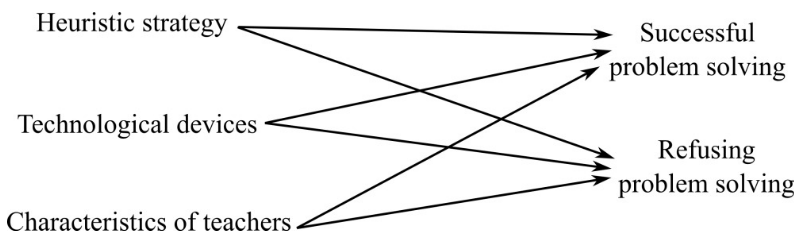

Figure 2: Diagram showing independent (left) and dependent (right) variables of the entire study

Independent variables include the characteristics of teachers. These were the basis for the selection of teachers for the study (see section 3).

The authors are not aware of research dealing with the change in pupils' attitude to problem solving using heuristic strategies with the use of technology.

The purpose of the study is to explore the impact of using technology while solving problems with the use of heuristic strategies on pupils' success rate in problem solving.

2.2. Theoretical background

2.2.1. Technology in mathematics education

The article is a contribution to the important issue of appropriate use of the potential that technology may offer to education. There are no longer any doubts about the benefits of using technology.

> Electronic technologies – calculators and computers – are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately. They can support investigation by pupils in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technological tools are available, pupils can focus on decision making, reflection, reasoning, and problem solving. (NCTM 2000, p. 24)

As far as problem solving is concerned, we believe one of the possible benefits of the use of technology in teaching is that technology eliminates certain technical difficulties in solving problems, namely, the difficulty connected with performing calculations or drawing and constructing objects. This allows the solver to gain insight into the task or solve it.

The literature offers a wide scope of work that focuses on the use of technology in mathematics education in general. A comprehensive overview of the possibilities of using technology in mathematics education is presented by Jezdimirović (2014).

Trouche and Drijvers (2010) examine the use of technology such as clay tablets, compasses, rulers, books, paper, pencils, and, in present times, calculators and computers in mathematics education. Although they focus mainly on handheld technology, their ideas are more broadly valid for all technologies.

Drijvers (2015) focuses on the question of "What is the potential of ICT for learning and teaching, and which factors are decisive in making it work in the mathematics classroom?" (p. 1) His conclusion is that"...crucial factors for the success of digital technology in mathematics education include the design of the digital tool and corresponding tasks exploiting the tool's pedagogical potential, the role of the teacher and the educational context." (p. 1)

The use of technology in teaching mathematics should not be random and should be governed by a definite rationale. Jančařík and Novotná (2011) suggest the following principles:

- The use of computers cannot be autotelic but must be linked to specific educational content.
- The computing power must be used effectively; the results should be presented in a comprehensible way.
- The results from the computer should be further interpreted; they should provide space for follow-up discoveries.

This is valid for computers. We claim that this also holds when using technology in general. Pupils' thinking should be developed, not substituted.

2.2.2. Problem solving, heuristic strategies and technological devices

We share the belief that ICT expands pupils' skills in solving mathematical problems (e.g Dirgha, 2017). His research does not focus on heuristic strategies, but the course of work in the experiments is in accordance with Pólya (1973). What is important for us is that the teachers' work in the experiments corresponds in some aspects (provision and distribution of materials) to the course of work of the teachers involved in our experiment.

The end of this subsection is devoted to implementing the four discussed strategies in the context of existing literature, and that from the point of view of using technology in their application.

The use of a spreadsheet precipitates a range of advantages in problem solving as well (e.g. Marley-Payne, Dituri, 2019). Let us note that Drier (2001) already points out that MS Excel can be used successfully while solving the so-called open-ended problems. Chaamwe and Shumba (2016) further advance and state that "The spreadsheet tool also promotes the development of problem solving skills and supports 'What if...' type questions." (p. 570)

The following conclusions based on experiments are presented about the strategy systematic experimentation by Doorman et al. (2013).

- By generating the results of a series of calculations for a variety of input values, the computer tool strengthened the pupils' notion of a function as a calculation procedure that transforms input values into output values.
- By generating output values for a series of input values, by generating tables of input and output values, and by enabling the pupils to move up and down the values in these tables, the computer tool supported the pupils in developing a dynamic notion of a variable that can move in a space of possible values, and the corresponding idea of co-variation.

The first incentives of solution drawing have been brought by Pólya (1973). He takes drawing a figure as an encouragement, motivating the pupil to use more effective problem solving. He demonstrates this in a variety of model situations. Most frequently, he exploits drawing a picture to illustrate a particular situation, which enables the pupil to gain insight into the problem or to affirm the direction of their thought.

Unlike Pólya, we understand "drawing a figure" and especially manipulating the objects in it as a self-contained heuristic strategy, which we call solution drawing.

Not every drawing counts as a heuristic strategy. The strategy solution drawing bears the characteristics of the above-mentioned manipulation of objects.

Introducing a dynamic geometry system or software in teaching gives pupils the opportunity to stop concentrating on drawing as such. Pupils do not have to follow their accuracy but can focus on relations among objects in the task, which enables them to reach the solution more easily. We agree with Kuzle (2017, p. 38) that "The interface of a DGS [dynamic geometry software] creates an opportunity to transform a mathematics classroom into an environment of investigation, in which pupils engage in predicting, manipulating, observing, conjecturing, testing, and developing explanations for observed phenomena."

GeoGebra is an interactive mathematics software for learning and teaching not only geometry on every level of mathematics education. In this article we focus on the geometric part of GeoGebra. Basic geometric constructions work with objects such as points, lines and circles and other objects rises oh these. Some of the basic constructions are already built into the SW, such as constructing a perpendicular from a given point to a given line, constructing tangents from a given point to a given line, and so on. There are even built-in functions that allow you to perform basic planar transformations such as axial symmetry, homothety, and rotation. What we must emphasize at this point is that the main advantage of interactive geometry is that all relationships are preserved when manipulating objects. Iranzo and Fortuny (2011) studied the general relation of paper-and-pencil skills in the environment of GeoGebra, with a strong pro-GeoGebra approach, and, among other things, they indicate: "We have also observed that the use of GeoGebra fosters a more geometrical thinking in the context of the given tasks." (p. 102)

There is no doubt about the suitability of introducing GeoGebra in teaching mathematics and the opportunities it suggests are shown, for example, on the web

GeoGebra Tube. We focused our research on mathematical problem solving with heuristic strategies with a little help of dynamic geometry systems. The encouragement of the heuristic approach to problem solving through GeoGebra is cited in (Abdu et al., 2015). In their particular case, the focus has been put on whole-class scaffolding. Guerrero-Ortiz et al. (2015) say: "We obtained evidence that the use of a DGS [dynamic geometry system] can enhance the emergence of a wide range of heuristics, some of which are specifics to this type of environment,..." (p. 294) The opportunities of GeoGebra in class, corresponding with our research, are to be found by Koyuncu et al. (2014), as they state: "After solving each problem, the pupils attempted to find alternative strategies, and they usually found alternative solutions." (p. 860).

As far as the introduction of an auxiliary element and analogy are concerned, the authors of this article are not aware of any research linking the strategies with technologies. The authors are of the opinion that it is due to two main reasons: firstly, both strategies are of non-algorithmic character, and their use demands a certain level of mental involvement on the part of the solver (Přibyl, Eisenmann, 2014), which makes the use of technology in terms of accelerating the computing more difficult. Secondly, a number of solvers do not recognize the very presence of the auxiliary element in the graphical depiction of the solution and such a way remains for them on the level of the solution drawing strategy.

3. Design of the study

The research was designed as a quasi-experiment in the sense of repeated measures (pre-test and post-test) of one group. The interval between the first and second measurements was 12 months. The research sample did not change in this interval.

3.1. Research questions and hypotheses

The following questions were posed at the beginning of our experiment:

RQ1: *Will the participating pupils' success rate in problem solving increase as a consequence of being taught the four selected heuristic strategies while using technology actively?*

RQ2: *Will the number of participating pupils who do not even start solving the problem decrease as a consequence of being taught the four selected heuristic strategies while using technology actively?*

On the basis of the findings from the second stage of the study and on our research (Eisenmann et al., 2015), we formulated the following primary and secondary hypothesis:

Primary hypothesis: There will be a statistically significant increase in the success rate of problem solving between the pre-test and the post-test due to the use of heuristic strategies (when using technology actively).

Secondary hypothesis: There will be a statistically significant decline in the frequency of no response (situation when the pupils do not even start solving the problem) between the pre-test and the post-test.

3.2. Discussed heuristic strategies

In this section, we present a brief description of the four heuristic strategies that were subjects of our study. They are described in more detail in (Eisenmann et al., 2015; Přibyl, Eisenmann, 2014).

3.2.1. Systematic experimentation

The principle behind the strategy Systematic experimentation (SE) is the idea that the result may be reached in a finite number of attempts. These attempts are carried out systematically. Each subsequent attempt will be "slightly" modified with respect to the previous one. The principle is based on the fact that the solver selects an initial value and then gradually gets closer to the sought solution. This strategy seems to be very efficient in connection with the use of technological devices as these make it possible to conduct the experiments in real-time. Let us illustrate this strategy with the following problem.

Problem: The numbers that are read the same from left to right and right to left, for example 452 254, are called palindromes. My friend claims that all four-digit palindromes are divisible by 11. Is it true?

Solution: Let us now take several four-digit palindromes at random and divide them by eleven. We will get, for example, the following:

$$4\ 554 = 414 \times 11$$
$$1\ 001 = 91 \times 11$$
$$8\ 338 = 758 \times 11$$

It seems my friend could be right. However, we will not be one hundred percent sure unless we check all 90 existing four-digit palindromes. This can be done efficiently using a spreadsheet (see table 1).

Table 1: Solving of the problem – shortened table (the so-called brute force method) (source: own calculation)

a	***b***	**"=*b2*"**	**"=*a2*"**	**"=1000*A2+100*B2+10*C2+1*D2"**	**"=E2/11"**
1	0	0	1	1 001	91
1	1	1	1	1 111	101
...	...	...	...	...	...
2	9	9	2	2 992	272
...	...	...	...	...	...
9	1	1	9	9 119	829
...	...	...	...	...	...
9	9	9	9	9 999	909

The last column shows that all quotients are whole numbers.

Answer: My friend is right.

3.2.2. Introduction of an auxiliary element

The basic idea of this strategy is that the introduction of an auxiliary element (IAE) makes the solution much more easily accessible to the solver. We define an auxiliary element as an object not included in the question of a problem, which we insert into the problem, hoping it will make the solving procedure easier. In the case of geometrical problems, it will usually be a straight line, a line segment, a point, or a figure, in the case of arithmetical or algebraic problems it will be a number or a function. Let us illustrate this strategy with the following problem.

Problem: Let ABC be an acute-angled triangle. Let us construct such a square $KLMN$ whose side KL lies on the side AB, vertex M lies on the side BC, and vertex N on the side AC.

The problem is from (Pólya, 1973).

Solution: Let us now visualize the situation of the problem (see Fig. 3).

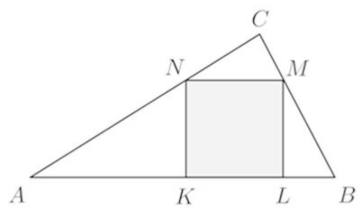

Figure 3: Visualization of the problem

Several attempts will make it obvious that we cannot construct such a square. It is difficult to meet all the three conditions of the assignment – condition (kl): the side KL lies on the side AB, condition (m): the vertex M lies on the side BC, and condition (n): the vertex N lies on the side AC.

Let us omit the condition (m). This will allow us to construct a square that meets the conditions (kl) and (n). This is not so difficult as the origin problem. The construction done by the software GeoGebra is illustrated in figure 4.

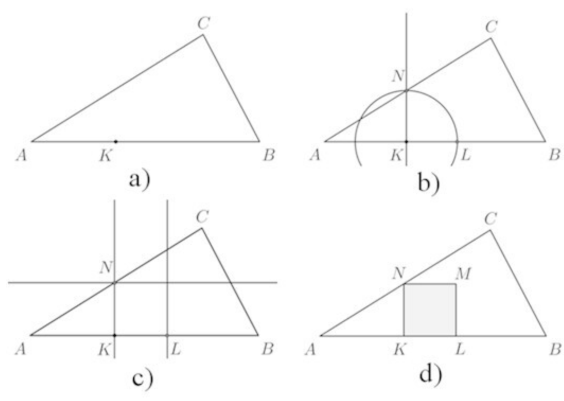

Figure 4: Construction of the square with one omitted condition

Point K is the variable in the construction. If it is moved along the line segment AB, a number of different squares will be generated. We can track the movement of the point M (using the function trace, see Fig. 5).

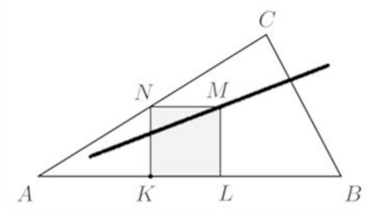

Figure 5: Track of the movement of the point M

Obviously, if we want to solve the problem, we have to introduce an auxiliary element – the line p that goes through points A and M (see Fig. 6). The intersection point of line p with the side BC is the sought point that meets the condition (m).

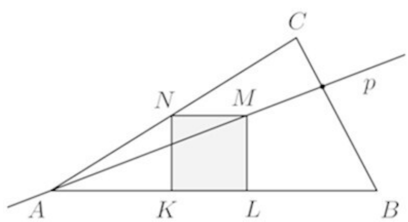

Figure 6: Auxiliary element – line p

Let us now make use of the fact that dynamic geometry software allows us to hide some objects. The newly created point will be identified as the sought point M and we can construct the sought square as is shown below (see Fig. 7).

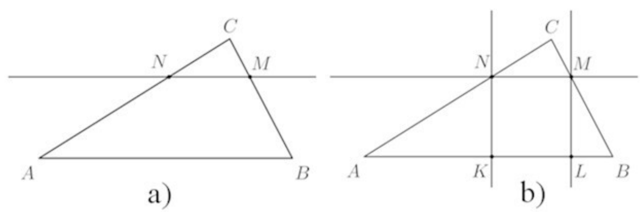

Figure 7: Construction of the square $KLMN$

Answer: Construction of the square $KLMN$ is described above.

3.2.3. Analogy

The use of the strategy Analogy (AN) is based on the fact that the solution of a problem "analogical" to the original problem is simpler. The solver discovers the way to the solution and then applies the solution procedure to the original problem. What is important here is that the solver poses this analogical problem on their own. This is usually done by substituting objects in the problem by other objects. Some of the qualities of the objects remain intact, but the others are changed. For example, a sphere is substituted by a circle, large numbers are substituted by small numbers, and an 8×8 chessboard is substituted by a 3×3 chessboard. Let us illustrate this strategy with the following problem.

Problem: Given are the straight line p and a regular octahedron. Determine the plane that goes through line p and divides the given octahedron into two parts of equal volume (see Fig. 8).

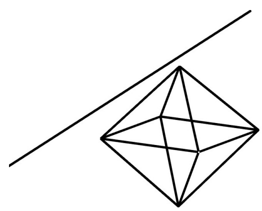

Figure 8: Visualization of the problem

Solution: The analogy is based on going one dimension lower. Instead of a regular octahedron, we will work with a square (the plane projection of the octahedron) and instead of the straight line, we take a point. We formulate an analogical problem.

The analogical problem: Given is a square and the point P, which is not the centre of the square. Determine the line that goes through the point P and divides the given square into two figures of the same area.

The solution of the analogical problem: We experiment using the software GeoGebra. If we draw several lines through the point P intersecting the square, we discover that we get figures of the same area when the line goes through the centre of the square (see Fig. 9). These polygons are surely congruent, which proves the equality of the areas of the figures.

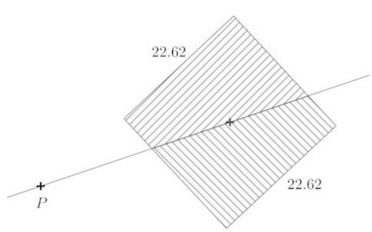

Figure 9: Sought line intersecting the centre of a square and the point P

Answer to the analogical problem: The sought straight line must intersect the centre of the square.

Answer: The sought plane must go through the centre of the octahedron.

3.2.4. Solution drawing

The graphic method is very often a very efficient way of solving a problem. We usually visualize the problem using an illustrative drawing – this is quite common, for example, in the case of geometric constructions in geometry. The drawing allows us to describe what is given and also what we are looking for. The drawing we create is called an illustration and is a graphical representation of what is being solved. Sometimes, it can help us realize how to solve the given problem. However, we often manipulate with the drawing (e.g. we add the needed auxiliary elements). This extended drawing often helps us solve the problem. This drawing will be referred to as the solution drawing (SD). This way of finding a solution confirms the known proverb that one good drawing saves one thousand words. Let us illustrate this strategy with the following problem.

Problem: We have a square inscribed in a circle and this circle is inscribed in another square (see Fig. 10). Determine how much of the larger square is occupied by the smaller square.

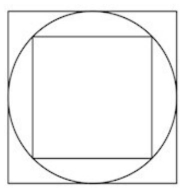

Figure 10: Visualization of the problem

Solution: Adding the diagonals and using a suitable rotation of the smaller square (see Fig. 11) enables us to solve the problem easily. The square is divided into four equal parts – smaller squares. Each small square is divided by diagonal into two equal parts – triangles. Therefore, the inscribed square occupies exactly one half of the circumscribed square.

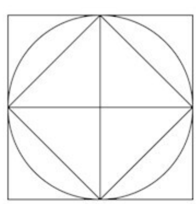

Figure 11: Rotation of the smaller square

Answer: The smaller square occupies one half of the larger square.

3.3. The timeline of the study

The timeline (see Fig. 12) of the study was the following:

1. Training teachers in in-service teaching courses (11 months)
2. Selection of problems (5 months)
3. Sampling (1 month)
4. Pre-test (1 month)
5. Experimental teaching (10 months)
6. Post-test (1 month)

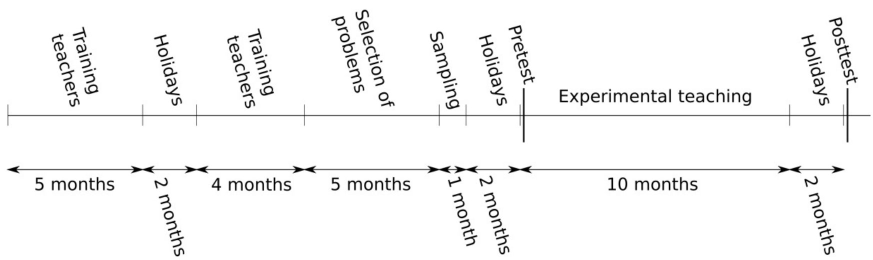

Figure 12: Time line of the study

The goal of the first stage was to train teachers for the following experimental teaching. In the nine months of in-service teacher training courses, 19 teachers were introduced to the use of heuristic strategies in problem solving. In total, there were four two-hour workshops. At this stage, the feedback from the teachers served as the basis for selection of the four here discussed strategies. These strategies proved to have the potential of allowing the effective use of technology while solving problems. Namely, the following expectations emerged in the discussions when solving tasks with teachers. The SE strategy can be used efficiently only when a spreadsheet programme such as MS Excel is used. In the case of SD and IAE strategies, it proved to be useful when creating a solution drawing or finding an auxiliary element to start with making illustrative drawings at the beginning of the solution using a dynamic geometry system, for example GeoGebra. For some problems solved using AN strategy, teachers experimented with GeoGebra 3D and solved the problem, for example, by reformulating the problem to an analogical problem by projecting an object into a plane, thus working with one less dimension.

The goal of the second stage was to prepare the appropriate problems for each of the four selected strategies to be used in experimental teaching. This stage lasted 5 months. The above-mentioned 19 teachers with their 13–17 year old pupils piloted the tasks in their classes. From the 30 problems prepared by us for each of the strategies, the 20 most suitable were selected according to the following criteria:

- the solution of the problem using the given strategy is an efficient way, if the solver cannot solve the problem straight way manner;
- the problem wording is clear to the selected age category of pupils;
- pupils' skills in the area of technology allow the pupils to use technology;
- a successful solution of the problem takes about 5–10 minutes.

3.4. The participating teachers and pupils

In the third stage, 14 of the above-mentioned 19 teachers were selected for the experimental teaching. The first criterion for their selection was that they had at least 5 years of teaching experience and were willing to commit to responsible cooperation during the experimental teaching. Another criterion was that the teachers should be able to provide a class for the experimental teaching, that would meet the following criteria:

- there are 20–26 pupils in the class;
- it is an ordinary class with no specialization;
- the class does not integrate pupils with major learning difficulties;
- the class as a whole may be seen as an average with respect to its performance and discipline;
- the class was not involved in the second stage of the study, i.e. pupils were not involved in selection of the most suitable problems.

The assessment of the class for the criterion of performance and discipline (the fourth criterion) was based on two sources: the grades in mathematics on the school report and evaluation by the participating teachers. The participating teachers agreed that according to their experience, the classes were average, there were no outstanding numbers of either low achieving or high achieving pupils, and there were no severe disciplinary issues.

Table 2 presents information about the teachers involved in the experimental teaching.

The selection of teachers meant also the selection of pupils involved in the experimental teaching. 14 classes with 342 pupils took part in the study:

- 5 classes of 13–14 year old pupils, there were 22, 25, 27, 23 and 22 pupils
- 4 classes of 15–16 year old pupils, there were 26, 25, 24 and 26 pupils
- 5 classes of 16–17 year old pupils, there were 25, 26, 26, 22 and 23 pupils

The age of the pupils given in the previous paragraph is the pupils' age at the beginning of the experimental teaching. There were 180 boys and 162 girls (342 pupils in total) in the lower and upper secondary schools in the Czech Republic, namely, in Prague and Ústí nad Labem. None of the classes integrated physically

Table 2: Information about the teachers involved in the experimental teaching (source: own calculation)

Group	Sex	Age	Teaching experience (years)
13–14	F	38	10
13–14	F	31	6
13–14	F	40	11
13–14	M	29	5
13–14	M	32	7
15–16	F	50	21
15–16	F	32	8
15–16	F	35	11
15–16	M	59	28
16–17	F	56	27
16–17	F	32	6
16–17	M	33	7
16–17	M	40	15
16–17	M	32	8

or mentally disabled pupils or pupils with low socioeconomic status. All pupils were of Czech nationality except for two Vietnamese and seven Roma pupils. All pupils were native speakers of Czech.

Table 3 shows the arithmetic mean of the grade in mathematics on end-of-year school report and the standard deviations of each age group. Grades in the Czech Republic are on the scale from 1 to 5, where 1 is the best and 5 is the worst.

Table 3: Arithmetic mean of the grade in mathematics (source: own calculation)

Age of the group	The arithmetic mean of the grade in mathematics	The standard deviation
13–14	2.42	1.08
15–16	2.32	1.07
16–17	2.37	1.10

A certain comparison may be seen in the fact that the arithmetic mean of grade in mathematics was 2.34 in the 20 selected elementary schools in Prague and Ústí nad Labem.

The pupils' experience and skills in technology were mostly on the level of ICT curriculum for lower secondary schools in the Czech Republic. To be more explicit, the pupils were able to work with MS Excel in the extent enabling them to make simple formulas and use them in additional cells by automatic filling.

The pupils were also able to use GeoGebra, namely, the part that works with planimetric constructions.

The participating teachers had mastered MS Excel at a good level. They were also proficient in the use of GeoGebra, not only in the area of planimetric construction but also in the area of symbolical calculations using CAS. None of the teachers was an ICT teacher.

3.5. Experimental teaching

Within the frame of the study, the pupils were taught the following four strategies: SE, IAE, AN, SD. As stated above, a set of about 20 problems for each strategy was prepared for the experimental teaching. The problems related to the following areas of school mathematics: arithmetic, word problems on movement, joint work and percentages, elementary and solid geometry, equations, functions. About three quarters of these problems were suitable for all age categories involved. The rest of the problems could only be used in the group of 16–17 year old pupils. Each problem was supplemented by a solution using the standard way, then using the given strategy and also using at least one other heuristic strategy.

The experimental teaching took 10 months in every class, starting in September and finishing in June. During the whole period of the experimental teaching, the selected heuristic strategies were not taught separately in individual teaching blocks. The selected problems were included in the current topic taught. On average, the pupils solved two problems using heuristic strategies each week.

The teachers' working procedure in the lessons was the following: They presented the problem to their pupils (mostly in the written form, on a worksheet). They let them work and for some time (when at least one-half of the pupils had solved the problem) they asked one pupil to present the solution to the others. Then they checked the rest of the class, had understood the presented solution, and invited the pupils to show their (own) solutions if they were different. If the solution required by the heuristic strategy, which was the goal of setting the problem, did not appear among the presented solutions, the teacher showed it to the pupils. The pupils were always encouraged to look for more ways of solving a particular problem, and to record their solving procedure. In the discussions, they were asked to justify their solving procedures as we are convinced that this not only develops pupils' communication skills but also their ability to solve problems.

A computer, a laptop, or a tablet were available to all pupils in the lessons. About half of the pupils were sitting in pairs at the computers. In problems requiring the use of a spreadsheet (especially when using the strategy SE), the pupils were working with MS Excel, in problems requiring the use of a dynamic geometry system (SD, IAE and partly also AN) they were using GeoGebra and in two classrooms the Cabri II Plus. It should be noted here that all the pupils who participated in the experimental teaching had the opportunity to use a desk computer or a laptop or a tablet in their home preparation.

Cooperation with the participating teachers was intense and systematic. The teacher always collected the pupils' worksheets with the solutions and evaluated them. The worksheets, individual problems, strategies used, and the individual pupils were then discussed in regular meetings with one member of the research team. Each of the 14 teachers had one partner from the research team. They cooperated closely for the whole period of the experiment. Members of the research team had access to the pupils' worksheets during the whole experiment. The cooperating members of the research team also went to observe a lesson in the experimental class once every four months.

3.6. Pre-test and Post-test

Before the start of the experimental teaching, the pupils took the pre-test, after the experimental teaching was completed, the pupils took a post-test. The problems in the pre-test and the post-test were identical. The interval between the first and second measurements was 12 months. The interval was long enough to eliminate the influence of the pre-test on the post-test. The test did not contain problems whose uniqueness could have made the pupils remember them.

Each of the three groups of pupils (divided with respect to age) had a different test corresponding to their knowledge. In all three cases, it was a set of four problems that were most effectively solvable by one of the four here discussed strategies using technology. Each problem represented only one of the four strategies discussed. In all tests, problem 1 corresponded to the strategy SE, problem 2 to the strategy IAE, problem 3 to the strategy AN and problem 4 to the strategy SD. These problems were not solved with the pupils during the experimental teaching.

The tests were taken in lessons of mathematics. The teacher was present. Each pupil was given one sheet of paper with all four problems. There was an empty space below each problem for the solution. When taking the pre-test/post-test, computers, laptops or tablets were available to all pupils.

Each teacher was given the following instruction:

- The teacher must not give any advice or explain the assignment.
- Use of any textbook, tables, or collection of formulas is prohibited.
- The pupils must write the solving procedure or record the reasoning that leads to the result, not just the result.
- If pupils use technology, having solved the problem, they must raise their hand so that the teacher can come to them, have a look at the monitor, and put a stamp on the test to confirm that the technology was really used in its solution.
- The pupils must write a full answer (where needed) or double underline the result.
- We leave it up to the teacher to ask the pupils to sign or do not sign the test. The goal is maximum effort on the pupils' part.
- The pupils have a maximum of 45 minutes for the test.

In each test task, the pupil had to select an answer Yes/No, indicating whether they had used a computer, laptop, or tablet while solving the problem. In this study, the use of a calculator or a similar device is not considered to be the use of technology. If the computer was used only for calculations as a calculator, the pupil's answer to the question was No.

All tasks in the test were analyzed and assessed in detail. Each solution was coded. The following phenomena were coded:

- way of solving the problem (straight way or heuristic strategy);
- success of problem solving (yes/no);
- no response (the pupil did not even try to solve the problem); and
- use of technology.

The pre-test and post-test comparisons were conducted using McNemars' test for homogeneity of the marginal distributions. This test was chosen because in this case we analyze the dependent dichotomic variables. Software STATISTICA was used for all statistical evaluations.

In all cases, the studied phenomenon is described by the values solved/did not solve. The value solved corresponds to a complete required solution of the problem.

Let us illustrate how the test problem was evaluated. The problem comes from the test for 13–14 year old pupils.

Problem: Determine two consecutive odd natural numbers whose product is 1023.

Solution: The sought numbers are 31 and 33.

The following correct solutions were used in the tests:

(1) Straight way, algebraic mode:

The first odd number................................$2n-1$, where $n \in N$
The second odd number............................$2n+1$, where $n \in N$

$$(2n-1)\cdot(2n+1) = 1023$$
$$n^2 - 256 = 0$$

For $n_1 = 16$

$$(2\cdot 16 - 1) = 31$$
$$(2\cdot 16 + 1) = 33$$

For $n_2 = -16$

$$2\cdot(-16) - 1 = -31$$
$$2\cdot(-16) + 1 = -33$$

The numbers -31, -33 are negative integers. The only solutions to this problem are thus numbers 31 and 33.

(2) Straight way, arithmetical mode I

There is one even number between two consecutive odd numbers.

$$\sqrt{1023} \doteq 32$$

$$32 - 1 = 31$$

$$32 + 1 = 33$$

These numbers really work:

$$31 \cdot 33 = 1023.$$

(3) Straight way, arithmetical mode II

Let us write number 1023 as a product of prime numbers. The decomposition makes the solution clear:

$$1023 = 3 \cdot 11 \cdot 31$$

$$1023 = 33 \cdot 31$$

All these solutions were coded as straight way, successfully.

(4) Systematic experimentation

The following shortened table 4 solves the problem by making a list of all possibilities (the so-called brute force method).

Table 4: Solving of the problem (the so-called brute force method) (source: own calculation)

First odd number	Second odd number	Product of numbers
1	3	3
3	5	15
5	7	35
...	...	...
29	31	899
31	33	1023

This solution was coded as the use of a heuristic strategy, use of technology, successfully.

To conclude this subsection, let us give by way of illustration the test for the age category 16–17 years.

1. The numbers that are read the same from left to right and right to left, e.g. 452 254, are called palindromes. My friend claims that all four-digit palindromes are divisible by 11. Is it true?

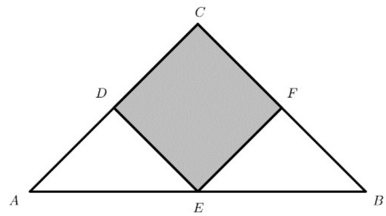

Figure 13: Visualization of the problem

2. Square $CDEF$ is inscribed into an isosceles triangle ABC (see Fig. 13). What is the area of the square if line segment AB is 8 cm long?
3. A car covered the distance of 420 km and used up 29 l of petrol. What was its average petrol consumption per 100 km?
4. The triangle ABC in Figure 14 has unit area. The points P, Q, R, S divide the sides AC and BC into three equal parts. What is the area of the colored quadrilateral?

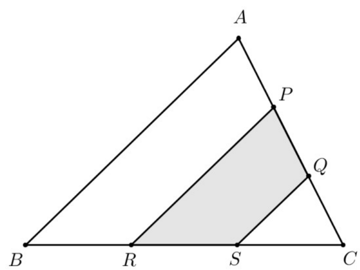

Figure 14: Visualization of the problem

4. Results

This section provides the results received from the analysis of pupils' written tests.

4.1. Primary hypothesis

The studied phenomena are the following:

- the success rate in solving the problems;
- the overall success rate in solving the problems using a heuristic strategy;
- the overall success rate when solving the problems in a straight way;
- the overall success rate in solving the problems using technology.

Let us note here that the last studied phenomenon is meaningful only when connected to the use of a heuristic strategy because a solution in a straight way corresponds to an algorithmic solution presented at school without using technology. We can state here that none of the pupils used the technology when solving the problem in a straight way.

4.1.1. Success rate in solving the problems

In this case, the observed phenomenon takes value from 0 to 342. The first bound corresponds to the situation when the problem is solved by none of the pupils. The latter bound corresponds to the situation when all pupils solved it successfully. Summary table 5 shows the overall success rate in the solution of the problem (i.e. regardless of whether the problem was solved in a straight way or using a heuristic strategy), χ^2 for McNemars' test with Yates correction (with 0.01 level of significance) and the corresponding p. Tables 5.1–5.4 show the data needed for table 5 construction.

Table 5: Overall problem solving success rate (source: own calculation)

	Pretest (Number)	Posttest (Number)	χ^2	p
Problem 1 (SE)	139	302	109.8	0.0000
Problem 2 (IAE)	171	205	10.27	0.0013
Problem 3 (AN)	142	201	36.17	0.0000
Problem 4 (SD)	40	177	114.88	0.0000

Table 5.1: Data for the problem 1 (SE) (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	101	38	139
Pretest – unsuccessful	201	2	203
Total	302	40	342

Table 5.2: Data for the problem 2 (IAE) (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	135	36	171
Pretest – unsuccessful	70	101	171
Total	205	137	342

Table 5.3: Data for the problem 3 (AN) (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	125	17	142
Pretest – unsuccessful	76	124	200
Total	201	141	342

Table 5.4: Data for the problem 4 (SD) (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	28	12	40
Pretest – unsuccessful	149	153	302
Total	177	165	342

We can formulate the corresponding null hypothesis H_0 for each problem as follows: The probability of improvement along with a decline between pre-test and post-test are equal.

The data in table 5 clearly show that the null hypothesis is rejected in all four cases. Thus, we can state that for each of the problems, there is a statistically significant increase in the success rate in problem solving. This fact will now be explained using the three phenomena mentioned above.

4.1.2. Overall success rate in solving problems using a heuristic strategy

In this case, the observed phenomenon takes value from 0 to 1368. The first bound corresponds to the situation when none of the 342 pupils was solving any of the assigned problems using a heuristic strategy, the other describes the situation when all problems were successfully solved by all pupils by some heuristic strategy. At the moment, we are watching the phenomenon as a whole, while we are not dealing with individual pupils or individual problems. Table 6 shows the data of the solutions using a heuristic strategy. χ^2 for McNemars' test with Yates correction (with 0.01 level of significance) is 352.07 and the corresponding p is 0.0000.

Table 6: Data of the success rate of solutions of problems using a heuristic strategy (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	44	2	46
Pretest – unsuccessful	360	962	1322
Total	404	964	1368

We can formulate the corresponding null hypothesis H_0 for the given phenomenon as follows: The probability of improvement along with a decline between pre-test and post-test is equal.

The data in table 6 clearly show that the null hypothesis must be rejected. Thus, we can state that the use of heuristic strategies has an impact on a statistically significant increase in the success rate in problem solving.

Table 7 presents the frequencies of use of different strategies and their success.

Table 7: Numbers of occurrence and successfully used heuristic strategies (source: own calculation)

	Pretest		Posttest	
	Occurrence	Successfully used	Occurrence	Successfully used
Problem 1 (SE)	58	27	178	174
Problem 2 (IAE)	80	8	124	67
Problem 3 (AN)	17	5	81	57
Problem 4 (SD)	59	6	191	106

Let us note here that the second (and third) columns in table 7 give spontaneous occurrence of heuristic strategies that had been used before actually starting the experimental teaching. Since the test tasks were selected in such a way that the selected heuristic strategy should be efficient way to solve them, the spontaneous occurrence of this way of solving had to be expected.

4.1.3. Overall success rate in solving the problems in a straight way

In this case, the observed phenomenon takes a value from 0 to 1368. The first bound describes the situation when none of the 342 pupils were solving any of the assigned problems in a straight way, and the latter bound describes the situation when all pupils were successfully solving all problems in a straight way. At the moment, we are watching the phenomenon as a whole, while we are not dealing with individual pupils or individual problems. Table 8 shows data of success rate in solving the problems in a straight way, χ^2 for McNemars' test with Yates correction (with 0.01 level of significance) is 5.09 and the corresponding p is 0.0240.

Table 8: Data of successful solution of problems in a straight way (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	350	96	446
Pretest – unsuccessful	131	791	922
Total	481	887	1368

The following null hypothesis H_0 can be formulated for this phenomenon: The probability of improvement along with a decline between pre-test and post-test is equal.

The data in table 8 do not allow rejection of the null hypothesis.

4.1.4. Overall success rate in solving the problems using technology

The last discussed phenomenon refers to the situation when a pupil was solving a problem using a heuristic strategy and used the technology while solving it. In addition, in this case, the studied phenomenon takes a value from 0 to 1368. The first bound describes the situation when none of the 342 was solving any of the assigned problems using technology, and the latter refers to the situation when all pupils were solving all problems using technology. At the moment, we are watching the phenomenon as a whole, while we are not dealing with individual pupils or individual problems. Table 9 shows the data for success rate in the solution using some heuristic strategy and technology, $\chi^2$for McNemars' test with Yates correction (with 0.01 level of significance) is 233.00 and the corresponding p is 0.0000.

Table 9: Data of successful solution of problems using technology (when a heuristic strategy was used) (source: own calculation)

	Posttest – successfull	Posttest – unsuccessful	Total
Pretest – successful	27	235	262
Pretest – unsuccessful	0	1106	1106
Total	27	1341	1368

The following null hypothesis H_0 can be formulated for this phenomenon: The probability of improvement along with a decline between pre-test and post-test is equal.

The data in table 9 clearly show the null hypothesis must be rejected. Thus, we can state that the use of technology (when a heuristic strategy is used) has an impact on a statistically significant increase in the success rate in problem solving.

Let us note here that the increase in the use of technology was most significant in the case of the strategies SE and SD. Table 10 shows the occurrence of using technology in relation to different strategies.

Table 10: Occurrence of technology (source: own calculation)

	pretest	Posttest
Problem 1 (SE)	25	172
Problem 2 (IAE)	0	7
Problem 3 (AN)	0	1
Problem 4 (SD)	2	82

4.2. Secondary hypothesis

The studied dependent variable is the phenomenon of no response. This phenomenon takes value from 0 to 1368. The first bound describes the situation when all 342 pupils started to solve all four assigned problems, and the latter describes the situation when none of the pupils started to solve any of the four assigned problems. At the moment, we are watching the phenomenon as a whole, while we are not dealing with individual pupils or individual problems. Table 11 presents the values from the pre-test and post-test.

Table 11: Number (and percentage) of no response (source: own calculation)

	Number	Percentage
Pretest	492	36%
Posttest	233	17%

The pre-test and the post-test comparisons were conducted using McNemars' test (with 0.01 level of significance) again.

The null hypothesis H_0 in this case reads as follows: The probability of improvement along with a decline between the pre-test and the post-test is equal.

Statistical evaluation allows us to state that $\chi^2 = 97.17$ if $p = 0.0000$ and so the null hypothesis H_0 has been rejected.

The secondary hypothesis holds and we can state that the pre-test and the post-test have proved a statistically significant decline in the share of pupils who do not even try to solve the problem.

5. Discussion

5.1. Interpretation of our results

5.1.1. Primary hypothesis

The primary hypothesis of our study states:

There will be a statistically significant increase in the success rate of problem solving between the pre-test and the post-test thanks to the use of heuristic strategies (when using the technology actively).

The results show that we cannot state that a straight way of solving has an impact on a statistically significant increase in problem solving.

This is caused by the fact that the number of straight way solutions increased, but only by 7%. We think this increase had to be expected and corresponds to the natural development of pupils over the given period of time.

Let us now interpret the phenomenon of using heuristic strategies and technology together because they were interrelated in the test to a certain extent.

Let us first compare the results from tables 7 and 10. The first that can be stated is a significant increase in frequency and success rate in the use of the strategies SE, AN and SD. If we focus on table 10, we can also state that in the case of strategies SE and SD this goes hand in hand with an increase in the use of technology while solving the problems.

Within the strategy SE, the reason would be its algorithmic character. The post-test analyses have shown that all solvers used a spreadsheet. The reason for the high success rate of solvers while using the strategy SE is the fact that tasks solved using this strategy can be identified more easily in comparison to other types of tasks.

Considering the strategy SD, we assume the increase in the use of this strategy may signal a deeper understanding of the problem in the sense of visual thinking, as used by Nelsen (1993; 2000). It is the use of technology in problem solving through strategy SD, thanks to which the pupils were able to experiment easily using a dynamic geometry system, that we believe could have been the reason for a deeper understanding of the problem.

The strategy AN was used by the fewest pupils both in the pre-test and the post-test. We assume the reason for this to be the fact that the assignments together with the character of the effectively solvable problem through the use of a strategy AN are not particularly invited to apply the strategy, which is the case of the other strategies. The set of problems effectively solvable through the use of the strategy AN is much more variable compared to the problem effectively solvable through the use of the three remaining strategies (Přibyl, Eisenmann, 2014). It can be stated that the use of technology while solving problem 3 was marginal. The analysis of written tests showed that pupils were able to use the strategy AN successfully (tab. 4) without using technology (tab. 7), which contradicts our expectations.

It can also be stated that the use of technology while solving problem 2 was marginal. The increase in the successful use of the strategy IAE (tab. 4) may be the outcome of the experimental teaching where the active use of a dynamic geometry system when solving problems using the strategy IAE enabled pupils to get a feel for the active use of the proposed auxiliary elements as was shown in the analysis of the post-test.

5.1.2. Secondary hypothesis

The secondary hypothesis of our study states:

There will be a statistically significant decline in the frequency of no response (a situation when the pupils do not even start solving the problem) between the pre-test and the post-test.

This hypothesis corresponds to our assumption that a pupil's success, when solving problems using a heuristic strategy will bring about a positive change in the pupil's attitude toward problem solving in general. The pupil will not give up the solving process at the beginning even in case they do not know the needed solving algorithm because they gain experience with heuristic strategies. Technology gives them the chance to experiment and to at least attempt to solve the problem using one of the here discussed strategies.

The results show (see subsection 4.2) that the secondary hypothesis holds. However, it is hard to say whether this was as a consequence of the experimental teaching or through the natural development of the pupils. Still, it can be assumed that the long-term experience of using suitable heuristic strategies played a role in the pupils' decision at least to try the solution.

5.2. Relations of study results with theory and practice

Understanding the fact that solving problems plays an irreplaceable role in teaching mathematics, this issue has been reflected in other individual fields of didactics of mathematics. Each of the fields accentuates a different sphere of education and each approaches the outcomes of our research discussed here in a different way.

Constructivism also covers the *Theory of didactical situations*, which is supported by the authors. The changes in approaches that have been mentioned to have had a positive impact on the experimental teaching have become a part of the didactical contract and they are, among others, described in (Artigue et al., 2014).

The introductory subsection refers to the curricular document (NCTM 2000), where the potentials of using technology to accelerate and to conduct calculations along with visualizing ideas have been described. The results of the conducted study show that one of the possibilities of exploiting the potential of technology is to use the technology while solving problems by some selected heuristic strategies.

Another important curricular document (ACARA, 2015) has formulated the targets of using technology in teaching mathematics: "Pupils develop ICT capability when they investigate, create and communicate mathematical ideas and concepts using fast, automated, interactive and multimodal technologies. They employ their ICT capability to perform calculations, draw graphs, collect, manage, analyse and interpret data." (p. 12) Our research outcomes prove that problem solving by selected heuristic strategies is an appropriate means to fulfil the above-mentioned targets.

The National Research Council has specified that the term Mathematical proficiency is traditionally formed by four components: "procedural fluency – knowing how and when to apply procedures, conceptual understanding – holding deep and rich connections among ideas, adaptive reasoning – the capacity to reason logically and to justify one's reasoning, and strategic competence – formulating, representing, and solving problems". (Philipp, Siegfried, 2015, p. 489) The results presented here show that the use of technology while solving problems with the use of the strategies SE and SD has contributed to the development of mathematical proficiency in pupils, especially in the field of procedural fluency.

Using technology while solving problems with heuristic strategies may be considered as one of the possible forms of scaffolding. It is a field given special attention in the present days (e.g. Bakker et al., 2015; van Oers, 2014). Our research has brought another dimension into the area, encouraging positive attitudes in pupils to solving problems in such cases when they are not clear about the solution procedure (the algorithm to be used to deal with the task successfully). Using technology allows them to benefit from applying some heuristic strategies. Pupils can overcome, for example, the obstacles caused by the need to tackle technically challenging difficult mathematics calculations or a large number of analogy calculations together with the necessity to draw accurately or measure the length of the line segment, the sizes of angles or the area of geometric forms. If we understand scaffolding as a set of supportive measures efficiently helping pupils gain new knowledge and skills, then use of technology is one of these means of support allowing to overcome the above-mentioned obstacles.

It is necessary to emphasize that we do not expect that using technology while solving mathematical problems should become a universal tool for solving word problems. However, we recommend giving students the opportunity to become familiar with different methods and to choose a suitable method or combination of methods for themselves. In accordance with (Jančařík, Novotná, 2011) the results gained using a computer must be further interpreted and, in many cases must be set into a broader framework using other solving methods.

The results of this research can be used in teacher education. It is essential that future teachers become familiar with problem solving using heuristic strategies as well as the use of suitable technology in solving problems using these strategies. Without this experience, teachers are not likely to encourage their pupils to use technology.

5.3. Limitations of our research

To conclude this chapter, let us have a look at some limitations we are aware of. These limitations have to be taken into account when interpreting the results, or when planning a new research study.

5.3.1. Selected heuristic strategies

A certain limitation of the presented study is its restriction to the discussed four heuristic strategies. These strategies were selected because, in our opinion, they have the greatest potential for the use of technological devices within the solving process. In the case of the strategy SE, the spreadsheet seems to be the choice number one to make a list of all possible solutions. The nature of the strategies IAE and SD in the case of geometrical problems encourages experimentation with dynamic geometry systems. One of the most efficient uses of the strategy AN is the formulation of a problem with more "user-friendly" objects, for example, smaller numbers.

The nature of other heuristic strategies (guess – check – revise, problem reformulation, working backwards, generalization and specification, use of false assumptions, decomposition into simpler cases, omitting a condition) makes them less suitable for the use of technological devices within the solving process (Přibyl, Eisenmann, 2014).

5.3.2. Research sample

The 14 classes were selected according to the teacher, who had been chosen using the above-listed criteria (see subsection 3.4). Let us now discuss the limitations this selection implies:

The participating teachers are regarded as comparable. This "comparability" is based on the fact that we got to know them in the in-service teacher training courses. However, there might be differences between them – both in the degree of their involvement and the quality of teaching. We regard these differences as the first limitation of our study.

Another limitation is the "comparability" of the fourteen classes. However, they are all ordinary classes with no specialization, the classes do not integrate pupils with major learning difficulties, and each class as a whole may be seen as average with respect to its performance and discipline.

5.3.3. The research instrument

Selection of only one tool for the evaluation of our research hypotheses may also be seen as a limitation. This tool was the written test. However, we are convinced that for the needs of verification of our research hypotheses, this tool was sufficient.

Furthermore, the choice of a quasi-experiment without a control group might be perceived as a limitation to our study. However, having thought about it thoroughly at the beginning of the research and taking into account our previous experience, we decided on this form of experiment, mainly because of the difficulty to secure the comparability of independent variables during the 10-month teaching experiment in both groups.

5.3.4. Used technology

The selection of three types of technological devices used in the experimental teaching (computer, laptop, and tablets) does not comply with the currently frequently used technologies (e.g. smartphones).

6. Conclusions

Let us focus on the pupils' success rate in the solution of problems effectively solvable using the strategies systematic experimentation and solution drawing. The results of the study show that as a consequence to the 10-month experimental teaching of the four here discussed heuristic strategies, the success rates improved significantly if the use of these strategies went hand in hand with the active use of technological devices.

In the case of the strategy IAE, the problem solving success rate was supported by the experimental teaching within whose frame technology was used. However, the use of technology in the testing itself was marginal.

The changes could also be observed in pupils' attitude to problem solving. The fact that the number of cases when pupils gave up the solution decreased to one half, is undoubtedly very positive.

The results of our research (Eisenmann et al., 2015) have shown the opportunities for successful use of heuristic strategies in mathematics classrooms in primary and secondary schools and the results given in this article stress the positive impact of the use of technology with four selected heuristic strategies. We assume that pupils should have a chance to actively use the opportunities that are brought into classrooms through technology.

In the final part of this section, we propose how this study can be further developed. A few other directions opening in front of us may be traced. The first opportunity is to focus on other heuristic strategies than those that have been the target of our research. It is feasible to see that heuristic strategies offer such types where the use of technology is not only possible, but also appropriate.

The second point of view comes from a compact research oriented to pupils with disabilities (Hasselbring et al., 2006). Technology is an irreplaceable means enabling education with a number of disabilities, speaking of both, sensual disability (e.g. sight impairment) and functional disability (e.g. dyspraxia, dyscalculia).

The third possible direction of further research is the verification of results concerning the description of changes in attitudes of pupils towards problem solving and that through other means focusing on the pupils as such – an analysis of observing pupils. This analysis could be supplemented by structured interviews with teachers and a questionnaire survey for pupils.

The last direction opens in the field of using mobile technological devices in teaching mathematics, considering classical classroom teaching or out-of-school education.

To conclude, we can say that the limited results of the here presented research study show that the use of technology in mathematics lessons as such, specifically at different levels of the whole educational process, is meaningful. In the first place, it should be used in primary and secondary school lessons not only in problem solving but also in inquiry-based learning. It proved to be very useful to introduce in-service and pre-service teachers into the potential of the use of technology in their connection with the use of heuristic strategies in problem solving. If in-service and pre-service teachers have needed information about the potential of the use of technology in education, they may put pressure on policymakers to support actively efficient use of technology in school practice.

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References

ACARA.: 2015, *The Australian curriculum: Version 7.5.* Retrieved from: https://australiancurriculum.edu.au/f-10-curriculum/mathematics/

Abdu, R., Schwarz, B., Mavrikis, M.: 2015, Whole-class scaffolding for learning to solve mathematics problems together in a computer-supported environment. *ZDM Mathematics Education, 47*, 1163–1178. https://doi.org/10.1007/s11858-015-0719-y

Artigue, M., Haspekian, M., Corblin-Lenfant, A.: 2014, Introduction to the theory of didactical situations (TDS). In A. Bikner-Ahsbahs, S. Prediger

(Eds.), *Networking of theories as a research practice in mathematics education: Authored by the networking theories group* (pp. 47–65). Cham Heidelberg New York Dordrecht London: Springer International Publishing Switzerland. https://doi.org/10.1007/978-3-319-05389-9_4

Bakker, A., Smit, J., Wegerif, R.: 2015, Scaffolding and dialogic teaching in mathematics education: introduction and review. *ZDM Mathematics Education, 47*, 1047–1065. https://doi.org/10.1007/s11858-015-0738-8

Brousseau, G., Novotná, J.: 2008, La culture scolaire des problèmes de mathématiques. In: B. Sarrazy (Ed.), *Les didactiques et leurs rapports à l' enseignement et à la formation. Quel statut épistémologique de leurs modèles et de leurs résultats?* Bordeaux : AFIRSE, IUFM d'Aquitaine - Université Montesquieu Bordeaux IV, LACES – Université Victor Segalen Bordeaux 2. [CD ROM]

Bu, L., Haciomeroglu, E.S., Mumba, F.: 2011, Mathematical problem solving in dynamic learning environments: The case of GeoGebra. In D. Martinovic, Z. Karadag, D. McDougall (Eds.), *GEOGEBRA – NA 2011, Proceedings of the second North American GeoGebra conference: Where mathematics, education and technology meet* (pp. 20–31). Toronto (ON): University of Toronto.

Chaamwe, N., Shumba, L.: 2016, Spreadsheets: A tool for e-Learning – a case of matrices in Microsoft Excel. *International Journal of Information and Education Technology, 6*, 570–575. https://doi.org/10.7763/ijiet.2016.v6.753

Cherry, K.: 2021, March 15. *How does self-determination theory explain motivation?* Verywell Mind. Retrieved February 1, 2022, from https://www.verywellmind.com/what-is-self-determination-theory-2795387

Dirgha, R.J.: 2017, Influence of ICT in Mathematics Teaching. *International Journal of Innovative Research, Growth*, 3, 7–11.

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., Reed, H.: 2013, Design research in mathematics education: The case of an ICT-rich learning arrangement for the concept of function. In T. Plomp, N. Nieveen (Eds.), *Educational design research – Part B: Illustrative cases* (pp. 425–446). Enschede, the Netherlands: SLO.

Drier, H.S.: 2001, Teaching and learning mathematics with interactive spreadsheets. *School Science and Mathematics, 101*, 170–179. https://doi.org/10.1111/j.1949-8594.2001.tb18020.x

Drijvers, P.: 2015, Digital technology in mathematics education: Why it works (or doesn't). In S.J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). https://doi.org/10.1007/978-3-319-17187-6_8

Eisenmann, P., Novotná, J., Přibyl, J., Břehovský, J.: 2015, The development of a culture of problem solving with secondary students through heuristic strategies. *Mathematics Education Research Journal*, *27*(4), 535–562.

European Commission, Thematic Working Group on Mathematics, Science and Technology (2010 – 2013). *Final report addressing low achievement in mathematics and science*. Available from http://ec.europa.eu/dgs/education_culture/repository/education/policy/strategic-framework/archive/documents/wg-mst-final-report_en.pdf [July 22, 2018]

Eysenck, M.W.: 2001, *Principles of cognitive psychology*. Hove: Psychology Press.

Eysenck, M.W., Keane, M.T.: 2010, *Cognitive psychology: A student's handbook*. Hove: Psychology Press.

Fan, L., Zhu, Y.: 2007, Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics, 66*, 61–75. https://doi.org/10.1007/s10649-006-9069-6

Guerrero-Ortiz, C., Reyes-Rodriguez, A., Espinosa-Perez, H.: 2015, Integrating synthetic and analytical aspects of geometry through solving problems using a DGS. In L. Uden, D. Liberona, T. Welzer (Eds.), *Learning technology for education in cloud: 4th international workshop, LTEC 2015 Maribor, Slovenia, August 24–28, 2015 Proceedings* (pp. 283–297). Cham Heidelberg New York Dordrecht London: Springer International Publishing Switzerland.

Hasselbring, T.S., Lott, A.C., Zydney, J.M.: 2006, *Technology-supported math instruction for students with disabilities: Two decades of research and development*. Washington, DC: CITEd, Center for Implementing Technology in Education.

Hertwig, R., Pachur, T.: 2015, Heuristics, History of. In *International encyclopedia of the social & behavioral sciences*, 2nd edition, Volume 10 (pp. 829–835). https://doi.org/10.1016/B978-0-08-097086-8.03221-9

Huang, N.-T.N., Chiu, L.-J., Hong, J.-Ch.: 2015, Relationship among pupils' problem-solving attitude, perceived value, behavioral attitude, and intention to participate in a science and technology contest. *International Journal of Science and Mathematics Education, 14*(8). 1419–1435. http://dx.doi.org/10.1007/s10763-015-9665-y

Iranzo, N., Fortuny, J.M.: 2011, Influence of GeoGebra on problem solving strategies. In L. Bu, R. Schoen (Eds.), *Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra* (pp. 91–103). AW Rotterdam, The Netherlands: Sense Publishers. https://doi.org/10.1007/978-94-6091-618-2_7

Jančařík, A., Novotná, J.: 2011, "For show" or efficient use of ICT in mathematics teaching? In M. Joubert, A. Clark-Wilson, M. McCabe (Eds.), *Proceedings of the 10th International Conference on Technology in Mathematics Teaching* (pp. 166–171). Portsmouth: University of Portsmouth.

Jeřábek, J., Lisnerová, R., Smejkalová, A., Tupý, J. (Eds.).: 2013, *Rámcový vzdělávací program pro základní vzdělávání [The framework educational programme for basic education].* Prague: The Ministry of Education, Youth and Sports.

Jezdimirović, J.: 2014, Computer based support for mathematics education in Serbia. *International Journal of Technology and Inclusive Education, 3*(1), 277–286. https://doi.org/10.20533/ijtie.2047.0533.2014.0036

Koyuncu, I., Akyuz, D., Cakiroglu, E.: 2014, Investigating plane geometry problem-solving of prospective mathematics teachers in technology and paper-and-pencil environments. *International Journal of Science and Mathematics Education, 13*, 837–862. https://doi.org/10.1007/s10763-014-9510-8

Kuzle, A.: 2017, Delving into the nature of problem solving processes in a dynamic geometry environment: Different technological effects on cognitive processing. *Technology, Knowledge, and Learning, 22*(1), 37–64. https://doi.org/10.1007/s10758-016-9284-x

Marley-Payne, J., Dituri, P.: 2019, Spreadsheets as an Effective Use of Technology in Mathematics Education. *Spreadsheets in Education, 12*(1), 2–26.

Mavrou, K., Meletiou-Mavrotheris, M.: 2014, Flying a Math Class?: Using Web-Based Simulations in Primary Teacher Training and Education. In S. Mukerji, P. Tripathi (Eds.), *Handbook of Research on Transnational Higher Education* (pp. 391-417). IGI Global. https://doi.org/10.4018/978-1-4666-4458-8.ch020

NCTM.: 2000, *Principles and standards for school mathematics.* Reston, VA: The National Council of Teachers of Mathematics, Inc.

NCTM.: 2014, Principles to Actions: Ensuring Mathematical Success for All. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Nelsen, R.: 1993, *Proofs without words: Exercises in visual thinking.* Washington, D.C: Mathematical Association of America.

Nelsen, R.: 2000, *Proofs without words II: More exercises in visual thinking.* Washington, DC: Mathematical Association of America.

Novotná, J.: 2000a, Students' Levels of Understanding of Word Problems. In H. Fujita (Ed.), *ICME 9. Abstracts of Plenary and Regular Lectures* (pp. 96–97). Tokio: ICME 9. https://doi.org/10.1007/1-4020-7910-9_43

Novotná, J.: 2000b, *Analýza řešení slovních úloh [Analysis of word problems solutions].* Prague: Charles University in Prague.

OECD.: 2006, *Education at a glance 2006: Highlights*. Available from https://www.oecd.org/education/skills-beyond-school/37376068.pdf [July 20, 2018]

OECD.: 2014, *PISA 2012 results: What pupils know and can do: Pupil performance in mathematics, reading and science: Volume I*. Paris: OECD Publishing. https://doi.org/10.1787/9789264208780-en

Philipp, R.A., Siegfried, J.M.: 2015, Studying productive disposition: The early development of a construct. *Journal of Mathematics Teacher Education, 18*(5), 489–499. https://doi.org/10.1007/s10857-015-9317-8

Pólya, G.: 1973, *How to solve it: A new aspect of mathematical method*. Princeton: Princeton University Press.

Přibyl, J., Eisenmann, P.: 2014, Properties of problem solving strategies. In M. Houška, I. Krejčí, M. Flégl (Eds.), *Proceedings of Efficiency and Responsibility in Education 2014* (pp. 623–630). Prague: Czech University of Life Sciences.

Přibyl J., Eisenmann P., Gunčaga, J.: 2018, The Phenomenon of False Assumption in Historical and Educational Texts. *Science & Education, 27*(7), 737–767.

Reeve, J.: 2002, Self-determination theory applied to educational settings. In E.L. Deci, R.M. Ryan (Eds.), *Handbook of self-determination research* (pp. 183–203). University of Rochester Press.

Samuelsson, J.: 2008, The impact of different teaching methods on students' arithmetic and self-regulated learning skills. *Educational Psychology in Practice, 24*(3), 237–250. https://doi.org/10.1080/02667360802256790

Sarama, J.A., Clements, D.H.: 2009, *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Stacey, K.: 1991, The effects on students' problem solving behavior of long-term teaching through a problem solving approach. In F. Furinghetti (Ed.), *Proceedings of the 15th conference of the International group for the psychology of mathematics education: Vol. 3* (pp. 278–285). Assisi: IGPME.

Tao, T.: 2006, *Solving mathematical problems: A personal perspective*. Oxford: Oxford University Press.

Trouche, L., Drijvers, P.: 2010. Handheld technology for mathematics education: flashback into the future. *ZDM, 42*(7), 667–681. https://doi.org/10.1007/s11858-010-0269-2

van Oers, B.: 2014, Scaffolding in mathematics education, In S. Lehrman (Ed.), *Encyclopedia of mathematics education*, (pp. 535–538). Dordrecht Heidelberg New York London: Springer Science+Business Media Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_136

Uzunboylu, H., Tugun, V.: 2016, Validity and reliability of tablet supported education attitude and usability scale. *Journal of Universal Computer Science, 22*(1), 82–93.

Villavicencio, F.T., Bernardo, A.B.I.: 2015, Beyond math anxiety: Positive emotions predict mathematics achievement, self-regulation, and self-efficacy. *The Asia-Pacific Education Researcher, 25*(3), 415–422. https://doi.org/10.1007/s40299-015-0251-4

Yeh, C.Y., Cheng, H.N., Chen, Z., Liao, C.C., Chan, T.: 2019, Enhancing achievement and interest in mathematics learning through Math-Island. *Research and Practice in Technology Enhanced Learning, 14*(5). https://doi.org/10.1186/s41039-019-0100-9

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