

Piotr Błaszczuk, Anna Petiurenko

**Commentary to Book I of the *Elements*.
Hartshorne and beyond***

Abstract. (Hartshorne, 2000) interprets Euclid’s *Elements* in the Hilbert system of axioms, specifically propositions I.1–34 covering the foundations of Euclidean geometry. We develop an alternative interpretation that explores Euclid’s practice concerning the relation *greater-than*. Discussing the Fifth Postulate, we present a model of non-Euclidean plane in which angles in a triangle sum up to π . It is a subspace of the Cartesian plane over the field of hyperreal numbers \mathbb{R}^* .

Contents

1	Introduction	44
2	Hilbert axioms for plane geometry	45
3	Review of Book I of the <i>Elements</i>	47
	3.1. Transportation of line segments. I.1–3	47
	3.2. Congruence of triangles: SAS to SSS. I.4–8	49
	3.3. <i>Greater-than</i> and <i>Common Notions</i>	53
	3.4. <i>Reductio ad absurdum</i> arguments. Beeson on Book I	56
	3.5. Perpendicular lines. I.9–12	57
	3.6. A comment on Postulate 2	59
	3.7. Vertex angles. I.13–15	60
	3.8. Side of straight-line	62
	3.9. Euclid and modern mathematics. Avigad, Dean, and Mumma on Book I	64

*2020 Mathematics Subject Classification: Primary 01A120; Secondary 01A45, 01A60, 51P99
 Keywords and phrases: *Euclid, Hartshorne, Hilbert, Parallel Postulate, semi-Euclidean plane*
 Anna Petiurenko is supported by the National Science Centre, Poland grant 2020/37/N/HS1/01989

3.10.	Triangle inequality. I.16–21	68
3.11.	Transportation of angles. I.22–23	71
3.12.	ASA and SAA rules. I.24–26	72
3.13.	Parallel lines. I.27–31	75
3.14.	Sum of angles in triangle. I.32	78
3.15.	Parallelograms. I.33–34	79
3.16.	Theory of equal figures. I.35–45	80
3.17.	From equal figures to parallel lines. I.39–40	82
3.18.	Pythagorean theorem. I.46–48	83
3.19.	Non-Euclidean proportions. Beeson on equal figures	85
3.20.	Euclid and modern geometry. Mueller on Book I	88
3.21.	Non-use of the Fifth Postulate	90
4	Semi-Euclidean plane	91
4.1.	The Cartesian plane over the field of hyperreal numbers	91
4.2.	Euclid’s propositions which do not hold in the plane $\mathbb{L} \times \mathbb{L}$	93
4.3.	Klein and Poincaré disks. Euclidean trigonometry	95

1. Introduction

Robin Hartshorne’s (Hartshorne, 2000) is the most insightful reading of the foundations of the Euclid system ever. It provides a coherent interpretation of the *Elements* Books I to IV within the Hilbert system of axioms and a thorough discussion of Euclid’s stereometry, i.e., Books XI to XIII. As for Books V and VI, Hartshorne replaces the theory of proportion with the Hilbert-style, non-Archimedean field of line segments and, in that vein, interprets Euclid’s theory of similar figures. Due to that theoretical bias, the book does not provide tools enabling one to explain the way modern mathematics absorbed Euclid’s geometry.¹ Nevertheless, it is a must-read supplement to modern axiomatic interpretations of Euclid geometry, such as (Tarski, 1959) and (Borsuk, Szmielew, 1960).

In this paper, we adopt Hartshorne’s idea of the construction tools to compare the Hilbert and Euclid system in developing geometry without the parallel postulate, i.e., propositions I.1–28, and the theory of parallel lines, I.29–34. These include copying line segments and angles, as Hilbert tools, and Euclidean straight-edge and compass.

Euclid’s tools enable one to copy line segments (I.3) and angles (I.23), meaning they are at least as effective as Hilbert’s. On the other hand, Euclid’s proofs of propositions I.1 and I.22 require a compass and cannot be recovered in the Hilbert system. Nevertheless, Hartshorne views the construction of an equilateral triangle (I.1) and a triangle from given sides (I.22) simply as means to copy line segments and angles, while in propositions such as I.9–11, where Euclid needs an equilateral triangle, he applies an isosceles triangle instead. Owing to that procedure, he managed to interpret most of Euclid’s propositions from Book I within the Hilbert

¹(Błaszczyk, 2021) presents Descartes’ *Geometry* as transforming Euclidean proportion into the arithmetic of line segments.

system; however, in most cases, their thesis rather than proof techniques. In contrast to Euclid's I.1, Hartshorne proves the existence of an isosceles triangle but does not provide its construction. His proofs of I.9–11, thus, are not constructive (Hartshorne, 2000, 99–101). Yet, the crucial discrepancy concerns the role of the relation *greater-than* in both systems. It is a primitive concept in the *Elements*, while Hartshorne, like Hilbert, introduces it by definitions and, last but not least, reduces its role in the deductive structure of Book I. In this paper, we reveal the prominent position of the relation *greater-than* in Book I of the *Elements*.

Every commentary to Book I has to discuss the Fifth Postulate (P5). In the famed proposition I.32, Euclid shows that P5 implies that angles in any triangle sum up to π . Hartshorne shows (Hartshorne, 2000, 321–322) that the reverse implication obtains in Archimedean planes. Max Dehn, in (Dehn, 1900), presented a model of the absolute geometry where P5 is not satisfied, yet angles in triangles sum up to π , the so-called semi-Euclidean plane. It was a subspace of non-Archimedean, Pythagorean field. We present another model of semi-Euclidean plane. It is a subspace of non-Archimedean, Euclidean plane over the field of hyperreal numbers \mathbb{R}^* . In our model P5 is not satisfied, angles in triangles add up to π , but also the circle-circle axiom is satisfied, as well as the standard (modern) Euclidean trigonometry.

2. Hilbert axioms for plane geometry

Hilbert's *Grundlagen der Geometrie*, from (Hilbert, 1899) to (Hilbert, 1972) got eleven editions. (Hartshorne, 2000) includes its modern version adjusted to educational practice. Hilbert axioms, as presented therein, differ from the original only in applying modern symbols.² Here they are, grouped by Hilbert due to primitive concepts of his system: point, straight line, the relation of betweenness, congruence of line segments, and angles.

Axioms of Incidence

- I1. For any two distinct points A, B , there exists a unique line l containing A, B .
- I2. Every line contains at least two points.
- I3. There exist three noncollinear points (that is, three points not all contained in a single line).

Axioms of Betweenness

- B1. If B is between A and C , (written $A * B * C$), then A, B, C are three distinct points on a line, and also $C * B * A$.
- B2. For any two distinct points A, B , there exist points C, D, E such that $A * B * C$, $A * D * B$, and $E * A * B$.³
- B3. Given three distinct points on a line, one and only one of them is between the other two.

²(Greenberg, 2008, 597–602), provides a concise account of Hilbert axioms, while (Hartshorne, 2000) introduces them in pace as theory develops.

³Hartshorne adopts the existence of C and shows how to prove the existence of D and E (Hartshorne, 2000, 73–79); Hilbert adopts the existence of C and D (Hilbert, 1899, 82); Greenberg – C, D , and E (Greenberg, 2008, 597). The third version explicitly states that points on a line are *dense*, as well as the line can be *extended to the right* or *to the left*.

B4. (Pasch). Let A, B, C be three non collinear points, and let l be a line not containing any of A, B, C . If l contains a point D lying between A and B , then it must also contain either a point lying between A and C or a point lying between B and C .

Axioms of Congruence for Line Segments

C1. Given a line segment AB , and given a ray r originating at a point C , there exists a unique point D on the ray r such that $AB \cong CD$.

C2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

C3. (Addition). Given three points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F on a line satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

Axioms of congruence for Angles

C4. Given an angle $\angle BAC$ and given a ray \overrightarrow{DF} , there exists a unique ray \overrightarrow{DE} , on a given side of the line DF , such that $\angle BAC \cong \angle EDF$.

C5. For any three angles α, β, γ , if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

C6. (SAS) Given triangles ABC and DEF , suppose that $AB \cong DE$ and $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely, $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.

Archimedes' axiom (A)

Given line segments AB and CD , there is a natural number n such that n copies of AB added together will be greater than CD .

Parallel axiom (P)

For each point A and each line l , there is at most one line containing A that is parallel to l .

Absolute (neutral) geometry consists of axioms of the groups of incidence, betweenness, and congruence of line segments and angles.⁴

Other 20th century systems of elementary geometry, such as (Tarski, 1959), (Borsuk, Szmielew, 1960), adopt the general scheme of the Hilbert system and review Euclidean geometry in terms of incidence, betweenness, and congruence. While they include the axiom on copying line segments (C1), they managed to eliminate the congruence of angles. Instead of copying angles (C4), (Borsuk, Szmielew, 1960) introduces an axiom on triangle construction, and instead of the SAS criterion for congruent triangles, (Borsuk, Szmielew, 1960), and (Tarski, 1959) adopt the so-called five-segment axiom. These modifications show that the concept of angle can be reduced to a triangle. Indeed, in the *Elements*, the construction of a triangle with given sides is a technique of copying angles.

⁴(Hartshorne, 2000, 305), (Greenberg, 2008, 161), (Pambuccian, 2021, 114) justify that terminology.

3. Review of Book I of the *Elements*

(Hartshorne, 2000, 102), introduces the term *Hilbert construction tools*, meaning transportation (copying) of line segments and angles. Hilbert axioms C1 and C4 decree these tools and also state the uniqueness of respective line segments and angles – the stipulation, uncommon in the *Elements*, plays a key role in Hilbert-style demonstrations.

Hilbert tools reduce to the first (Hartshorne, 2000, 185–186). A tool enabling the copying of line segments is called a divider, a gauge, or a rigid compass. Hilbertean constructions, thus, are accomplished with straightedge and divider, Euclidean with straightedge and compass. Since Euclid shows how to copy line segments (I.3), his construction tools are at least as efficient as Hilbert’s.

In this paper, we adopt a perspective of construction tools to contrast Euclid and Hilbert approaches and seek to identify Euclidean constructions that surpass the capabilities of straightedge and divider.

Finally, let us observe that Hilbert construction tools require a grown theory to justify constructions. In the *Elements*, on the contrary, a theory develops step by step with new constructions, meaning they constitute a deductive structure of the system. And indeed, whereas *Postulates* 1–3 introduce straightedge and compass, *Postulate* 5 is the famous parallel axiom. Thus, from the perspective of the *Elements*, construction tools and the parallel axiom belong to the same category of basic rules.

3.1. Transportation of line segments. I.1–3

I.1 *To construct an equilateral triangle on the given line AB.*

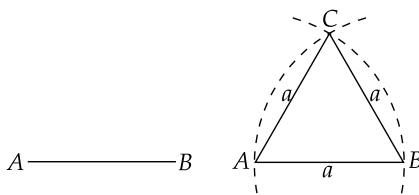


Figure 1: *Elements*, I.1 – schematized.

Given that a stands for the line-segment AB , point C , the third vertex of the wanted triangle is an intersection of circles (A, a) and (B, a) , i.e., circles with centers at A , and B , and radius equal a .

In tables like the one below, we lay out points resulting from intersections of straight lines and circles.⁵

$$\frac{(A, a), (B, a)}{C}$$

⁵The idea of such tables originates from (Martin, 1998).

Obviously, there are two solutions to that problem, yet at that stage, there are no means in the Euclid system to show these two triangles are equal.

In the sequel, we use the following abbreviations explained one after another while going through the subsequent propositions of Book I of the *Elements*.

AB^{\rightarrow}	extension of line segment AB	<i>Postulate 2</i>
(A, a)	circle with center A and radius a	<i>Postulate 3</i>
Ab	transportation of line segment b to point A	I.2
mid AB	midpoint of line segment AB	I.10
$AB \perp C$	perpendicular to line AB through point C	I.11, 12
$AB\alpha$	transportation of angle α to line segment AB at point A	I.23
$A \parallel BC$	parallel to BC through point A	I.27, 31
sq on AB	square on line segment AB	I.46

I.2 *To place a straight-line at point A equal to the given straight-line BC.*

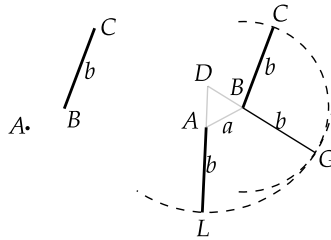


Figure 2: *Elements*, I.2 – schematized.

On the line-segment AB , we construct an equilateral triangle ABD with side a ; the accompanying diagram depicts it in gray (its *shadow*). Point G is the intersection of the circle (B, b) and the half-line DB^{\rightarrow} – the extension of the line segment DB according to *Postulate 2*. Now, DG represents the sum of line-segments a, b . Circle $(D, a + b)$ intersects the half-line DA^{\rightarrow} at point L . Due to the *Common Notions 3*, AL proves to be equal b .

$$\frac{(B, b), DB^{\rightarrow}}{G} \quad \Bigg| \quad \frac{(D, a + b), DA^{\rightarrow}}{L}$$

Owing to I.1–2, b is placed at A in a very specific position. Drawing a circle (A, b) , one can choose any other position at will, and that is the substance of proposition I.3.

I.3 *To cut off a straight-line equal to the lesser C from the greater AB.*

At first, line-segment b is transported to A into position AL ; the accompanying diagram depicts the *shadow* of that construction; let Ab be its symbolic representation. The intersection of the circle (A, b) and the line-segment AB determines E such that $AE = b$.

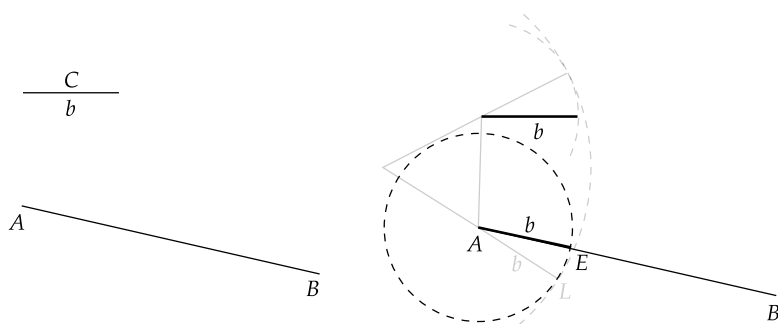


Figure 3: *Elements*, I.3 – schematized.

$$\frac{Ab \mid (A, b), AB}{\mid E}$$

Summing up, due to I.1–3, one can transport any line segment to any point and position. An equilateral triangle is a tool to this end, while the existence of circle-circle and circle-line intersection points are taken for granted.

The Euclid system requires a circle-circle or circle-line axiom, both finding grounds in *Postulates* 1–3 that introduce straightedge and compass. Logically, these two tools reduce to compass alone (*vide* Mohr-Mascheroni theorem), yet, throughout the ages, the economy of diagrams prevailed, and no one questioned the rationale for Euclid instruments. There are, however, models of the Hilbert system that do not satisfy the circle-circle axiom (Hartshorne, 2000, 168); (Martin, 1998, 91). Hartshorne shows (Hartshorne, 2000, 147) that the counterpart of Euclid proposition I.22 (construction of a triangle out of the given sides) is not universally carried out in the Hilbert system. Moreover, in absolute geometry, it is not true that there exists an equilateral triangle with a given length (Pambuccian, 1998). Hartshorne, thus, shows the existence of the isosceles triangle (Hartshorne, 2000, 100) and applies it in Euclid’s propositions I.9–11 instead of the equilateral triangle. Anyway, already at the very first propositions of the *Elements*, we observe that Euclid’s and Hilbert’s systems follow alternative deductive tracks. These facts indicate that one cannot simply merge Hilbert’s axioms with Euclid’s arguments.

Yet another set of problems relates to an intersection of two lines. We will address that question in a commentary to proposition I.10, discussing the concept *side of line*.

3.2. Congruence of triangles: SAS to SSS. I.4–8

Throughout propositions I.1–34, *equality* means congruence, whether applied to line segments, angles, or triangles. Starting with I.35, equality applies to non-congruent figures. The 20th-century versions of elementary geometry introduce a concept of measure (area) to cover that part of Euclid’s geometry, e.g., a formula for the area of a triangle.

In I.5–8, Euclid pursues to show the SSS theorem (side-side-side congruence rule), then assumes I.4, *Common Notions*, and characteristics of the *greater-than* relation. In the Hilbert system, I.4 is axiom C6, addition and subtraction of *things* referred to in *Common Notions* are defined, and respective relations proved, similarly with relation *greater-than* between line segments and angles. However, in the *Elements*, *greater-than* refers to all *magnitudes*, i.e., line segments, angles, triangles, figures, and solids. In the 20th-century geometry, greater-than relation does not apply to triangles but to their areas.

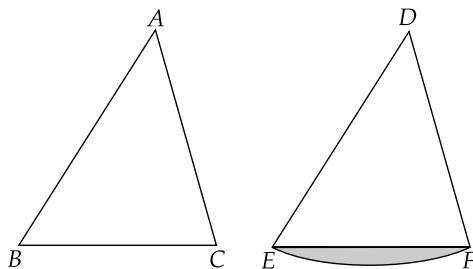


Figure 4: *Elements*, I.4 – grey area added.

I.4 *If two triangles have two corresponding sides equal, and have the angles enclosed by the equal sides equal, then they will also have equal bases, and the two triangles will be equal.*

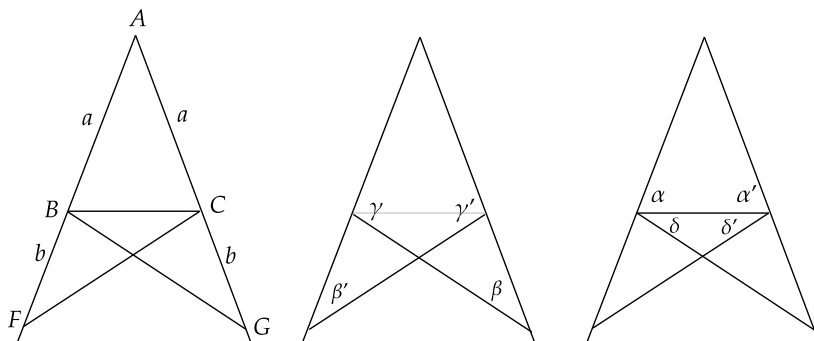
The proof of I.4 (SAS criterion) relies on the *ad hoc* rule: *two straight-lines can not encompass an area*. Figure 4 depicts an area encircled by the base EF of the triangle and a curve with ends E, F . By contrast, Hilbert axioms guarantee a unique straight line through points E, F and there is no room for diagram such as Fig. 4 in the Hilbert system.

Since Hilbert proved other axioms of his system do not imply I.4, there is no need to ponder Euclid's argument. Yet, it is worth mentioning that the 20th-century courses of Euclidean geometry, especially ones dedicated to secondary schools, still apply the method of superposition.

I.5 *Let ABC be an isosceles triangle. I say that the angle ABC is equal to ACB .*

The construction part is simple: F is taken at random on the half-line AB^{\rightarrow} , then G such that $AF = AG$ is determined on the half-line AC^{\rightarrow} .

$$\frac{AB^{\rightarrow}}{F} \mid \frac{(A, a + b), AC^{\rightarrow}}{G}$$


 Figure 5: *Elements* I.5 – scheme of the proof.

Now, due to SAS, $\triangle GAB = \triangle FAC$. Thus $FC = BG$ and

$$\beta = \angle AGB = \angle AFC = \beta',$$

$$\gamma = \angle ABG = \angle ACF = \gamma'.$$

Again by SAS, $\triangle BFC = \triangle BGC$, and

$$\delta = \angle CBG = \angle BCF = \delta'.$$

By CN 3, $\gamma - \delta = \gamma' - \delta'$. Since

$$\alpha = \gamma - \delta, \quad \alpha' = \gamma' - \delta',$$

the equality $\alpha = \alpha'$ holds. \square

I.6 Let ABC be a triangle having the angle ABC equal to the angle ACB . I say that side AB is also equal to side AC .

The proof reveals assumptions in no way conveyed through definitions or axioms. At first, it is the trichotomy law for line segments. Let $AB = b$, $AC = c$, $BC = a$ (Fig. 6). To reach a contradiction Euclid takes: if $b \neq c$, then $b < c$ or $b > c$. Tacitly he assumes that exactly one of the conditions holds

$$b < c, \quad b = c, \quad b > c.$$

Let $b > c$. Then the construction follows: “let DB , equal to the lesser AC , have been cut off from the greater AB ”. However, given that angles at B and C are equal, then $AB = c$, and the cutting off “the lesser AC from the greater AB ” cannot be carried out. On the other hand, if $AB = b$ and $b > c$, the triangle ABC is not isosceles, and angles at B, C are not equal. Throughout the proof, thus, the diagram *changes* its metrical characteristics and cannot meet the assumptions of the proposition; in the diagram, D is a random point on AB , rather than introduced *via* intersection of the circle (B, c) and the side AB .

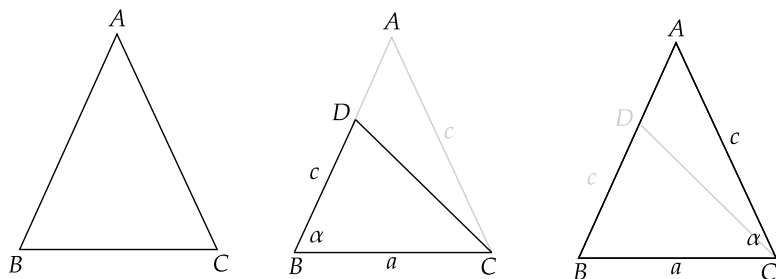


Figure 6: *Elements* I.6 – scheme of the proof.

Now, by SAS, the equality of triangles $\triangle DBC = \triangle ACB$ holds, and Euclid concludes *the lesser to the greater. The very notion is absurd.*

This time, the trichotomy law applies to triangles. The contradiction

$$\triangle DBC = \triangle ACB \ \& \ \triangle DBC < \triangle ACB$$

occurs, given the tacit rule: *For triangles, exactly one of the following conditions holds*

$$\triangle_1 < \triangle_2, \ \triangle_1 = \triangle_2, \ \triangle_1 > \triangle_2.$$

I.7 *On the segment-line AB, two segment lines cannot meet at a different point on the same side of AB.*

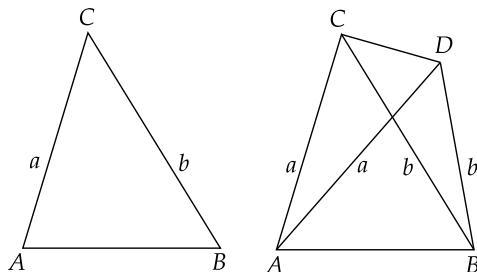
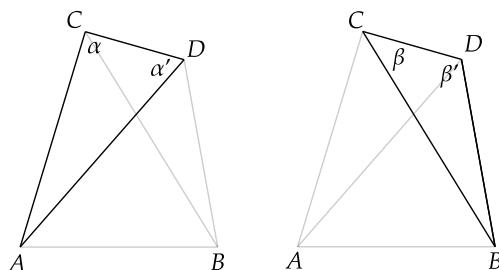


Figure 7: *Elements*, I.7 – letters a, b added.

The proof, atypically, includes no construction. To get a contradiction, Euclid assumes there are two points C, D such that $AC = a = AD$ and $BC = b = BD$ (Fig. 7).

Both triangles $\triangle ACD$ and $\triangle BCD$ are isosceles and share the common base CD . In the first, angles at the base are equal, $\alpha = \alpha'$. Similarly, in the second triangle, $\beta = \beta'$ (Fig. 8).


 Figure 8: *Elements*, I.7 – scheme of the proof.

At the vertex C , the inequality $\alpha > \beta$ is visualized, while at D , $\beta' > \alpha'$.⁶ Thus, $\beta' > \beta$ and, as stated earlier, $\beta' = \beta$. *The very thing is impossible* – clearly, because exactly one of the conditions holds

$$\beta' < \beta, \quad \beta' = \beta, \quad \beta' > \beta.$$

That proof assumes the trichotomy rule for angles and transitivity of *greater-than* relation. By modern standards, it is, thus, a total order.⁷

In I.8, Euclid literally states the SSS criterion. Since the proof depends on a superposition of triangles, we propose the following paraphrase:

If two triangles share a common side and have other corresponding sides equal, then their corresponding angles will also be equal.

In I.9–12, it is employed in that form as Euclid considers two equal triangles on both sides of the common side; in I.23, it is employed to copy angles, yet, the construction of perpendicular plus SAS would do to that end.

Proof of that modification of I.8 effectively reduces to I.7.⁸ Similarly, it does not include a construction part, meaning point G is only postulated rather than introduced through straightedge and compass (Fig. 9).

From I.8 on, Euclid considers two congruent triangles on both sides of a shared base.

3.3. *Greater-than and Common Notions*

Through §§ 10–11 of (Hartshorne, 2000), Hartshorne seeks to prove Euclid's propositions I.1–34 within the Hilbert system, except I.1 and I.22, as they require the circle-circle axiom. He observes that “Euclid's definitions, postulates, and common notions have been replaced by the undefined notions, definitions, and axioms” in the Hilbert system. Commenting on Euclid's proof of I.5–8, Hartshorne writes:

⁶(Błaszczuk, Mrówka, Petiurenko, 2020) expounds the term *visual evidence* in a broader context. (Beeson, Narboux, Wiedijk, 2019, 216), identifies Euclid's reliance on the *greater-than* relation and suggests that in I.7 one should consider the dimension of the space.

⁷Euclid applies the phrase “is much greater than” when referring to the transitivity.

⁸Proof of I.7 gets complicated when point D lies inside triangle ABC .

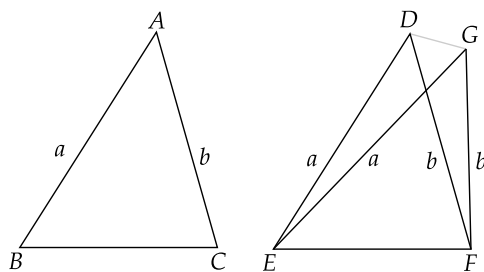


Figure 9: Proof of I.8 schematized.

Proposition I.5 and its proof is ok as they stand. [...] every step of Euclid's proof can be justified in a straightforward manner within the framework of a Hilbert plane. [...] Looking at I.6 [...] we have not defined the notion of inequality of triangles. However, a very slight change will give a satisfactory proof. [...] I.7 [...] needs some additional justification [...] which can be supplied from our axioms of betweenness [...]. For I.8, (SSS), we will need a new proof, since Euclid's method of superposition cannot be justified from our axioms (Hartshorne, 2000, 97–99).

The above comparison between Euclid's and Hilbert's axiomatic approach simplifies rather than expounds. Euclid implicitly adopts *greater-than* relation between line segments, angles, and triangles as a primitive concept; similarly to addition and subtraction. In the previous section, we have shown that he takes transitivity and the trichotomy law to be self-evident. Further characteristics follow from his theory of magnitudes developed in Book V – the only part of Euclid's geometry hardly discussed by Hartshorne (Hartshorne, 2000, 166–167). Here is a brief account based upon (Błaszczyk, Mrówka, 2013, ch. 3), (Błaszczyk, 2021), and (Błaszczyk, Petiurenko, 2019).

Euclidean proportion (for which we adopt symbol $::$: originated from the 17th-century) is a relation between two pairs of geometric figures (*megethos*) of the same kind, triangles being of one kind, line segments of another kind, angles of yet another. Magnitudes of the same kind form an ordered additive semi-group $\mathfrak{M} = (M, +, <)$ characterized by the five axioms given below:

$$\text{E1 } (\forall a, b \in M)(\exists n \in \mathbb{N})(na > b).$$

$$\text{E2 } (\forall a, b \in M)(\exists c \in M)(a > b \Rightarrow a = b + c).$$

$$\text{E3 } (\forall a, b, c \in M)(a > b \Rightarrow a + c > b + c).$$

$$\text{E4 } (\forall a \in M)(\forall n \in \mathbb{N})(\exists b \in M)(nb = a).$$

$$\text{E5 } (\forall a, b, c \in M)(\exists d \in M)(a : b :: c : d), \quad \text{where } na = \underbrace{a + a + \dots + a}_{n\text{-times}}$$

Clearly, E1–E3 provide extra characteristics of the *greater-than* relation; while E1 is a sheer rendition of definition V.4, currently called the Archimedean axiom.

A modern interpretation of *Common Notions* is straightforward: CN1 justifies the transitivity of congruence of line segments, triangles, and angles, CN2 and CN3 – addition and subtraction in the following form

$$a = a', b = b' \Rightarrow a + b = a' + b', \quad a - a' = b - b'.$$

The famous CN5, *Whole is greater than the part*, allows an interpretation by the formula $a + b > a$.⁹

In the Hilbert system, the *greater-than* relation is defined through the concept of *betweenness* and refers only to line segments and angles; similarly, addition of line segments and angles is introduced by definitions.¹⁰ Then counterparts of Euclid's axioms E2, E3, CN1–3 are proved as theorems.

In the sequel, we will be referring to our interpretation of Euclid's *greater-than* relation, therefore already at that stage, we juxtapose Euclid's and Hartshorne's proof of I.29 as a model clash of these alternative approaches. Its substance is as follows: *When a line n falls across parallel lines l, p , equality of angles obtains $\alpha = \beta$* (Fig. 10, left).

Euclid's proof goes like that: For, if they are not equal, one of the angles is greater, suppose $\alpha > \beta$. Then (implicitly by E3),

$$\alpha > \beta \Rightarrow \alpha + \alpha' > \beta + \alpha',$$

given that α, α' are supplementary angles.

Since $\alpha + \alpha' = \pi$, angles β, α' satisfy the requirement of the parallel axiom, i.e., $\beta + \alpha' < \pi$ and straight lines l, p meet, contrary to the initial assumption.

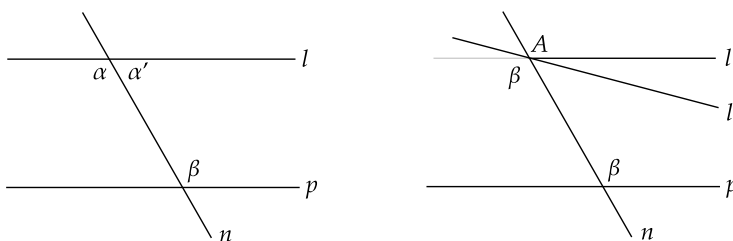


Figure 10: *Elements*, I.29 schematized (left). Hartshorne's version (right)

On the other hand, Hartshorne's proof of I.29 rests on the parallel axiom stating there is exactly one line through the point A parallel to p (Fig. 10, right). If $\alpha \neq \beta$, then, by I.27, a line l' through A making angles β with n is parallel to p , which contradicts the uniqueness of a parallel line through A .

⁹(Błaszczczyk, Mrówka, Petiurenko, 2020, 73–76). (Avigad, Dean, Mumma, 2009, 722), also adopt that interpretation of CN5, but without any reference to the *Elements*. On another occasion (p. 704), they interpret CN5 as an inclusion of areas. Indeed, it is controversial to imply that by checking a diagram, one could confirm that $a + b > a$. Yet, on page 744, they suggest that one can read off a diagram that parts make a sum $a + b$, and based on that observation – we guess – one can infer that $a + b > a$.

¹⁰(Hartshorne, 2000, 85, 93, 95, 168).

Euclid, thus, considers line l which meets or not p , Hartshorne – two parallels to p . The second argument clearly bears a hint of knowing the hyperbolic geometry. If two parallels to a given line occurred in the *Elements*, even though in a *reductio ad absurdum* proof, the history of the Fifth Postulate could take a different track.

3.4. *Reductio ad absurdum* arguments. Beeson on Book I

There are eleven indirect proofs in Book I; ten of them, namely I.6, 7, 14, 19, 25, 26, 27, 29, 39, 40, employ *greater-than* relation. It is – let us remind – a primitive concept characterized by the transitivity and the trichotomy law in the form: exactly one of the following conditions holds

$$x < y, \quad x = y, \quad x > y,$$

where x, y range over line segments, or angles, or triangles. *Reductio ad absurdum* proofs share the pattern: If $x \neq y$, then $x < y$ or $x > y$. Assuming $x < y$, a contradiction follows. Similarly, $x > y$ implies a contradiction. As a result, $x = y$.

Contradictions are of two types: (1) $v = u$ and $v > u$, (2) $v = u$ and $v = u + w$, where the range of v, u, w and x, y can differ. For example, in I.6, Euclid shows that inequality of line segments implies the contradiction of the second kind, moreover, it concerns triangles, rather than line segments. In I.26, inequality of line segments implies the contradiction of the second kind concerning angles. Under our interpretation of CN5 by the formula $a + b > a$, we can reduce contradictions of the second type to the first type.

(Beeson, 2010) develops an interpretation of Euclid's Book I in the Hilbert-style axioms combined with the intuitionistic logic; it reads:

We will take care to formulate our axioms without quantifiers and without disjunction, which will be key to our applications of proof theory. What we aim to do in this section is to formulate such a theory, which we feel is quite close in spirit to Euclid. In formulating this theory, we made use of the famous axioms of Hilbert [...]. The only question of serious interest is whether disjunction can be completely avoided. It can, as it turns out (Beeson, 2010, 17).

In this vein, he defines relation *greater than* for line segments through the relation of betweenness, yet does not prove the trichotomy law.¹¹ Indeed, the trichotomy law fails to hold in a constructive context. Beeson can prove in his system that $\neg(x = y) \Rightarrow (x < y \vee x > y)$, however, in intuitionistic logic, it does not entail the trichotomy law, i.e.,¹²

$$\neg(x = y) \Rightarrow (x < y \vee x > y) \not\equiv (x = y \vee x < y \vee x > y).$$

¹¹(Beeson, 2010, 17–22).

¹²Beeson adopts Markov's principle to show the implication $\neg(x = y) \Rightarrow (x < y \vee x > y)$, however, under some interpretations, INT+Markov's principle = Classical Logic.

When analyzing Euclid’s indirect proofs, specifically I.6, 26, Beeson observes:

it seems that the only classical arguments that occur in Euclid are applications of the principle ‘if ab and cd are unequal, then one of them is longer’ (Beeson, 2010, 28).

However, even assuming classical logic, Euclid’s indirect proofs do not satisfy the scheme: $x \neq y \Rightarrow x < y \vee x > y$. In proposition I.6, Euclid assumes that exactly one of the following conditions holds

$$x < y, \quad x = y, \quad x > y,$$

eliminates $x < y$, then $x > y$, and on these grounds concludes $x = y$. The trichotomy law is crucial when one seeks to mirror Euclid’s reasoning.

3.5. Perpendicular lines. I.9–12

Two subsequent propositions provide bisection of an angle and a line segment. Then Euclid constructs a perpendicular to a line through a point lying on it and outside it. The SSS rule is applied to justify these constructions.

I.9 *To cut a given rectilinear angle in half.*

Taking D on AB , point E is such that $AD = b = AE$ (Fig. 11). Point F is determined by I.1, taking $DE = a$. Then by SSS, $\triangle FDA = \triangle FEA$. Hence, $2\beta = \alpha$.

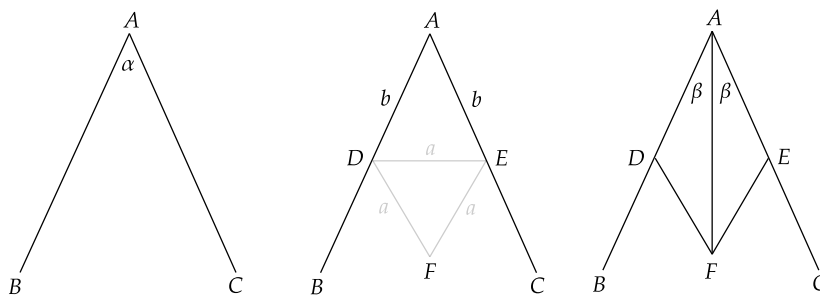


Figure 11: Proof of I.9 schematized.

$$\frac{AB \rightarrow}{D} \mid \frac{(A, b), AC \rightarrow}{E} \mid \frac{(D, a), (E, a)}{F}$$

Euclid’s diagram implies $a < b$. If $a > b$, the line AF also divides α in half. When $a = b$, the construction does not produce a new point, given that I.1 produces only one point. I.8 implies the construction of congruent triangles on both sides of DE .

I.10. *To cut a given finite straight-line in half.*

On both sides of AB , construct equilateral triangles $\triangle ABC$ and $\triangle ABF$ (Fig. 12). By I.9, CF divides in half angle $\angle ACB$. Then, by SAS, $\triangle ACD = \triangle BCD$, hence $AD = DB$.

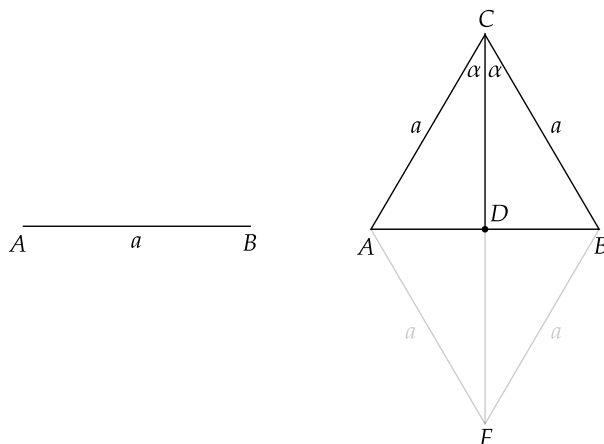


Figure 12: Proof of I.10 schematized.

$$\frac{(A, a), (B, a)}{C, F} \mid \frac{CF, AB}{D}$$

Note that point D occurs as an intersection of two straight lines. There is no explicit rule in the Euclid system to guarantee its existence. In the Hilbert system, it follows from the so-called cross-bar theorem, a simple follow-up of the Pasch axiom (Hartshorne, 2000, 77–78).¹³

I.11 *To draw a straight-line at right-angles to a given straight-line from a given point on it.*

Taking a random point D on the half-line CA^\rightarrow , E is determined such that $CD = CE$, and F is the vertex of equilateral triangle DFE (Fig. 13). By I.8, $\triangle DFC = \triangle EFC$. Hence, $\angle DCF = \angle ECF$. Since they are equal and supplementary angles, by I def. 10, they are both right angles.

$$\frac{CA^\rightarrow}{D} \mid \frac{(C, a), AB}{E} \mid \frac{(D, 2a), (E, 2a)}{F}$$

¹³One can also justify the existence of point D owing to the concept of side of straight line: the points C and F are known to lie on different sides of the line AB , and thus, the segment CF intersects the line AB ; see §3.8 below.

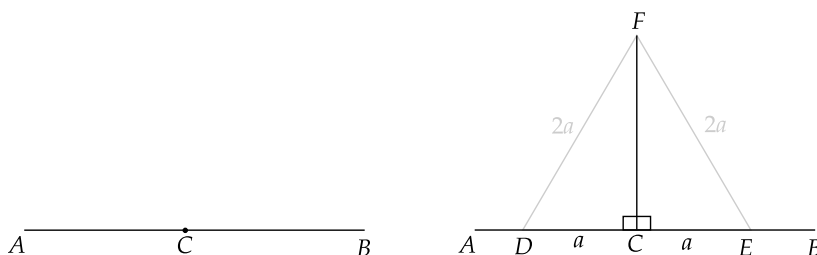


Figure 13: Proof of I.11 schematized

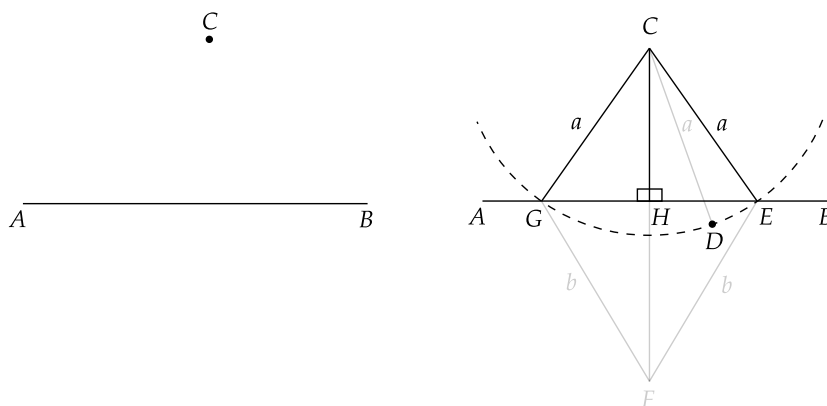


Figure 14: Proof of I.12 schematized

I.12 *To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.*

D is a random point “on the other side [to C] of the straight-line AB ” (Fig. 14). By I.10, H is determined such that $GH = HE$. By SSS, $\triangle GCH = \triangle ECH$ and by the same argument as in I.11, $\angle CHG = \angle CHE = \pi/2$.

$$\frac{AB, C}{AB, C} \mid \frac{AB, (C, a)}{D} \mid \frac{(G, b), (E, b)}{G, E} \mid \frac{CF, AB}{F} \mid \frac{}{H}$$

Euclid does not definite *the side of line* – the concept involved in I.8 and I.12. Modern systems reveal its importance.

3.6. A comment on Postulate 2

In the Hilbert system, due to B2, the straight line has no ends. In most of Euclid’s propositions, a straight line is a line segment with endpoints – a closed line segment, by modern standards. That is why the term *line standing on another*

line involved in propositions I.13–14 makes sense;¹⁴ in Fig. 16, lines EB and AB stand on DC . Such a line can stand at point B that is between D and C (Fig. 16), or at the endpoint, such as AB stands on CB in Fig. 17. Now, the line *standing on* another line enables Euclid to formulate iff-condition for a line to be an extension of another line, and that is a job of propositions I.13–14, while Postulate 2 characterizes the process of extending in a descriptive way, namely *To produce a finite straight-line continuously in a straight-line*.

Similarly, in propositions I.27–29, Euclid introduces an auxiliary line enabling him to eliminate a clumsy condition *being produced to infinity* included in definition of parallel lines (I def. 23). Given that symbol $l \dot{\div} p$ stands for *l is straight on with respect to p*, we can represent I.27–29 and I.14 as follows (Fig. 15)

$$l \parallel p \Leftrightarrow \alpha + \beta = \pi, \quad l \dot{\div} p \Leftrightarrow \alpha + \beta = \pi.$$

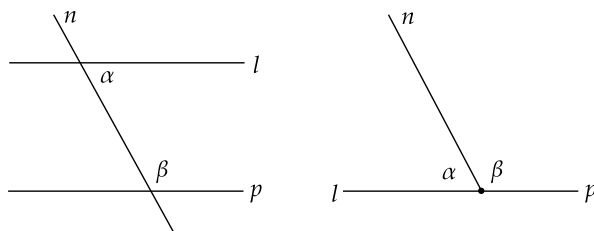


Figure 15: Iff-conditions for parallelism and extension of a line.

3.7. Vertex angles. I.13–15

I.13 *If a straight-line stood on a straight-line makes angles, it will certainly either make two right-angles, or equal to two right-angles.*

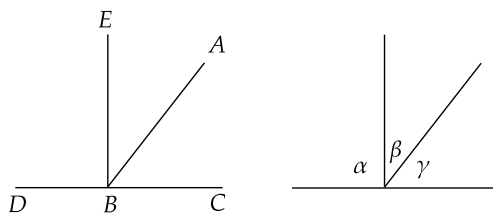


Figure 16: *Elements*, I.13 (left).

¹⁴The very concept is also employed in the definition of the right angle, I def. 10.

If $\angle CBA = \angle ABD$ (Fig. 16), by I def. 10, they are two right angles. If $\angle CBA \neq \angle ABD$, Euclid tacitly assumes $\angle ABD > \pi/2$, draws a perpendicular, $EB \perp DC$, and argues: since the following equalities of angles obtain

$$(\beta + \gamma) + \alpha = \gamma + \beta + \alpha,$$

$$(\alpha + \beta) + \gamma = \alpha + \beta + \gamma,$$

then, by CN1,

$$(\beta + \gamma) + \alpha = (\alpha + \beta) + \gamma.$$

Hence, $\angle ABD + \angle CBA = (\alpha + \beta) + \gamma = \pi$.¹⁵

□

Instead of Euclid's embroiled enunciation of proposition I.14, we offer the following paraphrase: If $\alpha + \beta = \pi$, then l is straight on to p , given that n stands on l at its end (Fig. 17, middle).

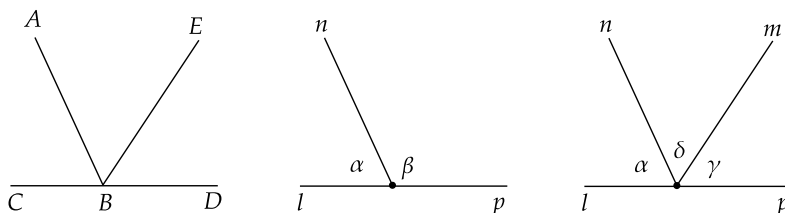


Figure 17: *Elements*, I.14 (left), thesis (middle), and its schematized proof (right).

The proof adopts *reductio ad absurdum* mode. For if not, let m be straight on to l ; Postulate 2 guarantees the existence of m (Fig. 17, right). Then, by I.13, $\alpha + \delta = \pi$. Since $\delta + \gamma = \beta = \angle(n, p)$, the following equality holds $\alpha + \delta + \gamma = \pi$. Hence

$$\alpha + \delta = \alpha + \delta + \gamma.$$

By CN3, $\delta = \delta + \gamma$, and Euclid continues: *the lesser to the greater. The very thing is impossible.*

□

With our interpretation of CN5, $\delta + \gamma > \delta$, and due to the trichotomy law, it can not be both $\delta = \delta + \gamma$ and $\delta + \gamma > \delta$.

Euclid applies criterion I.14 in I.15 and I.47.

I.15 *If two straight-lines cut one another then they make the vertically opposite angles equal to one another.*

By I.13 (Fig. 18),

$$\alpha + \alpha' = \pi = \alpha' + \beta.$$

By CN3, $\alpha = \beta$.

□

¹⁵However, there is no demonstration whatsoever that $\gamma + \beta + \alpha = \alpha + \beta + \gamma$.

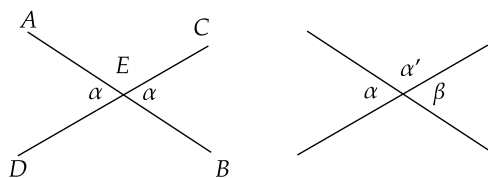


Figure 18: Proof of I.15.

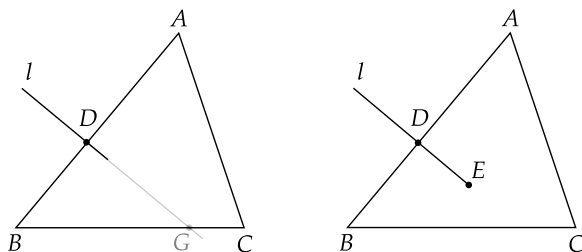
The condition $\alpha + \alpha' = \pi$ follows from the assumption AE stands on DC , while $\alpha' + \beta = \pi$ – from the assumption CE stands on AB . To be more Euclidean, α and α' make two right angles, similarly α' and β make two right angles. By Postulate 4, *all right-angles are equal to one another*. Hence $\alpha + \alpha' = \alpha' + \beta$, and the conclusion follows.

3.8. Side of straight-line

(Hartshorne, 2000, § 7), introduces the concept *side of line l* . It is an equivalence relation between points of plane not lying on l defined by: $A \sim B$ iff $A = B$ or segment AB does not meet l .

The transitivity of the relation is demonstrated as follows (Fig. 19, left). Let $A \sim C$, and $B \sim C$ and suppose $A \not\sim B$. Let $D = l \cap AB$. Then, by Pasch axiom, l intersects AC or BC , which contradicts $A \sim C$ or $B \sim C$, respectively.

Relation \sim determines two equivalence class, called sides of l , or half-planes determined by l . Euclid's straight-line is not that *long* and can not divide the plane into two halves. Postulate 2 guarantees that one can extend l to, say, point E , but can not guarantee an intersection l with AC or BC (Fig. 19, right).¹⁶

Figure 19: Transitivity of the relation the *same side of line* (left), Euclidean version (right).

¹⁶The symbol $AB \rightarrow$, which we have already applied in tables, evokes a modern half-line that extends the line segment AB .

Since there are three non collinear points, relation \sim determines at least one equivalence class. Hartshorne shows there are at most two classes. Let us consider a simpler question on whether there are two different classes. Let $A \notin l$, $E \in l$, then, by B2, there is F such that $A * E * F$ (Fig. 20, left). Hence, A and F lie on different sides of l .

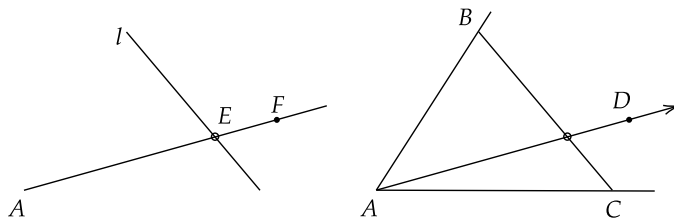


Figure 20: Finding points on different sides of l (left) and crossbar theorem (right).

Crossbar theorem (Hartshorne, 2000, 77–78), that builds on the Pasch axiom and the concept of side of the straight line, states that line through A and D meets the side BC (Fig. 20, right). It enables to infer the existence of points such as D or H in Euclid’s propositions I.10, 12. Furthermore, the concept of the side of straight-line is crucial in proposition I.12, as “ D have been taken somewhere on the other side (to C) of the straight-line AB ”. Indeed, if D is taken at random on the same side with C or on the line AB , Euclid’s construction would not work, for D could be on the perpendicular CH (Fig. 14).

In (Hartshorne, 2000), the concept of supplementary angles and the five-segment-lines theorem cover Euclid’s propositions I.13 and I.15 (Hartshorne, 2000, 92–93). As for I.14, Hartshorne finds it completely alien to Hilbert’s concepts and proposes an exercise “to rewrite the statement I.14 so that it makes sense in the Hilbert plane” (Hartshorne, 2000, 103).¹⁷

Let us review I.14 in the context of propositions I.13–15 and find what makes cutting lines look like in I.15 rather than in the proof of I.14. To this end, let m be a continuation of n , and p of l . Both n and m stand on the straight line $l \dot{\div} p$, also l and p stand on the line $n \dot{\div} m$ (Fig. 21). Furthermore, the angles between l and n , and n and p add up to π ;

$$\angle(l, n) + \angle(n, p) = \pi, \quad \angle(l, m) + \angle(m, p) = \pi.$$

These results fit to I.14. However, taking into account that p also stands on the line $n \dot{\div} m$, one obtains

$$\angle(m, p) + \angle(p, n) > \pi,$$

since $\angle(p, n) > \angle(p, l) = \pi$. It contradicts I.14.

¹⁷(Beeson, Narboux, Wiedijk, 2019, 218) finds it simply as a statement on the betweenness relation, (Mueller, 1981, 20) – a *converse* of I.13. Clearly, I.15 causes trouble for modern readers.

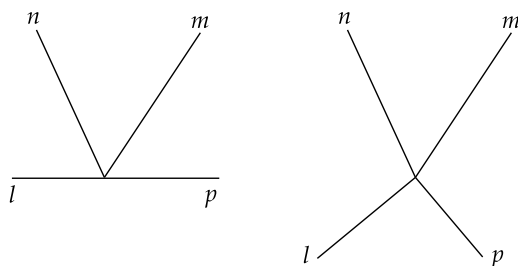


Figure 21: Weird straight-lines.

Although the above considerations suggest recovering the concept of the side of a straight line based on proposition I.14, there is no way in the Euclid system to introduce it: Euclid plane includes points that lie on a straight line l , those that do not lie on it, and those that could lie on the extension of l . In propositions that explicitly limit the scope of points considered, such as I.2, I.21, Euclid does not rely on Postulate 2 and simply introduces intersection points. In other words, Postulate 2 does not guarantee intersection points.¹⁸

Even though one may find the above speculations unfounded or detached from the historical context, Euclid could model a straight-line after a ray of light so it could take forms similar to those depicted in Fig. 22. Indeed, he devoted two separated volumes to geometrical optics, as we call that discipline nowadays, namely *Optics* and *Catoptrics* (Heiberg, 1895).

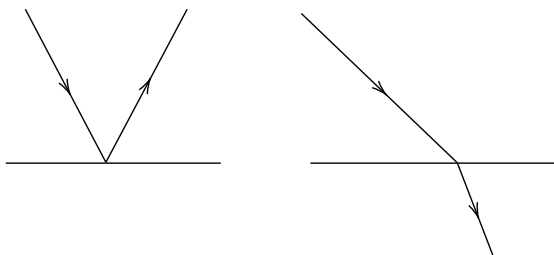


Figure 22: Ray of light reflected (left), refracted (right).

3.9. Euclid and modern mathematics. Avigad, Dean, and Mumma on Book I

Whereas (Hartshorne, 2000) interprets the *Elements* from the perspective of the Hilbert system, (Avigad, Dean, Mumma, 2009) seems to present a rival reading as it includes “a formal system providing a faithful model of the proofs in the *Elements*, including the use of diagrammatic reasoning” (Avigad, Dean, Mumma,

¹⁸In I.2, the extension of AD intersects circle $D(a + b)$, giving point L (Fig. 2); in I.21, the extension of BD intersects AC giving E (Fig. 28).

2009, 700–701). Throughout (Hartshorne, 2000) we can find a silent belief that Euclid’s diagrams encode only assumptions concerning intersecting lines made explicit due to the Pasch and circle-circle axiom. Avigad, Dean, and Mumma seek to show that Euclid’s diagrams hide much more – kind of deductive system which contributed to the historical triumph of the *Elements*:

For more than two millennia, Euclid’s *Elements* was viewed by mathematicians and philosophers alike as a paradigm of rigorous argumentation. But the work lost some of its lofty status in the nineteenth century, amidst concerns related to the use of diagrams in its proofs (Avigad, Dean, Mumma, 2009, 700).

Modern interpretations of the *Elements* – they continue – apply novel techniques and do not expound on ways of reading the *Elements* through the ages, specifically Euclid’s reliance on diagrams. In contrast, their formal system aims to decode the logic hidden in diagrams.

Without denying the importance of the *Elements*, by the end of the nineteenth century the common attitude among mathematicians and philosophers was that the appropriate logical analysis of geometric inference should be cast in terms of axioms and rules of inference. [...] This attitude gave rise to informal axiomatizations by Pasch (1882), Peano (1889), and Hilbert (1899) in the late nineteenth century, and Tarski’s (1959) formal axiomatization in the twentieth. Proofs in these axiomatic systems, however, do not look much like proofs in the *Elements*. Moreover, the modern attitude belies the fact that for over 2000 years Euclidean geometry was a remarkably stable practice. On the consensus view, the logical gaps in Euclid’s presentation should have resulted in vagueness or ambiguity as to the admissible rules of inference. But, in practice, they did not; mathematicians through the ages and across cultures could read, write, and communicate Euclidean proofs without getting bogged down in questions of correctness. So, even if one accepts the consensus view, it is still reasonable to seek some sort of explanation of the success of the practice (Avigad, Dean, Mumma, 2009, 700–701).

Let us pause for a while at these claims. Euclid’s geometry affected modern mathematics with mathematical techniques rather than rigorous inferences. These were, firstly, the theory of similar figures, secondly, the exhaustion method, say, a pre-calculus. In Greek mathematics, the former relied on proportions. Books V–VI cover rules of processing ratios and proportions of magnitudes (V.7–25), criteria for similar triangles (VI.4–7), areas of similar figures (VI.19, 20), or the so-called generalized Pythagorean theorem (VI.30). Yet, Euclid’s theory of proportion was neither rigorous nor exercised a diagrammatic reasoning (Błaszczuk, Petiurenko, 2019).

Modern mathematics replaced proportions with implicit rules of an ordered field, and then, in the 19th century, by the arithmetic of real numbers (Błaszczuk, 2021). In the 18th and 19th centuries, trigonometric identities encoded properties

of similar figures. By the middle of the 18th century, Euler managed to expand trigonometric functions into power series (Błaszczuk, Petiurenko, 2022). Thus, as disguised in trigonometric series, Euclid's theory of similar triangles is applied in contemporary real and complex analysis.

Modern axiomatic systems of geometry are motivated by methodology (Hilbert, Tarski, Borsuk, Szmielew) or education (Peano) of mathematics. They mirror the structure of Book I, first of all, the pattern: absolute geometry (I.1–28) plus the theory of parallel lines (I.29 and on), rather than Euclid's proof technique. Indeed, they identify gaps in a deductive structure viewed from the modern perspective. These are axioms relating to intersections of straight lines and circles, the status of SAS criterion (in modern systems, it is an axiom while in the *Elements* the proposition) and propositions I.13–15, which have to be covered alternatively due to the new concept of a straight line (explained below).

Another shift concerns construction tools. Instead of Euclid's straightedge and compass, (Hilbert, 1899) adopts straightedge and rigid compass. (Borsuk, Szmielew, 1960) and (Tarski, 1959) adopt copying of line segments (vide the axiom of segment construction) and refine Hilbert's approach by replacing line segments with equal distance between pair of points, the copying of an angle with a triangle construction, and SAS with the five segment axiom.

Now, let us review specifics of Avigad, Dean, and Mumma's approach in terms of a "faithful model" of Euclid's arguments. They write:

Our study draws on an analysis of Euclidean reasoning due to Manders (2008b), who distinguished between two types of assertions that are made of the geometric configurations arising in Euclid's proofs. [...] we present a formal axiomatic system, E, which spells out precisely what inferences can be 'read off' from the diagram (Avigad, Dean, Mumma, 2009, 701).

We skip that topic since (Błaszczuk, Petiurenko, 2022) discusses it in detail.

(Hartshorne, 2000) provides a discussion of every Euclid's proposition from I.1 through IV.16, detailing which one requires an extra axiom (e.g., I.1, I.22) or does not make sense in any modern system (e.g., I.14). Avigad, Dean, and Mumma suggest they can reconstruct every single proposition from Books I to IV:

We claim that our formal system captures all the essential features of the proofs found in Books I to IV of the *Elements* (Avigad, Dean, Mumma, 2009, 714).

In fact, they interpret only three propositions, namely I.1, 2, 10, and also indicate that Euclid's proofs of I.9 and I.35 do not accord with the rules of their system. However, since they focus on diagrammatic reasoning, it will not be an easy task to reconstruct Book II, where Euclid deals with objects not represented in diagrams at all, as shown in (Błaszczuk, Mrówka, Petiurenko, 2020).

Finally, Avigad, Dean, and Mumma employ the whole machinery of modern synthetic geometry, especially the Pasch axiom (the plane separation axiom) to reconstruct Euclid's reasoning. However, it makes the essential difference between

Greek and modern geometry: Euclid's straight line is finite while modern is infinite, meaning modern cuts the plane into separate half-planes, while Euclid's does not; Pasch clearly realized that by writing:

For Euclid, a 'straight line' is always bounded by two points. It may be 'lengthened' as needed, but even then remains bounded. For more recent mathematicians, a 'straight line' is unbounded: it would include all the points that can be reached when a bounded line is lengthened (Pollard, 2010, 100).

That is why modern systems reformulate the set of propositions I.13–15; in the *Elements* it builds on the alien concept of the straight line. Due to this fact, Euclid's proof of I.27 is not constructive, whereas those who use an infinite line view it as a constructive one (see § 3.13 below).¹⁹ That apostasy has a far-reaching consequence regarding Avigad, Dean, and Mumma's project.

Avigad, Dean, and Mumma adopt an unfounded interpretation of Euclid's Second Postulate: "Postulate 2 [...] allows any segment to be extended indefinitely" and continue

Distinguishing between finite segments and their extensions to lines makes it clear that at any given point in a proof, the diagrammatic information is limited to a bounded portion of a plane. But, otherwise, little is lost by taking entire lines to be basic objects of the formal system. So where Euclid writes, for example, 'let a and b be points, and extend segment ab to c,' we would write 'let a and b be distinct points, let L be the line through a and b, and let c be a point on L extending the segment from a to b.' Insofar as there is a fairly straight-forward translation between Euclid's terminology and ours, we take such differences to be relatively minor (Avigad, Dean, Mumma, 2009, 731–734).

Indeed, "the diagrammatic information is limited to a bounded portion of a plane" – but this is what makes the difference between Euclidean and semi-Euclidean planes we present in section § 4.²⁰ In that plane, all diagrams look like Euclidean diagrams, the Pasch axiom is satisfied, yet straight lines are too short to meet the Fifth Postulate. However, there are no means to read off that information from a diagram laid down on a piece of the plane; one can get it only due to the global perspective and the knowledge that the semi-Euclidean plane is a subspace of the Euclidean one.

¹⁹Cf. (Avigad, Dean, Mumma, 2009, 724).

²⁰(Avigad, Dean, Mumma, 2009, 745) adopts the Parallel Axiom after Tarski's formulation (Tarski, Givant, 1999, 184), Axiom 10. One can easily show that it is not satisfied in our model.

3.10. Triangle inequality. I.16-21

That set of propositions is a festival of *greater-than* relation, with proofs regularly referring to transitivity, trichotomy law, axioms E2, E3, and Common Notions 5.

I.16 *For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.*

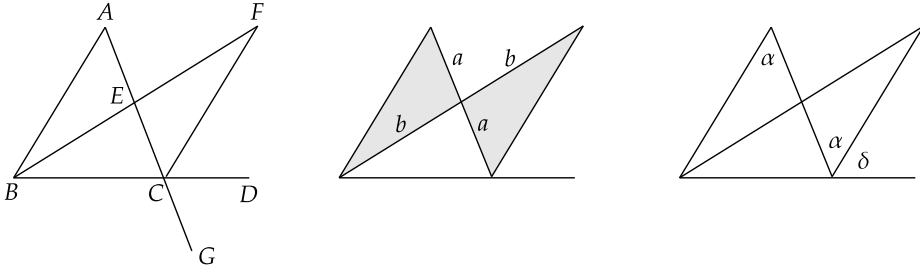


Figure 23: *Elements*, I.16 (left) and its schematized proof (middle and right).

	$BC \rightarrow$	mid AC	$BE \rightarrow, (E, b)$	$AC \rightarrow$
$\triangle ABC$	D	E	F	G

Point E is the middle of AC , $AE = a = EC$; F is such that $BE = b = EF$. By I.15, $\angle AEB = \angle FEC$. Hence, by SAS, $\triangle AEB = \triangle FEC$ (Fig. 23, triangles in grey), and angles at vertexes A and C are equal, $\angle A = \alpha = \angle C$. Now,

$$\alpha + \delta > \alpha,$$

meaning the exterior angle $\angle ACD$ is greater than the interior angle $\angle BAE$.

The same argument applies to angles $\angle ABC$ and $\angle BCG$, but $\angle BCG = \angle ACD$, thus the thesis obtains. □

I.17 *For any triangle, two angles are less than two right-angles.*

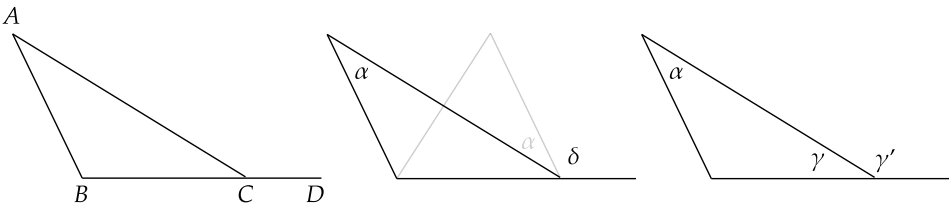


Figure 24: *Elements*, I.17 (left), shadow construction I.16 (middle), schematized proof (right).

By I.16, $\alpha < \gamma'$ (Fig. 24, right). Adding to both sides γ , we obtain

$$\alpha + \gamma < \gamma' + \gamma.$$

Since $\gamma' + \gamma = \pi$, the required inequality holds, $\alpha + \gamma < \pi$. □

I.18 *For any triangle, the greater side subtends the greater angle.*

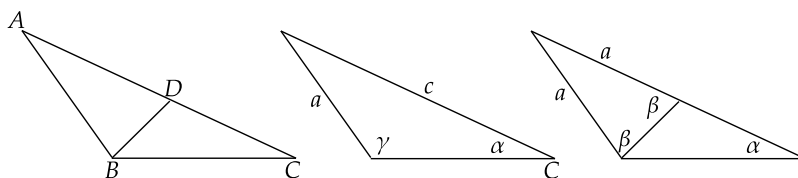


Figure 25: *Elements*, I.18 and its schematized proof.

In symbols (Fig. 25, middle)

$$c > a \Rightarrow \gamma > \alpha.$$

$$\frac{\triangle ABC}{AC, (A, a)} \Bigg| \frac{D}{D}$$

If $AC > AB$, there is point D such that $AD = a = AB$. In triangle $\triangle ABD$, angles at the base are equal (Fig. 25, right). By I.16, $\beta > \alpha$. By transitivity

$$\beta > \alpha, \gamma > \beta \Rightarrow \gamma > \alpha.$$

Inequality $\beta > \alpha$ is determined at point D ; inequality $\gamma > \beta$ – at the vertex B . □

While $c > a \Rightarrow \gamma > \alpha$ represents I.18, the reverse implication $\gamma > \alpha \Rightarrow c > a$, represents I.19 (Fig. 26). Hence, I.18-19 bring in the equivalence

$$c > a \Leftrightarrow \gamma > \alpha.$$

I.19 *For any triangle, the greater angle is subtended by the greater side.*

The proof builds on the trichotomy law. If c is not greater than a , then $c = a$ or $c < a$. From the first case, by I.5, equality follows $\gamma = \alpha$. From the second, by the previous proposition, $\gamma < \alpha$. Both cases contradict the supposition $\gamma > \alpha$. □

I.20 *For any triangle, two sides are greater than the remaining (side).*

$$\frac{\triangle ABC}{AB^{\rightarrow}, (A, c)} \Bigg| \frac{D}{D}$$

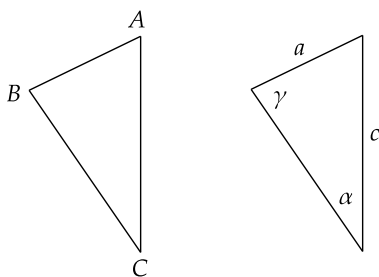


Figure 26: *Elements*, I.19 (left).

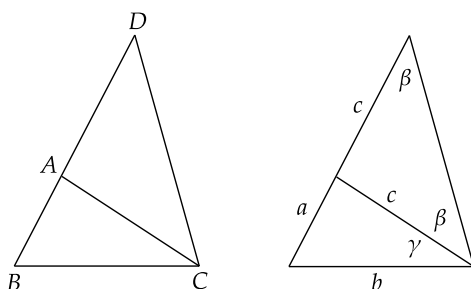


Figure 27: *Elements*, I.20 and its schematized proof.

Point D is constructed on BA^{\rightarrow} such that $AD = c = AC$. Hence in triangle $\triangle ADC$, angles at the base are equal (Fig. 27, right). Since $\gamma + \beta > \beta$, in triangle BDC , by I.19, $a + c > b$. With regard to other pairs of sides one proceeds similarly. □

I.21 *If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle. In symbols:*

$$BA + AC > BD + DC \text{ and } \angle A < \angle D \text{ (Fig. 28, left).}$$

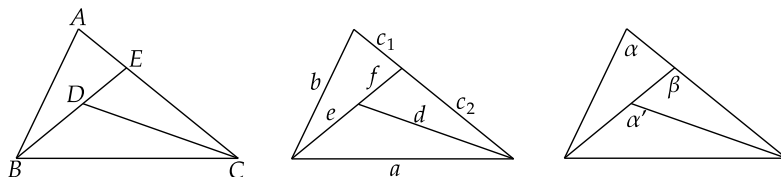


Figure 28: *Elements*, I.21 and its schematized proof.

The proof is an exercise in the already proved triangle inequality. Given that $c + c_1 + c_2$, we have

$$e + f < b + c_1 \Rightarrow e + f < b + c_1 + c_2,$$

$$d < f + c_2 \Rightarrow e + d < e + f + c_2.$$

Hence

$$e + d < b + c.$$

For the second part, Euclid applies twice I.16 and transitivity as follows

$$\alpha' > \beta, \beta > \alpha \Rightarrow \alpha' > \alpha.$$

□

3.11. Transportation of angles. I.22–23

In I.22, Euclid builds a triangle from three given line segments.²¹

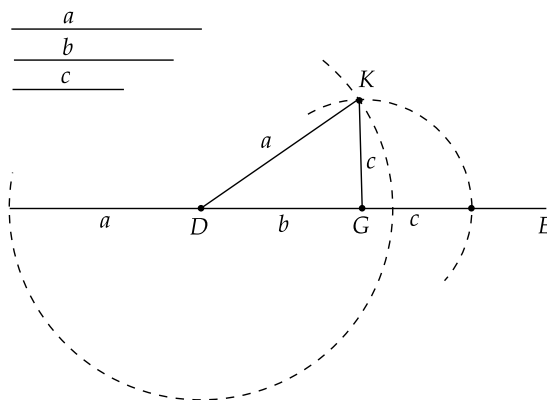


Figure 29: *Elements*, I.22 – small letters added.

The below table presents G, K as intersection points. Let us remind that symbol Da stands for transportation of line segment a to point D based on I.2.

	Db	$(D, b), DE$	Da	Gc	$(D, a), (G, c)$
DE		G			K

The remainder of the proof includes justifications of equalities $FK = a, FG = b, GK = c$. □

I.23 *To construct a rectilinear angle equal to a given rectilinear angle at a point on a given straight-line.*

The transportation of an angle α is reduced to a transportation of triangle $\triangle DCE$ (Fig. 30).

²¹(Greenberg, 2008, 173), observes it is equivalent to the circle-circle axiom.



Figure 30: *Elements*, I.23 – small letters added.

The construction part consists of picking two random points on arms of the angle.

$$\frac{CK \rightarrow}{D} \quad | \quad \frac{CE \rightarrow}{E}$$

Then triangle $\triangle DCE$ is copied at point A on the line AB (Fig. 31).

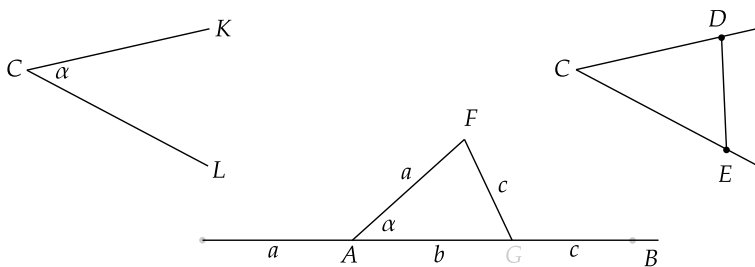


Figure 31: Copying an angle.

By the SSS, $\triangle CDE = \triangle AFG$, hence $\angle KCL = \alpha = \angle FAG$.

□

3.12. ASA and SAA rules. I.24–26

Propositions I.24–25 are companions to I.18–19, yet, this time, Euclid considers two separate triangles. We present their thesis in concise, symbolic forms.

I.24 Let $AC = a = DF$, $AB = b = DF$, and $\angle CAB = \alpha$, $\angle FDE = \beta$. If $\angle CAB > \angle FDE$, then $CB > EF$ (Fig. 32). In small letters mode

$$\alpha > \beta \Rightarrow c > d.$$

Since triangle $\triangle DGF$ is isosceles, the equality of angle holds $\angle DGF = \angle DFG$. Hence, in triangle $\triangle GFE$, $\delta < \gamma$. By I.18, $d < c$, meaning $FE < CB$.

□

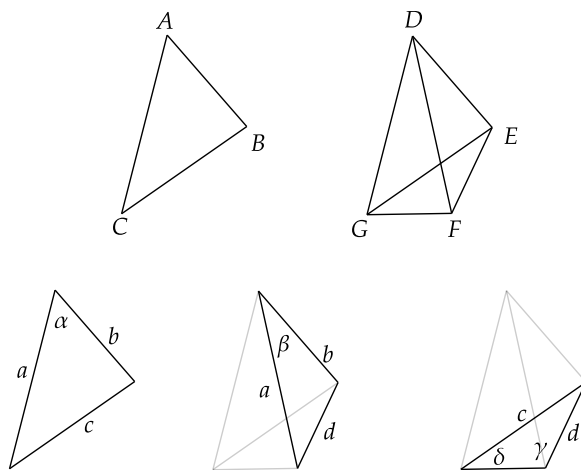


Figure 32: *Elements*, I.24 and its schematized proof.

I.25 Let $AB = a = DE$, $AC = b = DF$, and $\angle BAC = \alpha$, $\angle EDF = \beta$. If $CB > EF$, then $\angle CAB > \angle FDE$ (Fig. 33). In small letters mode

$$c > d \Rightarrow \alpha > \beta.$$

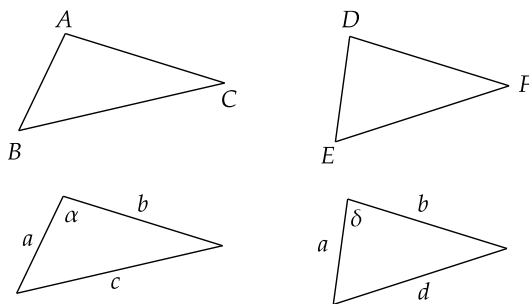


Figure 33: *Elements*, I.25.

The proof is an exercise in trichotomy law and goes like that. If it is not that $\alpha > \beta$, then $\alpha = \beta$ or $\alpha < \beta$. In the first case, $c = d$. In the second, by I.24, $c < d$. \square

I.26 *If two triangles have two angles equal to two angles, respectively, and one side equal to one side then (the triangles) will also have the remaining sides equal to the remaining sides, and the remaining angle (equal) to the remaining angle.*

It is ASA congruence rule: If $BC = EF$, $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, then $\triangle ABC = \triangle DEF$ (Fig. 34).

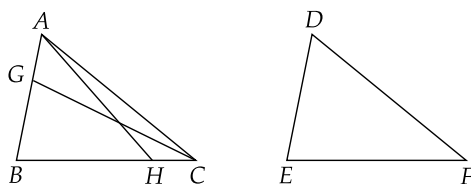


Figure 34: *Elements*, I.26.

Let $AB = b$, $DE = b'$ (Fig. 35). Supposing $b' < b$, Euclid lays down b' on AB , and by SAS rule, gets the equality of triangles $\triangle GBC = \triangle DEF$. Hence, the equality of angles follows $\angle GCB = \angle ABC$, *the lesser to the greater*.

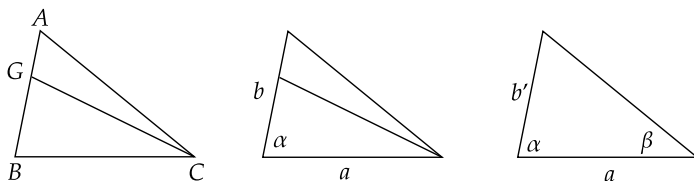


Figure 35: *Elements*, I.26 - scheme of the first case proof

In the second case, SAA, $AB = DE$, $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$. Suppose $BC > EF$. Let $BN = b' = EF$ (Fig. 36). Thus, $\triangle AHB = \triangle DFE$, and, on the one hand $\angle AHB = \angle DFE$, on the other, by I.16, $\angle AHB > \angle DFE$, *the very that is impossible*.

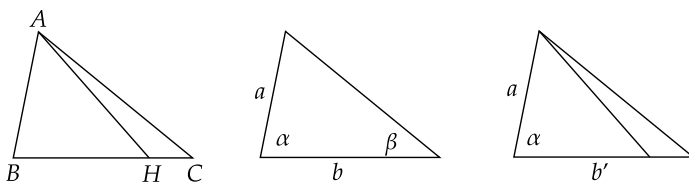


Figure 36: *Elements*, I.26 - scheme of the second case proof.

□

3.13. Parallel lines. I.27–31

Until proposition I.29, Euclid’s arguments do not rely on the parallel postulate, yet, in I.27, aiming to show $AB \parallel CD$, given that $\angle AEF = \angle EFD$ (Fig. 38, left) he invokes definition of parallel lines:

Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (I def. 23).

I.27 *If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the straight-lines will be parallel to one another.* In symbols, $\alpha = \beta \Rightarrow p \parallel l$ (Fig. 37).

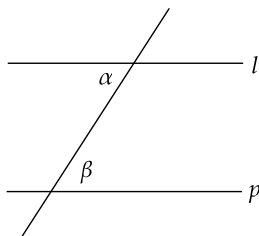


Figure 37: Simplified version of I.27.

The proof proceeds in *reductio ad absurdum* mode and starts with the claim: “if not, being produced, AB and CD will certainly meet together”. Suppose, thus, AB and CD are not parallel and meet in G (Fig. 38, right). Then, in triangle EFG , the external angle $\angle AEF$ is equal to the internal and opposite angle $\angle EFD$, but, by I.16, $\angle AEF$ is also greater than $\angle EFD$. Hence, $\angle AEF = \angle EFD$ and $\angle AEF > \angle EFD$. *The very thing is impossible.*

□

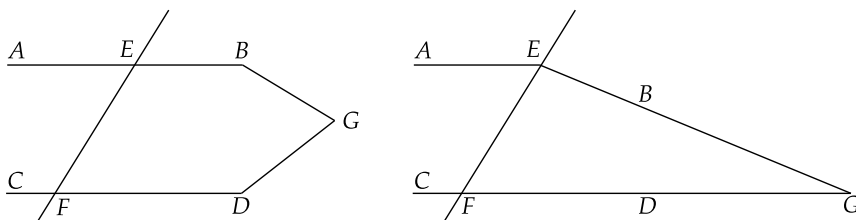


Figure 38: *Elements*, I.27 (left) and a triangle implied in its proof (right).

The rationale for point G lies in the definition of parallel lines rather than in a construction with a straightedge and compass. Thus, next to I.7, it is another non-constructive proposition of the *Elements*.

I.28 *If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, then the (two) straight-lines will be parallel to one another.* In symbols, $\alpha = \beta \Rightarrow l \parallel p$ (Fig. 39).

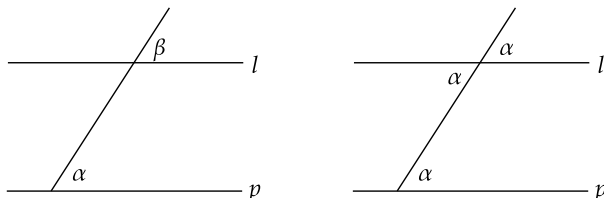


Figure 39: Simplified version of I.28 (left) and its proof (right).

The proof refers to I.27 and the equality of vertical angles (Fig. 39, right).

I.29 *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another.* In symbols (Fig. 40, right)

$$p \parallel l \Rightarrow \alpha = \beta.$$

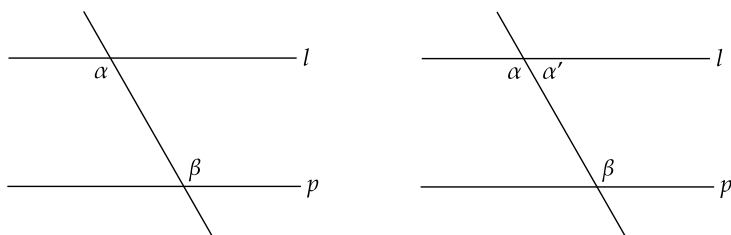


Figure 40: Simplified version of I.29 (left) and its proof (right).

To get a contradiction, suppose $\alpha \neq \beta$. Hence, one of the angles is greater. Let $\alpha > \beta$. Then,

$$\alpha > \beta \Rightarrow \alpha + \alpha' > \beta + \alpha'.$$

Since $\alpha + \alpha' = \pi$, angles β, α' satisfy the requirement of the parallel axiom, i.e., $\beta + \alpha' < \pi$ and straight lines l, p meet, contrary to initial assumption. \square

In I.29, Euclid applies the Fifth Postulate for the first time.²² It reads:

And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles.

²²(De Risi, 2016) enlists versions of the Postulate 5 throughout early modern and modern editions of the *Elements*.

In the Fig. 40, angles β , α' satisfy – let us repeat – the requirement of that Postulate, that is $\beta + \alpha' < \pi$. In definition I.23, parallel lines on a plane are characterized by the condition *being produced to infinity do not meet*; Postulate 5 includes such a condition that when it is satisfied, makes lines intersect when *being produced to infinity*.

Due to propositions I.27–29 we get the following equivalence (Fig. 41)

$$l \parallel p \Leftrightarrow \alpha = \beta.$$

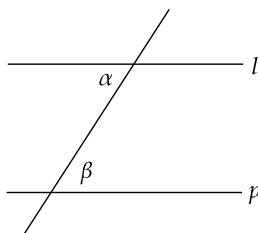


Figure 41: Characteristic of parallel lines.

From I.29 on, Euclid applies the above characteristics of the parallel lines and does not rely on the proviso *being produced to infinity* any more.

I.30 (*Straight-lines*) *parallel to the same straight-line are also parallel to one another.*

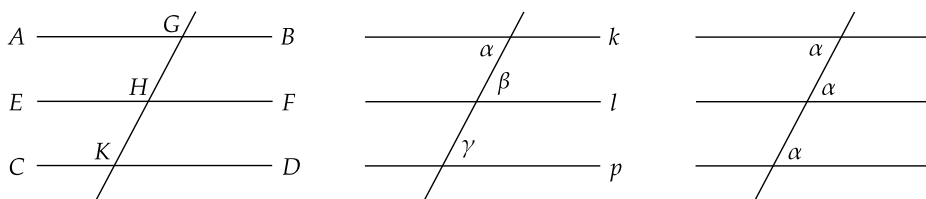


Figure 42: *Elements*, I.30 (left) and its schematized proof (middle and right).

Since $k \parallel l$, by I.29, $\alpha = \beta$ (Fig. 42, middle). Similarly, since $l \parallel p$, by I.29, $\beta = \gamma$. By CN1, $\alpha = \gamma$. Hence, finally, by I.27, $k \parallel p$. □

Propositions I.44, 45 and 47 involve three parallel lines.

The proof seems simple, built on I.27–29 and the transitivity of equality. However, with no discussion, Euclid assumes the existence of the line GK falling on the three parallel lines (Fig. 42, left).²³

²³That assumption, in the case in which all three lines are pairwise parallel, is equivalent to the *Lotschnittaxiom*, as shown in (Pambuccian, 2021), Theorem 5.2.

I.31 *To draw a straight-line parallel to a given straight-line, through a given point.*

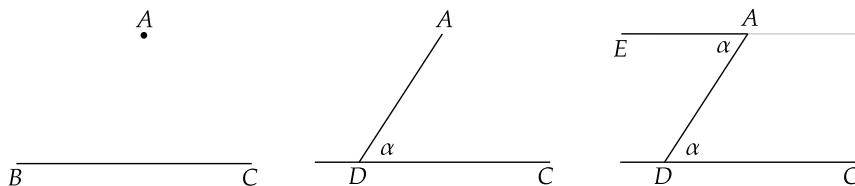


Figure 43: Proof of I.31 schematized.

It is a sheer construction-type proposition. On the straight-line BC , Euclid picks a random point D and copies the angle $ADC = \alpha$ at point A . By I.27, line EA is parallel to DC (Fig. 43 and the below table).

$$\frac{BC \rightarrow}{D} \quad \Bigg| \quad \frac{AD\alpha}{E}$$

In sum, proposition I.27 provides grounds for the existence of a line parallel to p through a point A not lying on p ; therein, ‘parallel’ means not intersecting p . Due to I.29, it is the only line through A not meeting p . These two propositions justify Hilbert’s version of Euclid’s axiom: *There is at most one line parallel to p through A* . Since I.27 holds in the absolute geometry, we can also use a more efficient version, namely: *There is exactly one line parallel to p through A* .

3.14. Sum of angles in triangle. I.32

I.32 *Three internal angles of the triangle, ABC , BCA , and CAB , are equal to two right-angles.*

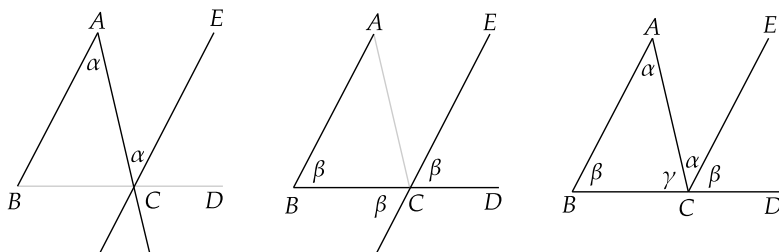


Figure 44: Proof of I.32 schematized.

To prove the thesis, Euclid transports angle α to point C , and draws CE , which, by I.27, is parallel to AB (Fig. 44). Hence, by I.29, $\angle ECD = \beta$, and angles at C sum up to “two right angles”, $\beta + \alpha + \gamma = \pi$.

The construction part comprises to drawing parallel line to AB through the point C .

$$\frac{AB \parallel C}{E}$$

Note that, concerning the angle α , AC is an auxiliary line falling on AB and EC , regarding $\beta - BD$ is such a line. Thus, line EC is parallel to the side AB , while other sides of the triangle play the role of auxiliary lines. It seems possible that in that proof lies Euclid's idea of *Postulate V*.

3.15. Parallelograms. I.33–34

I.33 *Straight-lines joining equal and parallel on the same sides are themselves also equal and parallel.*

In that and the subsequent proposition, straight-line stands for a line segments, thus rendered in symbols, I.33 reads (Fig. 45, middle and right),

$$l \parallel p, l = p \Rightarrow q \parallel s, q = s.$$

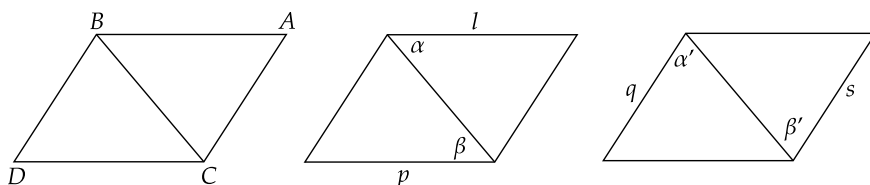


Figure 45: *Elements* I.33 and its schematized proof (middle and right).

From the assumption $l \parallel p$, by I.29, it follows that $\alpha = \beta$. Due to SAS, the equality of triangles obtains $\triangle ABC = \triangle DCB$. Hence, $\alpha' = \beta'$ and $q = s$. Finally, since $\alpha' = \beta'$, by I.27, $q \parallel s$. \square

I.34 *For parallelogrammic figures, the opposite sides and angles are equal to one another and a diagonal cuts them in half.* In symbols (Fig. 46, right),

$$l \parallel p, q \parallel s \Rightarrow l = p, q = s.$$

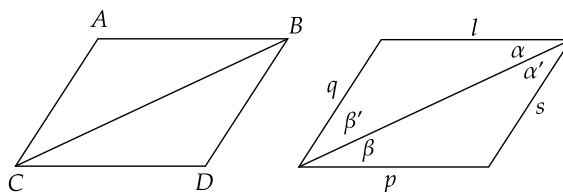


Figure 46: *Elements* I.34 (left) and its schematized proof (right).

Since $l \parallel p$, by I.29, $\alpha = \beta$. Similarly, from $q \parallel s$, the equality of angles follows $\alpha' = \beta'$. Due to ASA rule, the equality of triangles $\triangle ABC = \triangle DCB$ obtains. As a result $l = p$ and $q = s$. \square

3.16. Theory of equal figures. I.35–45

Euclid’s theory of equal figures is a set of propositions enabling the transformation of a (convex) polygon A into a square S , meeting the requirement $A = S$. While congruence of figures is based on *Common Notions* 4, the equality of non-congruent polygons is a procedural one: $A = B$ iff there is a series of figures A_1, \dots, A_n such that $A_1 = A$, $A_i = A_{i+1}$, $A_n = B$, while equalities $A_i = A_{i+1}$ are guaranteed by *Common Notions* and *Postulates* 1 to 3.²⁴ (Błaszczyk, 2018) details the theory, below we only sketch it.

It starts with proposition I.35, which states the equality of parallelograms $ADCB$ and $EFGB$ which are on the same base and between the same parallels (Fig. 47, upper left). Euclid reiterates the proviso *between the same parallels* throughout I.35–45; it means the respective figures are of the same height.

The proof I.35 proceeds as follows: By I.34 and SSS, triangles AEB , DFC are equal. Subtracting T_1 from each of them, the remainders $ADGB$ and $EFGB$ are equal (due to CN3). Adding T_2 to both $ADGB$ and $EFGB$, the whole parallelogram $ABCD$ is equal to the whole parallelogram $EFGB$ (due to CN2).

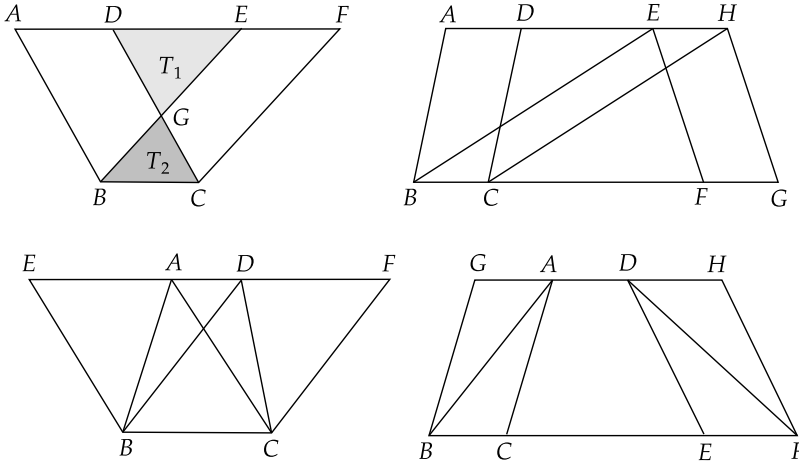


Figure 47: *Elements*, I.35–38.

In proposition I.36 (Fig. 47, upper right), Euclid shows the same result for parallelograms on equal bases. His argument relies on the transitivity of equality guaranteed by CN1 and equalities based on I.35, namely, $ADBC = EHCB$, and $EHCB = EHFGB$.

Propositions I.37–38 (Fig. 47, bottom row) reiterate the same results regarding triangles on the same base, and then, on equal bases. In both cases, triangles are considered halves of respective parallelograms.

²⁴The exhaustion method, developed in Book XII, brings in yet another meaning of equal figures.

I.41 demonstrates that a parallelogram on the same base and between the same parallels as a triangle is the *double of the triangle*. I.42 shows a parallelogram equal to a given triangle. Construction consists of finding the midpoint E of the base BC and drawing parallel lines through E and C (Fig. 48, right). Since angle FEC is arbitrary, that construction enables one to take a triangle into a rectangle.

I.43 demonstrates the equality of parallelograms $FEBG$ and $BMLA$ (Fig. 48, right). I.44 shows how to transform a parallelogram into an equal parallelogram, with an arbitrary height. It is the crucial construction and deserves a scrutiny.

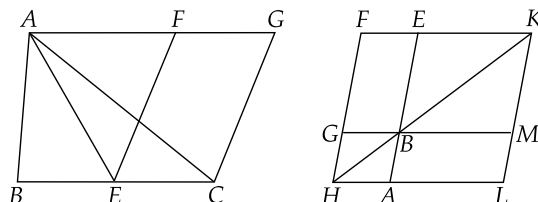


Figure 48: *Elements*, I.42, 44.

Let the parallelogram $FEGB$ be given (Fig. 48, right). The construction runs as presented in the below table, where one takes A on the half-line EB^{\rightarrow} at will, setting the height of the parallelogram $BMLA$.

EB^{\rightarrow}	$FG^{\rightarrow}, A \parallel GB$	$FE^{\rightarrow}, HB^{\rightarrow}$	$K \parallel FG, GB^{\rightarrow}$	$HA^{\rightarrow}, KM^{\rightarrow}$
A	H	K	M	L

The equality of parallelograms $FEGB = BMAL$ follows from a subtraction from equal triangles $\triangle FKH = \triangle LKH$, at first, equal triangles EBK and BMK , then GBH and ABH .

This construction is known as *applying* $FEGB$ to the given straight-line AB . The applied parallelogram has to fit angle $\angle BAL$.

Proposition I.45 summarizes preceding theorems (Fig. 49, left). Euclid's diagram depicts a quadrangle $ADCB$, nevertheless, the method applies to any polygon. The idea is this: cut the polygon $ADCB$ into adjacent triangles, say ADB , DCB ; transform each triangle into a parallelogram (I.42), say P_1 , P_2 ; let $FGHK$ be P_1 ; apply to the line GH a parallelogram equal to P_2 (I.44). It easily follows that $FLMK = ADCB$. Then, turn the resulting parallelogram $FLMK$ into a rectangle. In this way, any polygon A is transformable into an equal rectangle.

The theory culminates with a squaring of a rectangle introduced by the proposition II.14 (Fig. 49, right). Polygons A and B , being turned into squares, are easily compared in terms of greater-lesser.²⁵

²⁵In modern geometry, there are many tries to recover propositions I.35–45 based on an arithmetic of line segments or non-Euclidean proportions, to mention (Hilbert, 1899, ch. 4), (Hartshorne, 2000, ch. 5), or (Beeson, 2022).

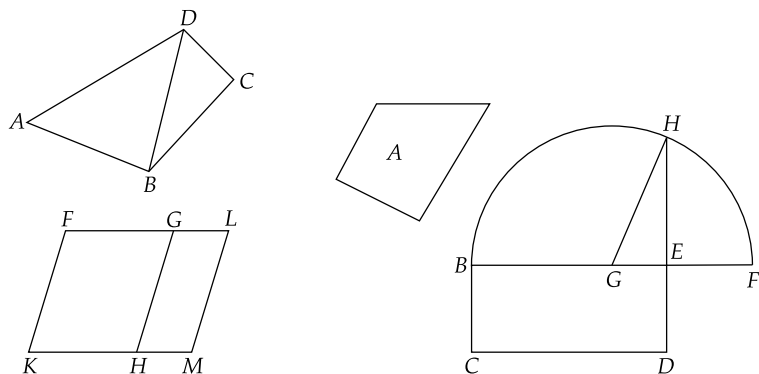


Figure 49: *Elements*, I.45 and II.14.

3.17. From equal figures to parallel lines. I.39–40

The theory of equal figures rests on the concept of a parallel line, meaning the unique line drawn according to I.31: all throughout propositions I.35–38, Euclid studies figures which are “between the same parallels”. In I.39–40, he takes the other way around: starting from equal figures he seeks to reach parallel lines.

I.39 *Equal triangles which are on the same base, and on the same side, are also between the same parallels.*

Let $\triangle ABC = \triangle DBC$ (Fig. 50, left). In I.8, Euclid shows that two equal, i.e., congruent, triangles on the same side of a line are not possible. With a new sense of equality, they are possible.

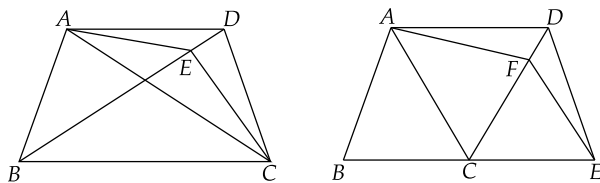


Figure 50: *Elements*, I.39, 40.

For the proof, suppose $\triangle ABC = \triangle DBC$ and AD is not parallel to BC . Then, the construction part follows: the parallel to BC through A meets BD at E .

$$\frac{A \parallel BC, BD \rightarrow}{E}$$

On the one hand, by I.37, $\triangle BEC = \triangle DBC$, on the other, $\triangle BEC < \triangle DBC$. *The very thing is impossible.* \square

The proof of proposition I.40 reiterates the same *reductio ad absurdum* argument.

3.18. Pythagorean theorem. I.46–48

I.46 *To describe a square on a given straight-line.*

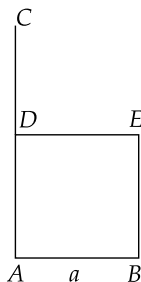


Figure 51: *Elements*, I.46.

Proposition I.40 reiterates the same result in regard to triangles on equal bases.

	$A \perp AB$	$AC, (A, a)$	$D \parallel AB$	$B \parallel AD$	
A, B	C	D			E

The above table mirrors Euclid’s construction. The last column highlights the fact that Euclid does not show that parallels to AB through the point D and to CD through B meet at all. Given that they meet, $ADEB$ is a parallelogram. Hence, by I.34, all of its sides are equal. Furthermore, since AD falls upon parallel lines AB, DE , and angle at A is right, then, by I.29, angle at D is also right. □

I.47 *In a right-angled triangle, the square on the side subtending the right-angle is equal to the squares on the sides surrounding the right-angle.*

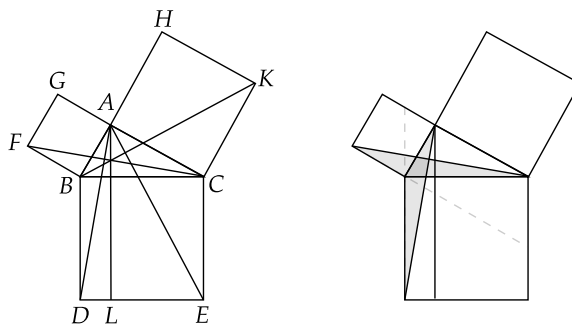


Figure 52: *Elements*, I.47 (left) and the crux step of its proof (right).

Euclid’s construction includes arguments to the effect CA is straight-on to AG that may surprise a modern reader. Indeed, since AG is a leg of a right-angle triangle and CA is a side of a square described on another leg, he has to

	<i>sq on BC</i>	<i>sq on AB</i>	<i>sq on AC</i>	$A \parallel BD, DE$
A, B, C	D, E	F, G	H, K	L

demonstrate these two line segments make a finite straight-line parallel to FB , another side of the square (Fig. 52). Actually, it follows from I.15.

There are two proofs of the Pythagorean theorem in the *Elements*: I.47 and VI.31. The first relies on the theory of equal figures, the second – on similar triangles. The former proceeds as explained below.

On the one hand, by SAS, $\triangle FBC = \triangle ABD$. By I.41

$$sq\ FGAB = 2\triangle FBC = 2\triangle ABD = rec\ DBL.$$

Similarly,

$$sq\ AHKC = rec\ CEL.$$

Hence

$$sq\ FGAB = sq\ AHKC = rec\ DBL + rec\ CEL = sq\ BCDE.$$

Wherein, the equality such as $rec\ DBL + rec\ CEL = sq\ BCDE$, is proved already in II.1 (Błaszczuk, Mrówka, Petiurenko, 2020). □

I.48 *If the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining sides of the triangle then the angle contained by the remaining sides of the triangle is a right-angle.*

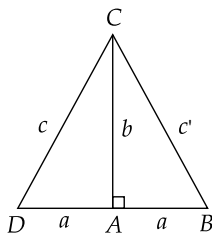


Figure 53: *Elements*, I.48; letters a, b, c , and c' added.

	$A \perp AC$	$AD \rightarrow, Aa$
A, C	D	B

In triangle $\triangle CAD$, the equality holds $c^2 = a^2 + b^2$, where $CD = c, AD = a, AC = b$. In the right triangle $\triangle CAB$, by I.47, the following equality holds $(c')^2 = a^2 + b^2$. And Euclid continues, “the square on DC is equal to the square on BC. So DC is also equal to BC”. Hence, by SSS, $\triangle CAD = \triangle CAB$, and $\angle CAB = \pi/2 = \angle CAD$. □

3.19. Non-Euclidean proportions. Beeson on equal figures

Proposition VI.2 (Thales theorem) is the starting point of Euclid’s theory of similar figures. Its proof relies on VI.1 – the only one in Book VI which applies the definition of proportion (V. def. 5). Euclidean proportion assumes that magnitudes of the same kind are comparable in terms of greater-than. It is explicit in the definition V.5 which we interpret as follows: $a : b :: c : d$ iff for every pair of natural numbers n, m the following conjunction obtains

$$(na >_1 mb \Rightarrow nc >_2 md) \wedge (na = mb \Rightarrow nc = md) \wedge (na <_1 mb \Rightarrow nc <_2 md),$$

where magnitudes of the same type being an ordered semi-group $(M, +, <)$, meaning a, b and c, d are elements of the structures $(M_1, +_1, <_1)$ and $(M_2, +_2, <_2)$ respectively. In VI.1, for example, a, b are line segments, and c, d – triangles (Błaszczyk, 2021).

In contrast, Descartes (Descartes, 1637, 298) takes the Thales theorem as a definition of the product of line segments $a \cdot b$ (Fig. 54, left); Hilbert (Hilbert, 1899, ch. 3) adopts the same strategy regarding a right-angled triangle (Fig. 54, middle), and reduces proportion to the equality of products, namely, $a : b = c : d$ iff $a \cdot d = c \cdot b$; Bernays (Hilbert, 1972, Supplement II) follows Hilbert, but turns his definition into a proportion of line segments, namely: $a : b = c : d$ iff a, b and c, d are legs of equiangular right-angled triangles (Fig. 54, right).²⁶

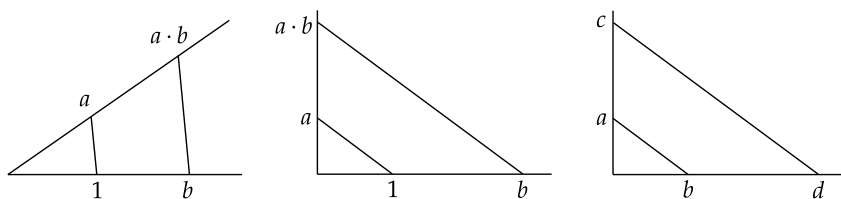


Figure 54: Starting with the Thales theorem: Descartes, Hilbert, Bernays.

Hilbert, Hartshorne, and Bernays, due to the Pappus’s, the cyclic quadrilateral, and Desargues’s theorem, respectively, managed to recover Euclid’s theory of similar triangles and proposition VI.16, relating proportion of line segments $a : b = c : d$ with the equality of products $a \cdot d = c \cdot b$ or rectangles $ad = cb$ – all that without referring to the relation greater-than.

However, non-Euclidean proportions, such as Bernays’s, apply only to line segments and can not even formulate acmes of Euclid’s theory which played a significant role in the early modern mathematics, namely V.19–20, relating areas of similar figures and the similarity scale.²⁷

²⁶More precisely, Bernays defines a ratio of legs a, b of a right-angled triangle in the following way: $a : b = \alpha$, where α is the angle opposite to the side a , and then, a proportion $a : b = c : d$ being equality of two ratios. Hartshorne takes Bernay’s ratio as a definition of the product $a \cdot b$ and simplifies Hilbert’s proof of the commutativity of the product and the distributive law (Hartshorne, 2000, 170–172).

²⁷See for example (Descartes, 1637, 104–105).

On the other hand, within the option of the segment arithmetic (Hilbert, Hartshorne), the formula for the area of a triangle, one-half of the product of a base and corresponding altitude, enables to formulate counterparts of VI.19–20. Though, the formula for the area of a triangle requires proof that it does not depend on the choice of a pair base and corresponding altitude. Indeed, right-angled triangles, first, with hypotenuse b and leg h_a , second, with hypotenuse a and leg h_b , are equiangular (Fig. 55, left), thus $a : h_b = b : h_a$, which within the arithmetic of line segments means $a \cdot h_a = b \cdot h_b$. Similarly, we can show that $a \cdot h_a = c \cdot h_c$.²⁸

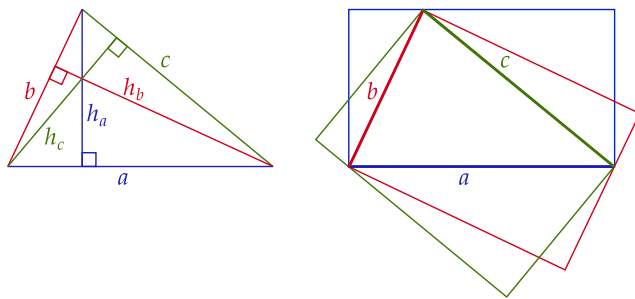


Figure 55: The invariance of the area of a triangle based on formula (left), equality of rectangles $ah_a = bh_b = ch_c$ (right).

Beeson (Beeson, 2022) seeking to revise Euclid’s theory of equal figures and its foundations, adopts Bernays’s definition of proportion and the following definition of equal rectangles: rectangles placed like yellow ones in Fig. 56 are equal *iff* points B, H, K are collinear. Thus, starting from the special case of I.43 (or I.44) taking a rectangle instead of a parallelogram, he gets both Bernays’s proportion $b : a = d : c$ and his own equality of rectangles (Fig. 56).²⁹

Regarding triangles, instead of the product of a base and the corresponding altitude, he considers a respective rectangle, e.g., with sides a, h_a . Since each base and the corresponding altitude determine different rectangles, Beeson has to show that they are equal (due to his definition). Indeed, he shows that rectangles with sides a, h_a, b, h_b , and c, h_c are equal (Fig. 55, rectangles blue, red and green, respectively). Yet, the proof is the same as the one on the formula for the area of a triangle (Beeson, 2022, Theorem 6.1). Finally, two triangles are equal *iff* the corresponding (*circumscribed*) rectangles are equal (Beeson, 2022, Definition 5.10).

²⁸Cf. “It is not the case that just because we can multiply, we can define areas, even of triangles. The problem of interpreting what Euclid meant by ‘equal area’ is not automatically solved by defining geometric arithmetic” (Beeson, 2022, 6). We can not guess any rationale for that claim.

²⁹That is how Beeson motivates his study: “The contribution of this paper is to eliminate the ‘equal figures’ axioms by defining the notions of ‘equal triangles’ and ‘equal quadrilaterals’” (Beeson, 2022, 6). Beeson’s sketchy analysis does not explain what he finds wrong with Euclid’s *equal figures axioms*, i.e., Common Notions, while some of his comments contradict textual evidence, for example, “it is worth noting that not only does he [Euclid] never mention the word ‘area’, but he also never speaks of one figure being greater than another. He never applied the common notions that mention ‘greater than’ to figures” (Beeson, 2022, 4).

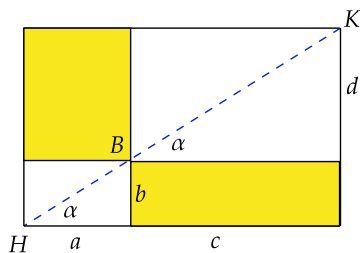


Figure 56: Beeson’s definition of equal rectangles.

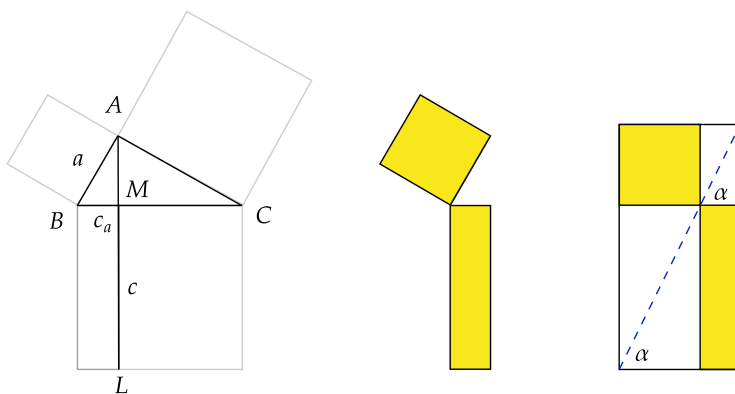


Figure 57: I.47 through Beeson’s method.

Now, let us check whether Beeson’s idea of equal rectangles provides new insights. Since I.47 is the first application of Euclid’s theory of equal figures, let us test Beeson’s method on that proposition.³⁰ In the right-angled triangle ABC , let us denote $BC = c = ML$, $AB = a$, $BM = c_a$ (Fig. 57, left). Within Bernays’s theory, one can show the counterpart of VI.8, i.e., proportion $a : c_a = c : a$, and then, mirror Euclid’s proof of the Pythagorean theorem based on similar triangles (VI.31). Euclid’s proof I.47 builds on his theory of equal figures, so let us try that option. To show that yellow square and rectangle are equal (Fig. 57, middle), one has to place them like in Fig. 57 (the diagram on the right). Then, their equality follows from the proportion, namely $a : c_a = c : a$ and VI.6. However, we need another axiom to guarantee that the square in the middle diagram equals the square in the diagram on the right. To that end, Beeson adopts the following rule: “Two rectangles are congruent if their sides are pairwise equal” (Beeson, 2022, 13).

In Euclid’s proof, equality of triangles $\triangle FBC$ and $\triangle ABD$ guarantees equality of yellow square and rectangle. It follows from proposition I.4; within the Hilbert system, let us remind, it is axiom C6. Beeson’s rule on equal rectangles, thus, is

³⁰Cf. Beeson’s account of I.47 (Beeson, 2022, 2).

just an extra axiom. We cannot figure out any benefits of such an extension of the Euclid system. The original technique of triangulation imprints on Euclid's absolute geometry and his theories of equal and similar figures.

Finally, there is a fundamental ambiguity in Beeson's project as he writes:

The contribution of this paper is to eliminate the 'equal figures' axioms by defining the notions of 'equal triangles' and 'equal quadrilaterals', by a definition that Euclid could have given, and proving the properties expressed in the 'equal figures' axioms, so that Euclid Books I to IV could be developed without the equal figures axioms. [...] Our method to achieve this aim is to define 'equal rectangles' using a figure much like the one Euclid uses for Prop. I.44. and use that to define 'equal triangles' and 'equal quadrilaterals' and use those defined notions to prove the propositions of Euclid Book I. [...] It follows that the equal-figures axioms are actually superfluous, in the sense that, using the new definition of 'equal figures', we could formalize Euclid Book I directly, without adding any equal-figures axioms. But we then take one step more, and show that the equal-figures axioms can in fact all be proved (Beeson, 2022, 6–7).

We have shown that in propositions I.1–34 Euclid applies Common Notions to congruent triangles and the theory of parallel lines enable him to define a rectangle. It is not clear, to say the least, how can Beeson develop *Elements* I.1–28 based on the concept of rectangle when he can not prove the existence of a rectangle.

3.20. Euclid and modern geometry. Mueller on Book I

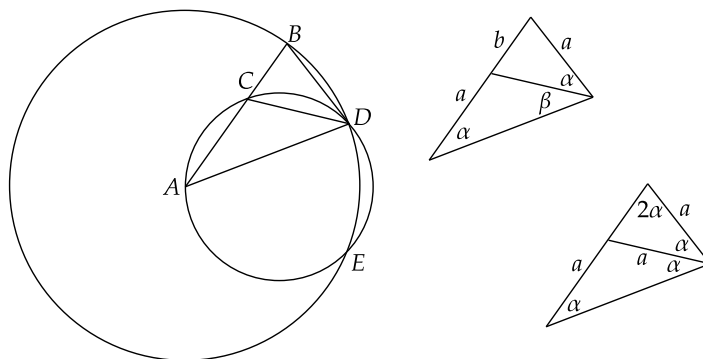
In the *Elements* (Στοιχεῖα), Euclid studies elements (στοιχεῖα) of primitive matter. According to Plato's cosmology, these were regular solids: tetrahedron, cube, octahedron, icosahedron (*Timaios*, 54d–55c). Euclid managed to construct the fifth one, unknown to Plato (*Timaios*, 55c), namely, dodecahedron. These elements, although real, were considered invisible (*Timaios*, 56c). Book XIII shows how to construct them, compares their sides when enclosed in the same sphere, and finally, includes the demonstration that there are no other regular (convex) polyhedra (XIII.18).

Viewed from that cosmological perspective, Book I through IV, have a clear objective: to construct an equilateral and equiangular pentagon (IV.11). Proposition IV.10 is essential in that plan. Digging through its proof, we find the entire resources of Euclid's plane geometry.

IV.10 *To construct an isosceles triangle having each of the angles at the base double the remaining (angle).*

In the Fig. 58, AB is the radius of the bigger circle; to simplify the subsequent account, let us put $AC = a$, $BC = b$. C is a point on AB such that $(a + b)b = a^2$ (II.11), D is such that $BD = a$ (I.3) and $AD = AB$, meaning, it is an intersection of two circles. ACD is the circumcircle on the triangle $\triangle ACD$ (IV.5)

By III.37, BD is tangent to the circle ACD , hence, by III.32, $\angle BAD = \alpha = \angle CDB$. Since the triangle $\triangle BAD$ is isosceles, the equality of angles holds


 Figure 58: *Elements*, IV.10.

$\angle ABD = \alpha + \beta = \angle BDA$ (I.5). By I.32, $\angle ACD = \pi - (\alpha + \beta)$. Hence, by I.13, $\angle BCD = \alpha + \beta$. Thus, the triangle $\triangle CDB$ is isosceles and $CD = a = BD$ (I.6). Again, the triangle $\triangle ACD$ is isosceles and $\alpha = \beta$ (I.5). Finally, in the triangle $\triangle ABD$ angles at the base BD double the angle at A .

□

Moreover, reviewing proofs of referred propositions, we find out that I.47 is the basic technique while proving II.9–14 and III.36–37, while propositions I.11–12 are essential when a tangent to a circle is studied. In short, Book IV, specifically IV.10, applies all sub-theories developed by Euclid in Books I through III.

In the 20th century, applications do not affect the understanding of geometry. Modern interpretations take a strictly formal perspective, focused on whether the theory is consistent (Hilbert), categorical (Borsuk and Szmielew), or decidable (Tarski). Accordingly, they seek to tame the formal language of geometry and expose its techniques such as congruent triangles, transportation of line segments, and angles. Indeed, it was basically our perspective while reviewing Book I. In the 21st century, that tendency pushes forward into automated theorem proving (Beeson, 2010), (Janičić, Narboux, Quaresma, 2012), (Beeson, Narboux, Wiedijk, 2019), (Błaszczuk, Petiurenko, 2019).

(Mueller, 1981) takes yet another position and reviews Book I from the perspective of the most important proposition. Mueller explains it as follows:

book I can be explained by reference to the construction of a parallelogram in a given angle and equal (in area) to a given rectilinear figure in proposition 45. [...] The point of view adopted here may be expressed by saying that this knowledge and the desire to prove I.45 by themselves suffice to account for much of book I. Obviously, even when this point of view is correct, it leaves much out of consideration. [...] More importantly perhaps, no explanation is offered for Euclid's interest in proving I.45 or II.14 rather than, e.g., justifying some formula or procedure for computing the area of an arbitrary figure (Mueller, 1981, 16–17).

Mueller, thus, believes proposition I.45 is the most important in Book I as it crowns the theory of equal figures, but he does not provide any rationale for that theory. From the perspective of regular polyhedra, the motivation for the theory of equal figures is the following: equality of non-congruent figures enables proving I.47, and then, both equality of non-congruent figures plus I.47 make tools in proving II.11 and III.37, which find their rationale in IV.10. We can speculate whether it could be any different. Hilbert's theory of plane area, as developed in (Hilbert, 1899, ch. 4), or Hartshorne's area function (Hartshorne, 2000, ch. 5), provide alternatives that, interestingly, rely on the arithmetic of line segments introduced by Descartes. The fact is, ancient mathematics did not develop any alternative to Euclid's theory of equal figures.

3.21. Non-use of the Fifth Postulate

In stark contrast to Euclidean practice, an opinion prevails that the Fifth Postulate is problematic, controversial, or unobvious. In proposition I.30 Euclid does not demonstrate the existence of the straight-line GK falling across three lines AB , EH , and CD (Fig. 42). In I.39, he takes for granted that lines BD and the parallel to BC drawn through A meet in E (Fig. 50).³¹ In I.46, he does not show that the parallel to AB drawn through D and the parallel to AD drawn through B meet in E (Fig. 51). In IV.5, when showing that the center of the circumcircle of a triangle is the intersection of perpendicular bisectors of its sides, he considers cases depending on whether they meet inside the triangle, or on its side, or outside it; he does not show that they meet at all. Indeed, in all these cases, the existence of intersection points follows from the Postulate 5. Interestingly enough, Euclid does not even mention there is a need to prove it. It is in contrast with the meticulousness of his other proofs. The only exception is I.44, where Euclid invokes the Postulate 5 to show that straight lines FE and HB meet at point K (Fig. 48).

Some of these propositions, e.g., IV.5, are equivalent forms of the Fifth Postulate; others, e.g., the relationship between I.30 and I.46, led to intriguing debate within the modern foundations of geometry (Pambuccian, 2021). In the subsequent section, we present a model of non-Euclidean plane and will discuss the assumptions underlying these propositions.

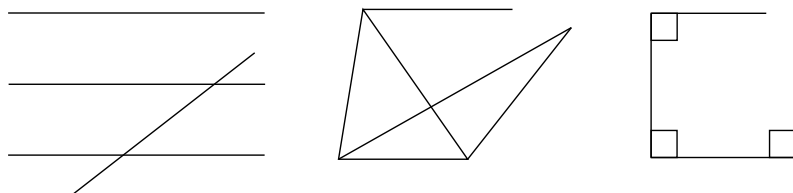


Figure 59: Seeking for intersection points in propositions I.30, 39, 46.

³¹Similarly in constructions such as I.42, 44, 45.

4. Semi-Euclidean plane

In this section, we present a model of a semi-Euclidean plane, i.e., a plane in which angles in a triangle sum up to π yet the parallel postulate fails. (Hartshorne, 2000, 311), introduces that term, but the very idea originates in (Dehn, 1900, §9). Dehn built such a model owing to a non-Archimedean Pythagorean field introduced in (Hilbert, 1899, §12); yet, it was a non-Euclidean field.³² We employ the Euclidean field of hyperreal numbers. In the Cartesian plane over hyperreals, the circle-circle and circle-line intersection axioms are satisfied, meaning one can mirror Euclid's straightedge and compass constructions. To elaborate, let us start with the introduction of the hyperreal numbers.

4.1. The Cartesian plane over the field of hyperreal numbers

An ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ is a commutative field together with a total order that is compatible with sums and products. In such a field, one can define the following subsets of \mathbb{F} :

$$\begin{aligned} \mathbb{L} &= \{x \in \mathbb{F} : (\exists n \in \mathbb{N})(|x| < n)\}, \\ \Psi &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| > n)\}, \\ \Omega &= \{x \in \mathbb{F} : (\forall n \in \mathbb{N})(|x| < \frac{1}{n})\}. \end{aligned}$$

They are called limited, infinite, and infinitely small numbers, respectively. Here are some relationships helpful to pursue our arguments.

$$\begin{aligned} (\forall x, y \in \Omega)(x + y \in \Omega, xy \in \Omega), \\ (\forall x \in \Omega)(\forall y \in \mathbb{L})(xy \in \Omega), \\ (\forall x \neq 0)(x \in \Omega \Leftrightarrow x^{-1} \in \Psi). \end{aligned}$$

To clarify our account, let us observe that the following equality $\Omega = \{0\}$ is a version of the well-known Archimedean axiom.

Since real numbers form the *biggest* Archimedean field, every field extension of $(\mathbb{R}, +, \cdot, 0, 1, <)$ includes positive infinitesimals. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} . The set of hyperreals is defined as a reduced product $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$. Sums, products, and the order are introduced pointwise. The field of hyperreals $(\mathbb{R}^*, +, \cdot, 0, 1, <)$ extends real numbers, hence, includes infinitesimals and infinite numbers; moreover, it is closed under the square root operation (Błaszczyk, 2016), (Błaszczyk, 2021). Fig. 60 represents in a schematized way a relationship between \mathbb{R} and \mathbb{R}^* , as well as between \mathbb{L} , Ψ , and Ω .

Due to the proposition (Hartshorne, 2000, 16.2), the Cartesian plane over the field of hyperreals is a model of Euclidean plane, with straight lines and circles given by equations $ax + by + c = 0$, $(x - a)^2 + (y - b)^2 = r^2$, where $a, b, c, r \in \mathbb{R}^*$ and due to the equation of a straight line, parameters a, b have to satisfy condition $a^2 + b^2 \neq 0$; angles between straight lines are defined as in the Cartesian plane over

³²See also (Hartshorne, 2000, §18). Example 18.4.3 expounds Dehn's model.

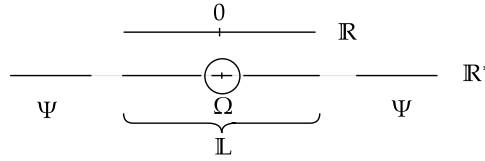


Figure 60: The line of real numbers and its extension to hyperreals.

the field of real numbers. Specifically, on the plane $\mathbb{R}^* \times \mathbb{R}^*$, angles in triangles sum up to π . Parallel lines are of the form $y = mx + b$ and $y = mx + c$, while a perpendicular to the line $y = mx + b$ is given by the formula $y = -\frac{1}{m}x + d$.

Now, let us take a subspace $\mathbb{L} \times \mathbb{L}$ of the plane $\mathbb{R}^* \times \mathbb{R}^*$. In that plane, circles are defined by analogous formula, namely $(x - a)^2 + (y - b)^2 = r^2$, where $a, b, c, r \in \mathbb{L}$, while every line in $\mathbb{L} \times \mathbb{L}$ is of the form $l \cap \mathbb{L} \times \mathbb{L}$, where l is a line in $\mathbb{R}^* \times \mathbb{R}^*$. Since we want plane $\mathbb{L} \times \mathbb{L}$ include lines such as $y_1 = \varepsilon x$, where $\varepsilon \in \Omega$, it also has to include the perpendicular $y_2 = -\frac{1}{\varepsilon}x$, but $-\frac{1}{\varepsilon} \notin \mathbb{L}$. Formula $l \cap \mathbb{L} \times \mathbb{L}$, where $l = ax + by + c$ and $a, b, c \in \mathbb{R}^*$ guarantees the existence of the straight line y_2 in $\mathbb{L} \times \mathbb{L}$. Finally, the interpretation of an angle is the same as in the model $\mathbb{R}^* \times \mathbb{R}^*$.

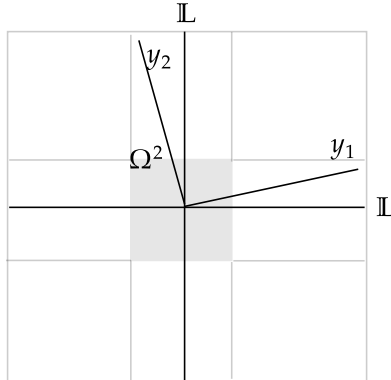


Figure 61: Perpendicular lines with infinitesimal and infinitely large slopes.

Explicit checking shows that the model characterized above satisfies all Hilbert axioms of non-Archimedean plane geometry plus the circle-circle and line-circle axioms, except parallel axiom; the more general theorem concerning Hilbert planes also justifies our model, namely (Hartshorne, 2000, 425), Theorem, 43.7 (a).

With regard to parallel lines, let us consider the horizontal line $y = 1$ and two specific lines through $(0, 0)$, namely $y_1 = \varepsilon x, y_2 = \delta x$, where $\varepsilon, \delta \in \Omega$. (Fig. 62). Since $\Omega\mathbb{L} \subset \Omega$, the following inclusions hold $y_1, y_2 \subset \mathbb{L} \times \Omega$. In other words, values of maps $y_1(x), y_2(x)$ are infinitesimals, given that $x \in \mathbb{L}$. The same obtains for any line of the form $y = \mu x$, with $\mu \in \Omega$. Since there are infinitely many infinitesimals, there are infinitely many lines through $(0, 0)$ not intersecting the horizontal line $y = 1$.

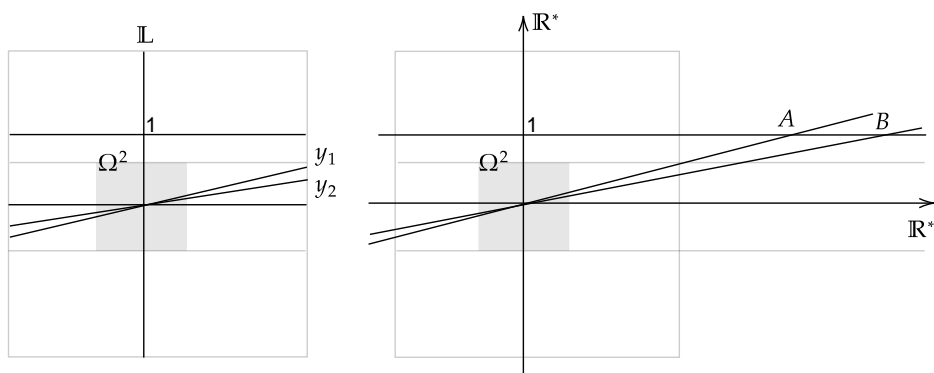


Figure 62: Non-Euclidean plane $\mathbb{L} \times \mathbb{L}$ vs. Euclidean plane $\mathbb{R}^* \times \mathbb{R}^*$.

Since every triangle in $\mathbb{L} \times \mathbb{L}$ is a triangle in $\mathbb{R}^* \times \mathbb{R}^*$, it follows that angles in a triangle on the plane $\mathbb{L} \times \mathbb{L}$ sum up to π (Fig. 63).

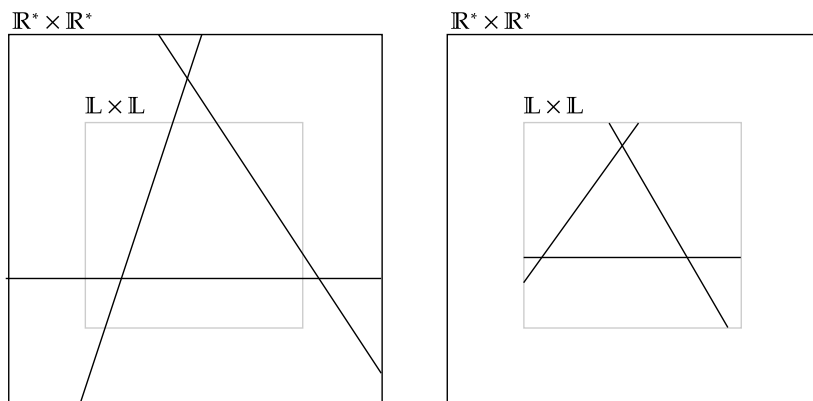


Figure 63: Triangles in Euclidean plane $\mathbb{R}^* \times \mathbb{R}^*$ and its subspace $\mathbb{L} \times \mathbb{L}$.

4.2. Euclid's propositions which do not hold in the plane $\mathbb{L} \times \mathbb{L}$

In proposition I.30, Euclid assumes the existence of a line falling on three parallel lines. In the plane $\mathbb{L} \times \mathbb{L}$ that assumption fails. Let us take three horizontal lines $y_1 = 0$, $y_2 = \varepsilon$, $y_3 = 1$. Line $y_4 = \varepsilon x$ meets y_1 at $(0, 0)$ and y_2 at $(1, \varepsilon)$. Yet the intersection of y_4 and y_3 is $(\frac{1}{\varepsilon}, 1)$ which does not belong to $\mathbb{L} \times \mathbb{L}$ (Fig. 64).

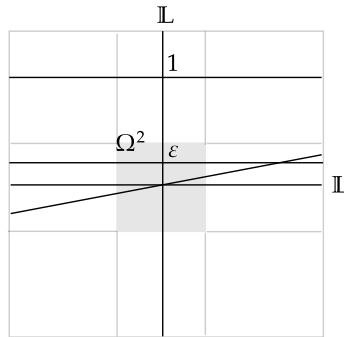


Figure 64: Transverse line crossing two of the three parallel lines.

In I.46, describing a square on a given straight line, Euclid draws parallels to sides of a right angle. As the construction applies I.31, they are perpendicular to the sides of the right angle. In the below examples, we take them to be literally parallels. In the plane $\mathbb{L} \times \mathbb{L}$, they have to meet, but the resulting figure is not a square (Fig. 65).

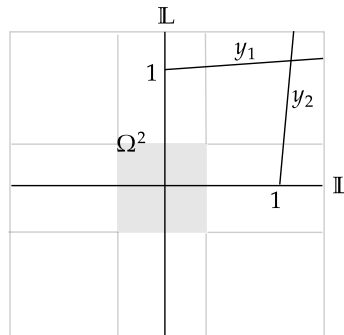


Figure 65: Parallels to sides of the right angle.

Let $y_1 = \varepsilon x + 1$ and $y_2 = \frac{x}{\varepsilon} - \frac{1}{\varepsilon}$, where $\varepsilon \in \Omega$. They do not intersect $x = 0$ and $y = 0$, respectively, meaning, $y_1 \parallel x = 0$, $y_2 \parallel y = 0$. Yet, they meet at the point $(\frac{1+\varepsilon}{1+\varepsilon^2}, 1 + \frac{\varepsilon^2+\varepsilon}{1+\varepsilon^2})$, and the resulting figure is not a square.

In VI.5, Euclid assumes that perpendicular bisectors of two sides of a triangle meet. Below we show it does not hold in the plane $\mathbb{L} \times \mathbb{L}$.

Let us take the line $y = -\varepsilon$ and points $A = (-1, -\varepsilon)$, $B = (1, -\varepsilon)$ on it. A line through the point $C = (0, 0)$ and A has the equation $y = \varepsilon x$. Perpendicular bisectors of the sides AB and AC have the equations $x = 0$ and $2x + 2\varepsilon y + \varepsilon^2 + 1 = 0$, respectively. They meet at the point $(0, -\varepsilon - \frac{1}{\varepsilon})$. However, $(0, -\varepsilon - \frac{1}{\varepsilon}) \notin \mathbb{L} \times \mathbb{L}$. Fig. 66 depicts three perpendiculars to the sides of the triangle ABC .

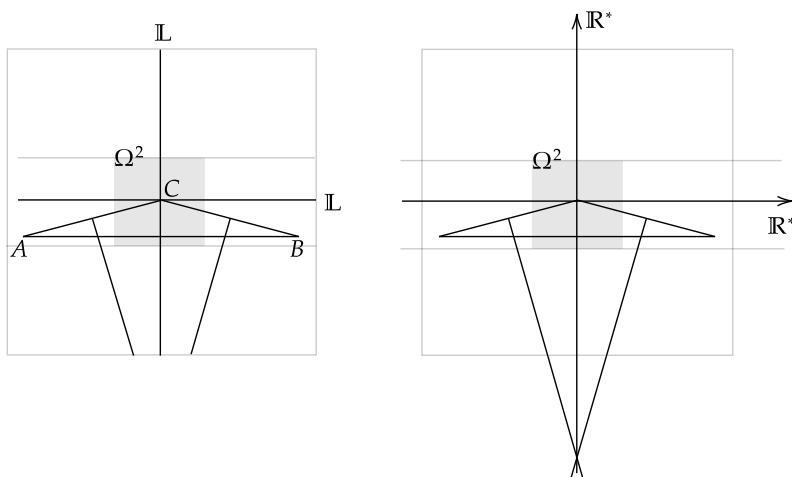


Figure 66: Triangle in $\mathbb{L} \times \mathbb{L}$ with no circumcircle (left) and its counterpart in $\mathbb{R}^* \times \mathbb{R}^*$ (right).

4.3. Klein and Poincaré disks. Euclidean trigonometry

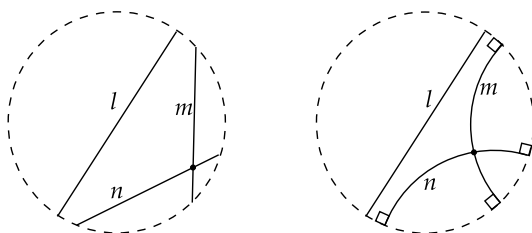


Figure 67: Straight lines in Klein (left) and Poincaré (right) disk.

Klein and Poincaré disks are classical models of non-Euclidean geometry (Fig. 67). Both consist of a fixed circle in the Euclidean plane, say Γ , representing the plane. In the Klein disk, chords of Γ are straight lines; in the Poincaré disk, straight lines are diameters of Γ or arcs of circles orthogonal to Γ (Fig. 67).

In the Poincaré model, an angle between intersecting circles is the Euclidean angle between tangents to these lines drawn at their intersection point (Fig. 68). In the Klein disc, an angle between intersecting straight lines is retrieved from the Poincaré model as presented in Fig. 69: for lines n, m , we draw circles orthogonal to Γ and determine the angle between them.

Standard models of the non-Euclidean plane, thus, change Euclid's concept of a straight line or angle. In the plane $\mathbb{L} \times \mathbb{L}$, they are both Euclidean. Moreover, we can also develop Euclidean trigonometry in that plane. To elaborate, let us step back to the construction of the field of hyperreals.

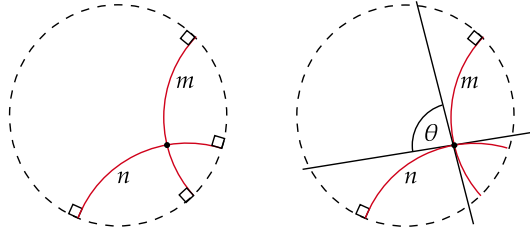


Figure 68: Angles in the Poincaré disc.

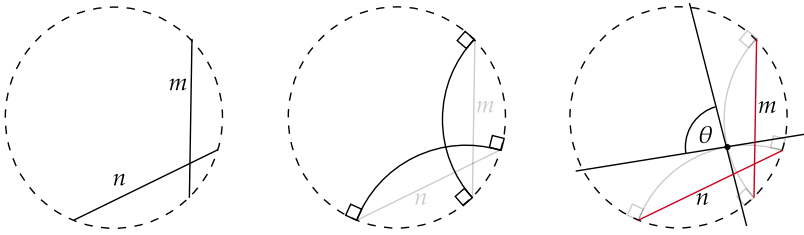


Figure 69: Angles in the Klein disc.

Let f be a real map, i.e. $f : \mathbb{R} \mapsto \mathbb{R}$. Its extension to a map on \mathbb{R}^* , $f^* : \mathbb{R}^* \mapsto \mathbb{R}^*$, is defined by

$$f^*([(r_n)]) = [(f(r_1), f(r_2), \dots)]. \tag{1}$$

If $r \in \mathbb{R}$, then $f^*(r) = [(f(r), f(r), \dots)]$. Since we identify real number r with hyperreal $[(r, r, \dots)]$, the equality $f^*([(r, r, \dots)]) = f(r)$ obtains, meaning, f^* extends f , $f^*|_{\mathbb{R}} = f$.

Putting $f = \sqrt{}$ in definition (1), we get

$$\sqrt{[(r_n)]}^* = [(\sqrt{r_n})] = [(\sqrt{r_1}, \sqrt{r_2}, \dots)], \text{ for } [(r_j)] > 0.$$

Similarity, under the definition (1), we have

$$\sin^*[(r_n)] = [(\sin r_1, \sin r_2, \dots)], \quad \cos^*[(r_n)] = [(\cos r_1, \cos r_2, \dots)].$$

Since for every n the identity $\sin^2 r_n + \cos^2 r_n = 1$ holds, we have

$$(\sin^* x)^2 + (\cos^* x)^2 = 1.$$

Similarly, every trigonometric identity translates into an identity involving the maps \sin^* and \cos^* , \tan^* , and \cot^* . From a *local* perspective, thus, the plane $\mathbb{L} \times \mathbb{L}$ is *similar* to a Euclidean plane, *globally* – to put it metaphorically – its straight lines are *too short* to meet the Parallel Postulate. Last but not least, that plane has a unique educational advantage: expounding crucial ideas of that model requires only the basics of Cartesian geometry and non-Archimedean fields.

References

- Avigad, J., Dean, E., Mumma, J.: 2009, Formal System for Euclid's *Elements*. *The Review of Symbolic Logic* **2**(4), 700–768.
- Beeson, M.: On the Notion of Equal Figures in Euclid; <https://www.researchgate.net/publication/343986568> [visited on 16 June, 2022]
- Beeson, M.: 2010, Constructive geometry. In: Arai, T. (ed.) et al., *Proceedings of the 10th Asian logic conference, Kobe, Japan, September 1–6, 2008*. Hackensack, NJ, World Scientific, 19–84; <https://www.researchgate.net/publication/264957027> [visited on 16 June, 2022]
- Beeson, M., Narboux, J., Wiedijk, F.: 2019, Proof-checking Euclid. *Annals of Mathematics and Artificial Intelligence* **85**, 213–257.
- Błaszczczyk, P.: 2021, Descartes' Transformation of Greek Notion of Proportionality. In B. Sriraman (ed.) *Handbook of the History and Philosophy of Mathematical Practice*. Springer, Cham; https://link.springer.com/referenceworkentry/10.1007/978-3-030-19071-2_16-1
- Błaszczczyk, P.: 2021, Galileo's paradox and numerosities. *Zagadnienia Filozoficzne w Nauce* **70**, 73–107.
- Błaszczczyk, P.: 2018, From Euclid's Elements to the methodology of mathematics. Two ways of viewing mathematical theory. *AUPC* **10**, 5–15; <https://didacticamath.up.krakow.pl/article/view/6613>
- Błaszczczyk, P.: 2016, A Purely Algebraic Proof of the Fundamental Theorem of Algebra. *AUPC* **8**, 6–22; <https://didacticamath.up.krakow.pl/article/view/3638>
- Błaszczczyk, P., Major, J.: 2014, Calculus without the Concept of Limit. *AUPC* **6**, 19–38; <https://didacticamath.up.krakow.pl/article/view/3654>
- Błaszczczyk, P., Mrówka, K.: 2013, *Euklides, Elementy, Księgi V–VI. Tłumaczenie i komentarz*, Copernicus Center Press, Kraków.
- Błaszczczyk, P., Mrówka, K., Petiurenko, A.: 2020, Decoding Book II of the *Elements*. *AUPC* **12**, 39–88; <https://didacticamath.up.krakow.pl/article/view/8462>
- Błaszczczyk, P., Petiurenko, A.: 2019, Euclid's proportion revised. *AUPC* **11**, 37–61; <https://didacticamath.up.krakow.pl/index.php/aupcsdmp/article/view/6901>
- Błaszczczyk, P., Petiurenko, A.: 2022, Euler's Series for Sine and Cosine. An Interpretation in Nonstandard Analysis. In: Waszek, D., Zack, M. (eds.). *Annals of the CSHPM*, Birkhauser (to appear).

- Błaszczyk, P., Petiurenko, A.: On diagrams accompanying *reductio ad absurdum* proofs in Euclid's *Elements* book I. Reviewing Hartshorne and Manders; <https://arxiv.org/pdf/2206.12213.pdf>; <https://www.researchgate.net/publication/361559448>
- Borsuk, K., Szmielew, W.: 1960, *Foundations of Geometry*, North-Holland, Amsterdam.
- Borsuk, K., Szmielew, W.: 1972, *Podstawy Geometrii*, PWN, Warszawa.
- Dehn, M.: 1900, Legendre'schen Sätze über die Winkelsumme im Dreieck. *Mathematische Annalen* **53**(3), 404–439.
- De Risi, V.: 2021, Euclid's Common Notions and the Theory of Equivalence. *Foundations of Science* **26**, 301–324.
- De Risi, V.: 2016, The development of Euclidean axiomatics The systems of principles and the foundations of mathematics in editions of the *Elements* in the Early Modern Age. *Archive for the History of Exact Science* **70**, 591– 676.
- Descartes, R.: 1637, *Discours de la Méthode pour bien conduire sa raison, et chercher la vérité dans les sciences, plus la Dioptrique, les Météores et la Géométrie qui sont des essais de cette Méthode*. Jan Maire, Lejda.
- Fitzpatrick, R.: 2007, *Euclid's Elements of Geometry translated by R. Fitzpatrick*; <http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf>
- Greenberg, M.: 2008, *Euclidean and non-Euclidean Geometries*, Freeman, New York.
- Hartshorne, R.: 2000, *Geometry: Euclid and Beyond*, Springer, New York.
- Heiberg, J.: 1883–1884, *Euclidis Elementa*. Vol. I–II. Teubneri, Lipsiae.
- Heiberg, J., Menge, H.: 1895, *Euclidis Opera Omnia*. Vol. VII. Teubneri, Lipsiae.
- Hilbert, D.: 2015, Grundlagen der Geometrie. Festschrift Zur Feier Der Enthüllung Des Gauss-Weber-Denkmal in Göttingen. Teubner, Leipzig 1899, 1–92. In: K. Volkert (Hrsg.), David Hilbert, *Grundlagen der Geometrie* (Festschrift 1899), Springer, Berlin.
- Hilbert, D.: 1972, *Grundlagen der Geometrie*. 11. Auflage, Stuttgart.
- Janičić, P., Narboux, J., Quresma, P.: 2012, The area method: a Recapitulation, *Journal of Automated Reasoning* **48**(4), 489–532.
- Martin, G.: 1998, *Geometric Constructions*, Springer, Berlin.
- Mueller, I.: 2006, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Dover, New York (first edition: MIT Press, Cambridge, Massachusetts 1981).

- Pambuccian, V.: 2021, Schacht, C. The Ubiquitous Axiom, *Results in Mathematics* **76**, 114.
- Pambuccian, V.: 1998, Zur Existenz Gleichseitiger Dreiecke in H-Ebenen, *Journal of Geometry* **63**(1–2), 147–153.
- Pollard, S. (ed.): 2010, *Essays on the Foundations of Mathematics by Moritz Pasch*, Springer, New York.
- Tarski, A.: 1959, What is elementary geometry? In: Henkin, L., Suppes, P., Tarski, A. (eds.) *The Axiomatic Method*, North-Holland, Amsterdam, 16–29.
- Tarski, A., Givant, S.: 1999, Tarski's System of Geometry, *Bulletin of Symbolic Logic* **5**, 175–214.

Piotr Błaszczyk
Pedagogical University of Krakow,
Poland
e-mail: piotr.blaszczyk@up.krakow.pl

Anna Petiurenko
Pedagogical University of Krakow,
Poland
e-mail: anna.petiurenko@up.krakow.pl