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Modernize the teaching of mathematics at the secondary level through R programming*

Abstract. Teaching mathematics using programming has cognitive benefits, improves practical skills, and provides powerful independent thinking and problem-solving strategies. R is a free software environment and one of the most popular programming tools, which allows for solving various mathematical problems through coding. In this work, we will use the R language to solve some problems posed in complex numbers and cover some exercises from the third-grade mathematics textbook in Algeria.

1. Introduction

Information and Communication Technologies (ICT) provide various important tools for teaching and learning in all disciplines, including mathematics, as they promote a more active pedagogical approach and provide significant additional skills to learners. The involvement of technological learning tools in mathematics education produces powerful new teaching and learning experiences for learners and simultaneously establishes active connections between mathematical content and reflection; in this way, technology, placed in an effective methodological framework, becomes a moderator of mathematics learning.

Programming started as a branch of mathematics in high school education, but all too quickly it took over and thus became a foundational discipline in the field of computer science (Ana, 2021; Campbell-Kelly, 2018). At its core, learning programming is about learning a computer language and learning a way of thinking. Developing a programming mindset is something that is missing in our traditional classrooms, and the role of a programming mindset is multifaceted. In addition to improving practical skills, and acquiring powerful independent thinking and problem-solving strategies, learning programming also develops computational

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thinking and strengthens mathematical skills from obtaining the solution to the problem and rethinking the solution in an alternative and accurate way. This instruction can also be used to provide students with a precise understanding of some key mathematical concepts (Briz-Redon, Serrano-Aroca, 2018). Many authors argue that learning mathematics using programming has cognitive benefits, starting with the book of Papert (Papert, 1980) who discusses the need for children to learn programming, he did however illustrate some of the peculiarities of LOGO programming, which is designed to provide very early and easy entry routes into programming for beginners without prior mathematical knowledge. Furthermore, there are several programming languages classified according to the levels or the age of the learner, for example, Scratch software for learners aged from about 8 or 9 to 13 or 14 years, it is a tool dedicated to beginners in programming, allowing the creation of educational content related to mathematics, ready to use as a teaching aid. Iskrenovic-Momcilovic (Iskrenovic-Momcilovic, 2020) has examined the relevance of the practice of Scratch in the teaching of mathematics, in particular the basic geometric shapes in the classroom, the experimental study shows that the results of the students who used the programs made in Scratch are better than those who learned the basics of geometry based on the traditional method. Finally, the author highlights the role of this environment in making mathematics more attractive to learners. Now, let us go to more advanced and complicated programming software as even, which is dedicated to a higher level and intervenes from the high school such as R, which we will use in this study, Matlab and Python. Obradovic et al (Obradovic, Mishra, Sharma, Mishra, 2021) highlighted the most commonly used commercial tools in mathematics education such as Mathematica, MatLab, GeoGebra, Octava, and Scilab. Brayan et al (Brayan, Natalia, Alberto, 2018) used MATLAB and GeoGebra software in the teaching of mathematics, considering the representations of the basic operations (addition, subtraction, multiplication, and division) and complementary operations (modulus and conjugate, roots, and polar form) of the numerical system of the complex numbers. However, complex numbers are a very important part of mathematics, it is a combination of algebra and geometry. In addition, understanding complex numbers, performing arithmetic operations, solving complex equations, and representing them geometrically in the complex plane is one of the essential standards of high school mathematics (Chavez, 2014). The usefulness of complex numbers lies in the fact that they have been widely used in a wide variety of fields for a long time.

2. Methodology

The central objective of this article is to provide teachers with techniques to strengthen the learning process of students, we will focus on those aspects that are most needed in secondary school, especially the calculus, the development of basic and complementary operations, solving equations, and geometric representation of complex numbers via the software R, the cause of selection this environment is that it plays a very important role in the academic career of the learner.

2.1. R environment (Micheaux, Drouilhet, Liquet, 2014)

R is a statistical software invented in 1996 (Ihaka, Gentleman, Liquet, 1996), that is a computer language and working environment, it is free and open-source. It runs under different operating systems. It is also an extremely powerful and extensive tool, suitable for computer implementation of the statistical approach.

2.2. Some commands R used in this note (Micheaux, Drouilhet, Liquet, 2014)

`plot()`: the standard function for plotting graphs.

`function()`: allows programming the new functions appropriate to our needs.

`text()`: allows to allocate a text to the point that was plotted.

`points()`: allows superimposing the new graph on the old one.

`abline()`: allows drawing horizontal and vertical lines.

`segments()`: allows joining points with line segments that are added to a pre-existing graphic.

`polygon()`: allows drawing polygons.

`for` loop: is a control structure intended to execute a portion of code several times in a row, as long as an output condition is not satisfied or a pre-specified number of loops has not been reached.

`polar()`: this function obtains polar coordinates, plots them in a Cartesian plane, and displays the polar structure on the plane.

`polyroot()`: is used to find a polynomial's roots.

3. Results

3.1. The complex numbers on R

```
\# To create a complex number z=-2+3i we use the function:
\# complex(length.out, real, imaginary, modulus, argument)
z ← complex(real = -2, imaginary = 3)
\# For the graphical representation of z we use the function
  plot(x, y, \^{a}\euro)
\# To get the coordinates (x, y) of z we use the functions Re,
  Im
plot(x = Re(z), y = Im(z), type = 'p', pch = 19, xlab = 'Re(
  z)', ylab = 'Im(z)', xlim = c(-3.5, 3.5), ylim = c(-3.5,
  3.5), main = 'Complex number z = -2 + 3i')
\# To plot the label of the complex number z we use the
  function:
\# text (x, y, labels, adj, pos, offset, vfont, cex, col, font
  , â€¦)
text(x = Re(z), y = Im(z), labels = 'z', pos = 4, offset = 0
  .7)
\# To draw the coordinate system axes we use the function:
\# abline(a, b, h, v, reg, coef, untf, ...)
abline(h = 0, v = 0)
```

[172]

Amine Amimour, Lynda Benbessai, Kada Becharef, Hafida Mokhenfer

```
\# To draw a segment between the center (0,0) and the complex
number z we use:
\# segments(x0, y0, x1, y1, col, lty, lwd, â€¦)
segments (x0 = 0, y0 = 0, x1 = Re(z), y1 = Im(z))
```

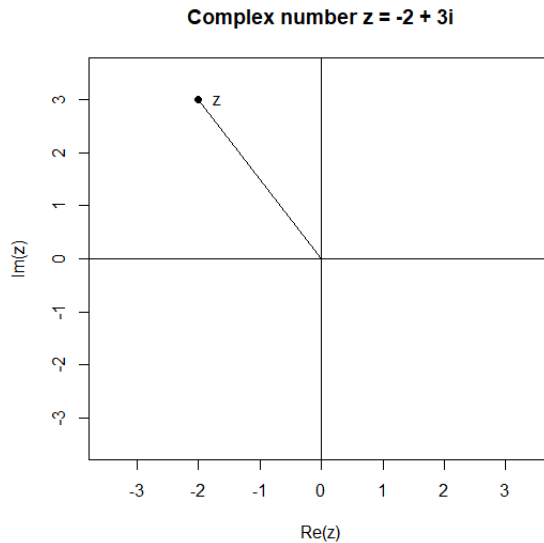


Figure 1: Plotting complex number z

3.2. Basic operations on complex numbers

In this section, we define and present how to manipulate the basic operations, the modulus, the conjugate, the roots, and the polar form.

3.2.1. Addition, subtraction, and multiplication

Let $z_1 = 2 - 3i$, $z_2 = 4 - 5i$, to subtract, add or multiply the two numbers z_1 and z_2 , we use directly the mathematical symbols ($- + * /$) for example for the subtraction :

```
z1 ← complex(real = 2, imaginary = 3)
z2 ← complex(real = -4, imaginary = 5)
\# To calculate the subtraction of the two numbers z1, z2 we
use z3 ← z1-z2
plot(x = Re(z1), y = Im(z1), type = 'p', pch = 19, xlab = ''
Re(z)'', ylab = ''Im(z)'', xlim = c(-10, 10), ylim = c(-10,
10), main = '' Substruction complex number z3 = z1-z2 '')
abline(h = 0, v = 0)
\# For the graphical representation of z1, z2 and z3 in the
same graph we use the function:
```

```

\# points(x, y = NULL, type = 'p',pch=19, col = 'red', ...
)
points(x = Re(z1), y = Im(z1),pch=19, col = 'red')
text(x = Re(z1), y = Im(z1), labels = 'z1', col = 'red',
     pos = 4, offset = 0.7)
points(x = Re(z2), y = Im(z2),pch=19, col = 'red')
text(x = Re(z2), y = Im(z2), labels = 'z2', col = 'red',
     pos = 4, offset = 0.7)
points(x = Re(z3), y = Im(z3),pch=19, col = 'blue')
text(x = Re(z3), y = Im(z3), labels = 'z3', col = 'blue',
     pos = 4, offset = 0.7)

```

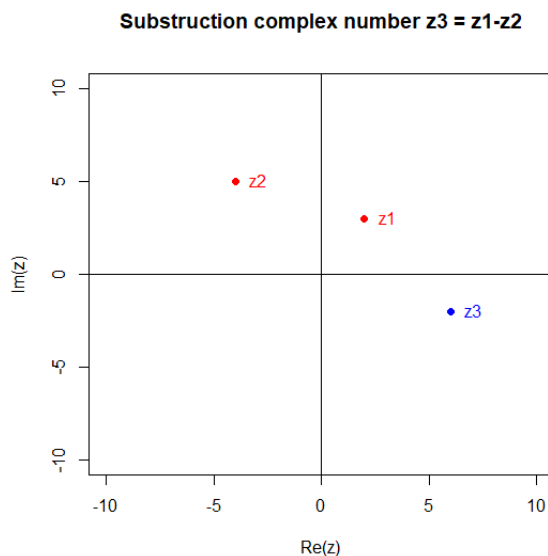


Figure 2: Subtraction of two complex numbers

3.3. Complementary operations on complex numbers

3.3.1. Modulus, conjugate, and argument

Modulus

```

z = -2+1i
plot(x = Re(z), y = Im(z), type = 'p', pch = 19, xlab = 'Re(
  z)', ylab = 'Im(z)', xlim = c(-3, 3), ylim = c(-3, 3),
     main = 'Modulus of complex number |z|')
abline(h = 0, v = 0)
segments(x0 = 0, y0 = 0, x1 = Re(z), y1 = Im(z))
points(x = Re(z), y = Im(z), pch = 19);
\#To draw a segment between the point of the complex number z
and the ordinate axis we use

```

```

segments(x0 = Re(z), y0 = 0, Re(z), Im(z), lty = 3)
\#To draw a segment between the point of the complex number z
and the ordinate axis we use
segments(x0 = 0, y0 = Im(z), x1 = Re(z), y1 = Im(z), lty=3)
\#To plot M in front of the number point z we use
text(x = Re(z), y = Im(z), labels = 'M', pos = 4, offset = 0
.7)
\# To plot Mod(z) in front of the module segment we use
text(x = -1.2, y = 0.4, labels = 'Mod(z)', pos = 4, offset
= 0.7, srt = Arg(z)*390/pi)

```

Conjugate

```

plot(x = Re(z), y = Im(z), type = 'p', pch = 19, xlab = 'Re(
z)', ylab = 'Im(z)', xlim = c(-3, 3), ylim = c(-3, 3),
main = 'Conjugate of complex number')
abline(h = 0, v = 0)
segments(x0 = 0, y0 = 0, x1 = Re(z), y1 = Im(z), lty=3)
points(x = Re(Conjz), y = Im(Conjz), pch = 19)
text(x = Re(Conjz), y = Im(Conjz), labels = 'M', pos = 4,
offset = 0.7)
\#To get the conjugate of the number z we use
Conjz=Conj(z)
segments(x0 = 0, y0 = 0, x1 = Re(Conjz), y1 = Im(Conjz))
points(x = Re(Conjz), y = Im(Conjz), pch=19)
text(x = Re(Conjz), y = Im(Conjz), labels = 'M', pos = 4,
offset = 0.7)
\#To draw points between the points M and M' which show the
symmetry of the two numbers z and it conjugate we use
segments(x0 = Re(z), y0 = Im(z), x1 = Re(Conjz), y1 = Im(
Conjz), lty=3)

```

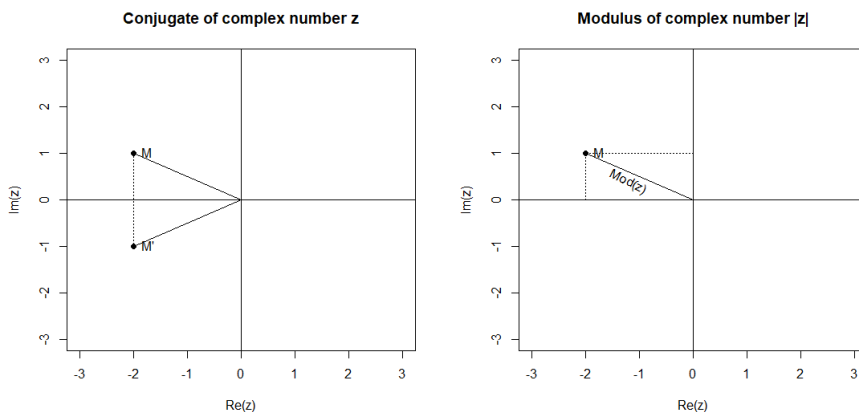


Figure 3: Modulus and conjugate of a complex number

Argument

For the argument of complex numbers, we can use the following commands:

```
plot(x = Re(z), y = Im(z), type = 'p', pch = 19, xlab = 'Re(z)',
     ylab = 'Im(z)', xlim = c(-2.5, 0), ylim = c(0, 2.5),
     main = '') Argument of complex numbers')
abline(h = 0, v = 0)
segments(x0 = 0, y0 = 0, x1 = Re(z), y1 = Im(z))
points(x = Re(z), y = Im(z), pch=19)
abline(h = 0, v = 0)
segments(x0 = Re(z), y0 = 0, x1 = Re(z), y1 = Im(z), lty=3)}
segments(x0 = 0, y0 = Im(z), x1 = Re(z), y1 = Im(z), lty
=3)}
text(x = Re(z), y = Im(z), labels = 'z', pos = 4, offset = 0
.7)}
r ← -0.5}
x ← seq(from = Re(-r*exp(1i*Arg(z))), to = r, length = 100)}
y ← sqrt(0.5^2-x^2) }
points(x,y,type = 'l')
text(-0.60,0.16,'Arg(z)',srt = 65,cex = 0.8)}
```

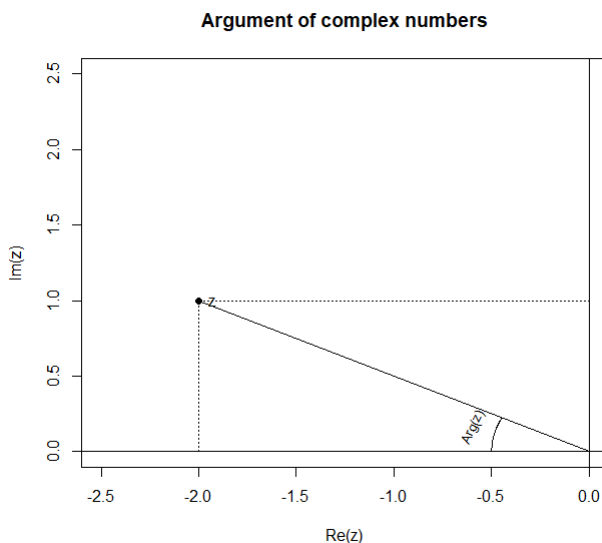


Figure 4: Argument of complex numbers

3.3.2. Polar form

The polar form of a complex number can be observed using the polar function of R. To do this, we will first install the "pracma" library which allows us to use the polar function as follows:

[176]

Amine Amimour, Lynda Benbessai, Kada Becharef, Hafida Mokhenfer

```
install.packages('pracma')
library('pracma')
z = -1 + 4i
r = Mod(z)
\# To find the argument and angle we use the function:
Theta = Arg(z)
\#To plot the polar coordinates in a Cartesian plane we use
the function:
polar(Theta,r, col = 'red')
points(r,Theta, pch = 19, col = 'red')
```

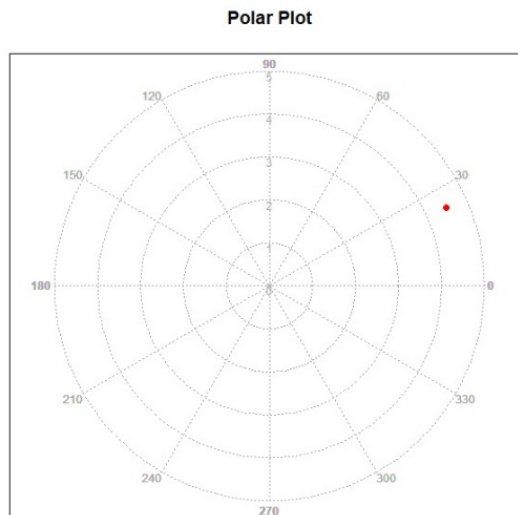


Figure 5: The polar form of complex number z

3.3.3. nth Root

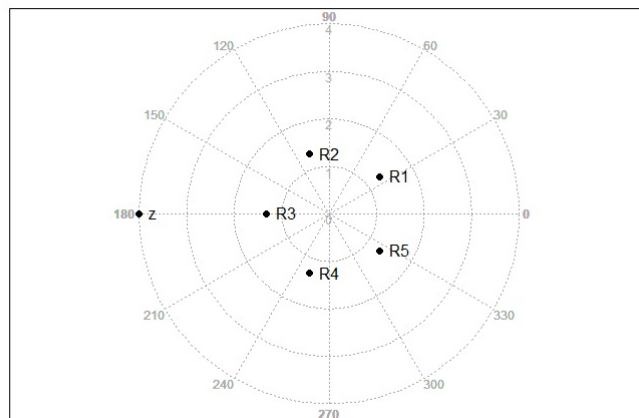
```
Roots = 0
z = -4 + 0i
n = 5
a = Mod(z)
b = Arg(z)
polar(b,a, col='red', main = 'The 5th Roots of z=-4+0i in
Polar Form')
points(x = Re(z), y = Im(z), pch = 19)
text(x = Re(z), y = Im(z), labels = 'z', pos = 4, offset = 0
.7)
for(k in 0 : (n-1)){
```



```

parameter = (b+2*k*pi)/n
Roots[k+1] = a^(1/n)*(cos(parameter) + 1i*sin(parameter))
points(Roots[k+1], pch = 19)
text(x = Re(Roots[k+1]), y = Im(Roots[k+1]),
\#To concatenate all strings in a character vector we use the
function
\#strcat(s1, s2 = NULL, collapse = '' '')
\# To generates a string representation of a provided argument
we use the function as.character()
strcat(c('R',as.character(k+1))), pos = 4){
\#Remark: strcat(c(collapse='R',as.character(k+1))) allow to
plot the roots R.

```

The 5th Roots of $z=-4+0i$ in Polar FormFigure 6: The 5th roots of complex number z

3.4. Application exercises

Exercise 19 page 145 (Ihaka, Gentleman, Liqueur, 2015)

Let A,B,C et D four points of affixes $-2+i$, $-1+4i$, $3+2i$ and $2-i$ respectively. Show that the quadrilateral ABCD forms a parallelogram.

In the R environment :

```
za = -2 + 1i
```

```
zb = -1 + 4i
```

```
zc = 3 + 2i
```

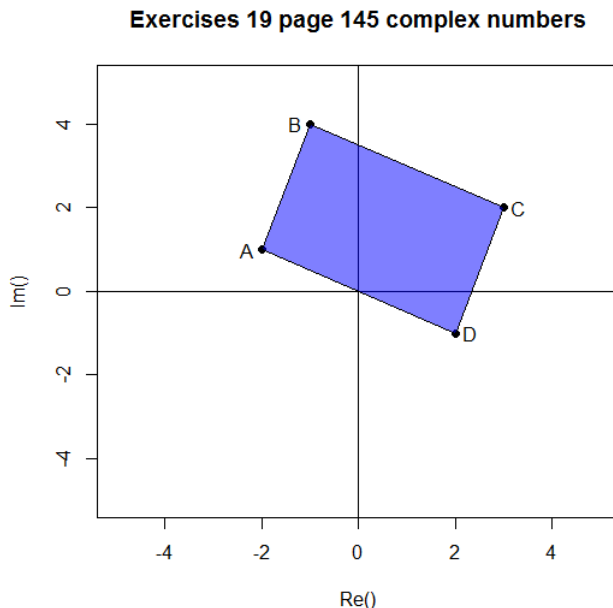
```
zd = 2 - 1i
```

```
\#To plot the points A, B, C and D we use:
```

```

plot(x = Re(za), y = Im(za), type = 'p', pch = 19, xlab = ''
      Re(z)', ylab = ''Im(z)', xlim = c(-5, 5), ylim = c(-5, 5)
      , main = ('' Exercises 19 page 145 complex numbers''))
points(x = Re(za), y = Im(za), pch = 19)
text(x = Re(za), y = Im(za), labels = ''A'', pos = 4, offset =
      0.7)
points(x = Re(zb), y = Im(zb), pch=19);
text(x = Re(zb), y = Im(zb), labels = ''B'', pos = 4, offset =
      0.7)
points(x = Re(zc), y = Im(zc), pch = 19);
text(x = Re(zc), y = Im(zc), labels = ''C'', pos = 4, offset =
      0.7)
points(x = Re(zd), y = Im(zd), pch = 19);
text(x = Re(zd), y = Im(zd), labels = ''D'', pos = 4, offset =
      0.7)
abline(h = 0, v = 0)
\#To draw the quadrilateral ABCD we use the function
\#polygon(x, y = NULL, density = NULL, angle = 45, border =
      NULL, col = NA, lty = par('lty'), \dots, fillOddEven =
      FALSE)
polygon(c(Re(za),Re(zb),Re(zc),Re(zd)),c(Im(za),Im(zb),Im(zc),
      Im(zd))), xpd = TRUE, col = rgb(0,0,1,0.5), lty = 1, lwd =
      1, border = ''black'')
AB = zb-za; DC = zc-zd

```

Figure 7: Parallelogram in \mathbb{R}

Exercise 27 page 145 (Ihaka, Gentleman, Liquet, 2015)

The complex plane is provided with a direct orthonormal reference frame (O, \vec{v}, \vec{u}) . Consider the points: A of affix $z_A = 2 + i$, B of affix $z_B = 2 + i$ and C of affix $z_C = -i$. Determine the affix z_D of point D, in order that the point A becomes the center of gravity of the triangle BCD.

A is the center of gravity of triangle BCD, this means that A is the barycenter of the weighted points $\{(B,1),(C,1),(D,1)\}$, i.e.

$$z_A = \frac{z_B + z_C + z_D}{3}, \quad (1)$$

In the R environment:

```
za = 2 + 1i
zb = 2 - 1i
zc = -1i
\#To plot the points A, B and C we use:
plot(x = Re(za), y = Im(za), type = 'p', pch = 19, xlab = 'Re(z)', ylab = 'Im(z)', xlim = c(-5, 5), ylim = c(-5, 5), main = (' Exercises 27 page 145 complex numbers'))
points(x = Re(za), y = Im(za), pch=19)
text(x = Re(za), y = Im(za), labels = 'A', pos = 4, offset = 0.7)
points(x = Re(zb), y = Im(zb), pch=19);
text(x = Re(zb), y = Im(zb), labels = 'B', pos = 4, offset = 0.7)
points(x = Re(zc), y = Im(zc), pch=19);
text(x = Re(zc), y = Im(zc), labels = 'C', pos = 4, offset = 0.7)
abline(h = 0, v = 0)
\#To calculate zd we use:
Zd = 3*za-zb-zc
points(x = Re(zd), y = Im(zd), pch = 19);
text(x = Re(zd), y = Im(zd), labels = 'D', pos = 4, offset = 0.7)
\#To draw and show the center of gravity of triangle BCD
polygon(c(Re(zb),Re(zc),Re(zd)),c(Im(zb),Im(zc),Im(zd))), xpd = TRUE, col = rgb(0,0,1,0.5), lty = 1, lwd = 1, border = 'black')
```

Exercise 67 page 148 (Ihaka, Gentleman, Liquet, 2015)

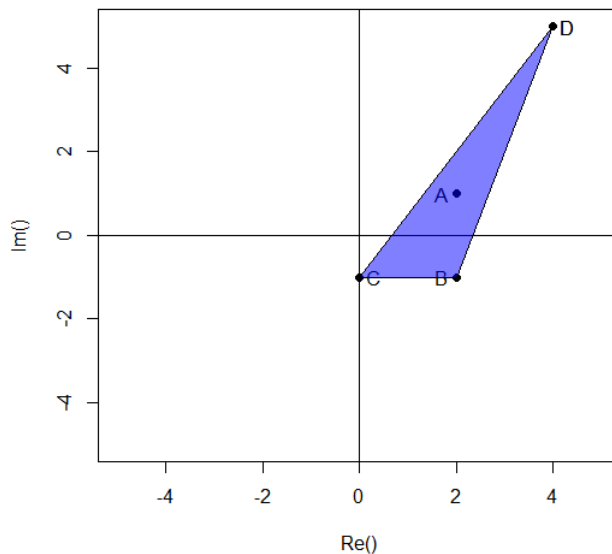
- 1) Find the square root of a complex number $3 + 4i$.
- 2) Solve the following equation for complex numbers z

$$z^2 - 2(1 + 2i)z + 9 + 20i = 0 \quad (2)$$

- 1) The square root

```
Roots = 0
z = 3 + 4i
n = 2
a = Mod(z)
```

Exercises 27 page 145 complex numbers

Figure 8: Center of gravity of triangle BCD in \mathbb{R}

```

b = Arg(z)
polar(b,a, col = 'red', main = 'Exercise 67 page 148')
points(x = Re(z), y = Im(z),pch=19);
text(x = Re(z), y = Im(z), labels = 'z', pos = 4, offset = 0
     .7)
for(k in 0 :(n-1)){
parameter = (b+2*k*pi)/n;
Roots[k+1] = a^(1/n)*(cos(parameter) + 1i*sin(parameter));
  points(x = Roots[k+1], pch = 19);
text(x = Re(Roots[k+1]), y = Im(Roots[k+1]),
strcat(c(collapse = 'R',as.character(k+1))), pos = 4)}

```

3) Solving the equation in the set of complex numbers \mathbb{C}

```

\# To find zeros of a complex polynomial az^2+bz+c=0, we use
the function:
\#polyroot(c(c,b,a))
S = polyroot(c(9+20i,-2-4i,1))
S

> S= polyroot(c(9+20i,-2-4i,1))
> S
[1] 3-2i -1+6i

```

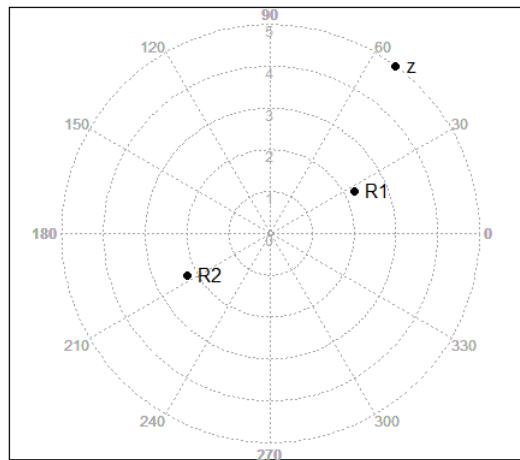
Exercise 67 page 148

Figure 9: Square root of a complex number

4. Discusson

The teaching and learning of complex numbers can often be very complicated due to the abstract mathematical system that contains these numbers and the concept of imaginary numbers. The latter can be taught interactively using R coding language, which allows mitigating the complexity of these numbers by graphical representations.

5. Conclusions and future work

In this work, we investigated the use of R programming activities to foster student learning of foundational complex numbers concepts. This work paves the way for further investigations, particularly to apply it in an experimental study showing the influence of R programming on the learning and understanding of complex numbers, also it will be important to use this language in other mathematical topics that are more difficult for learners.

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