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Veronika Hubeňáková, Ute Sproesser, Ingrid Semanišinová

Comparing German and Slovak teachers' knowledge of content and students related to functions *

Abstract. A crucial aspect of learning about (linear) functions is the ability to change between graph representation and equation and vice versa. Traditionally, German, and Slovak students are being exposed to different procedures for these representational changes. Within a sample of 49 German and 56 Slovak teachers, we analyzed if these different procedures can also be observed in the teachers' corresponding knowledge of content and students, i.e., if the teachers expected different student strategies and errors. The results confirm this assumption and emphasize the importance of considering this teacher's knowledge in a country-specific way and to be careful when comparing such knowledge of teachers from different countries.

1. Introduction

The purpose of this study is to investigate potential differences in particular components of teachers' professional knowledge between Germany and Slovakia. This should provide new insights about teachers' knowledge that will help to improve pre- and in-service teachers' education practices. The idea of this study arose as the authors exchanged ideas about what kind of procedures for representational changes between function graph and equation are helpful to support students' learning and to what extent they are common in teacher education and practice in these two countries. In order to frame this study in more detail, we will outline the theoretical background related to functions and teachers' professional knowledge.

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Keywords and phrases: *linear function, learning difficulties, country-specific teachers' knowledge*

2. Theoretical Background

Functional thinking is characterized as a specific and meaningful way of thinking in relationships, interdependencies, and changes (Vollrath, 1989). It is often required when working on mathematical problems with and without real-world contexts that connect two quantities (e.g., NCTM, 2000). Thus, students need it both in their everyday lives but also for different school subjects. In particular, functions are considered to be fundamental for the learning of mathematics (e.g., Selden Selden, 1992) and therefore play an important role during students' mathematics schooling in many countries, including in Germany and in Slovakia (KMK, 2004; ŠPÚ, 2014). In order to understand the abstract concept of a function, students can approach it via particular representations, for instance, tables, graphs and equations. As different properties of the function can be studied via different representations, it is important to be able to connect these representations and hence to change between them (e.g., Cooney, Beckmann, Lloyd, 2010). In particular, representational changes of functions support the corresponding concept formation and problem-solving (e.g., Vollrath, 1989). However, there is vast empirical evidence, that students have various difficulties with (linear) functions, especially with such representational changes (e.g., Nitsch, 2015). Teachers play a crucial role for their students' learning (e.g., Seidel, Shavelson, 2007), and hence, should know about typical student difficulties and errors, in order to adequately counteract them. Therefore, the next paragraphs address teachers' professional knowledge.

This paper focuses on specific facets of teachers' knowledge referring to the change between graph and equation of linear functions, as a prominent relevance is attributed to this particular representational change in both countries (e.g., Backhaus et al., 2017; Freudigmann et al., 2016; KMK, 2004; ŠPÚ, 2014). In order to approach teachers' knowledge for teaching functions, we use the theoretical framework of *Mathematical Knowledge for Teaching* (e.g., Hill, Ball, Schilling, 2008: MKT, Figure 1).

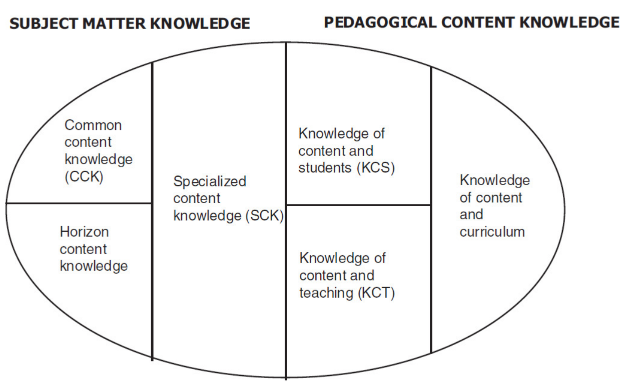

Figure 1: Domain map for Mathematical Knowledge for Teaching (Hill, Ball, Schilling, 2008, p. 377)

The model consists of six domains which are divided into two areas. The subject matter knowledge area contains three types of knowledge, namely: *Common content knowledge*, *Horizon content knowledge* and *Specialized content knowledge.* For the sake of this paper, more important is the other area called pedagogical content knowledge (PCK), "that special amalgam of content and pedagogy, that is uniquely in province of teachers" (Shulman, 1987, p.8). In the MKT model, PCK is comprised of *Knowledge of content and students*, *Knowledge of content and teaching* and *Knowledge of content and curriculum.*

In our paper, we focus on the facet *Knowledge of Content and Students* (KCS). According to Hill et al. (2008) four categories are subsumed in KCS. With regard to the scope of this contribution, we will describe them referring to changes between graph and equation of linear functions.

(1) **Teachers should be able to diagnose students' (stage of) understanding.** For instance, they should judge on the basis of talking about a corresponding task to what extent students are able to connect between graph and equation of a function (e.g., are the students able to recognize a function property in both representations; can students change directly between the representations; can students see that corresponding graph and equation describe the same real-world situation; etc.).

(2) **Teachers should be able to identify which tasks, topics and activities are suitable for a particular age group**, e. g., they should know what features might simplify or complicate a function task (e.g., using fractions for the slope; using an uncommon scaling; etc.).

(3) **Teachers should know about common strategies that students use when working on certain tasks.** When changing between graph and equation, different procedures can be used that should be familiar to the teachers. As the typical procedures to carry out these changes are different between Germany and Slovakia, we will explain them in more detail.

In line with common textbooks of grade 7 or 8 (e. g., Backhaus et al., 2017, p. 78; Freudigmann et al., 2016, p. 72), in Germany, the change between graph and equation of linear functions is usually taught by the following procedures (Figure 2):

In Slovakia, the topic is usually introduced in grade 9 (Šedivý, 2001, p. 32, 35) as follows (Figure 3):

(4) **Teachers should know about common student errors.** Obviously, common student errors are connected to the procedures commonly used. For instance, Nitsch (2015) reports in a German student sample of the common error of inverting numerator and denominator when determining the slope of a linear function. This error is obviously closely related to the German procedure of changing between graph and equation described above (see Figure 1), where students read the slope from a gradient triangle, and hence, might easily confuse the placing of numerator and denominator. Regarding the Slovak procedure (see Figure 2), where marking points in the coordinate system is an important step in order to change from equation to graph, a common error could be to incorrectly mark such a point in the coordinate system (see also result section).

Equation — Identify y-intercept — Mark y-intercept — Identify the slope — Use gradient triangle — **Graph**

Graph — Identify y-intercept — Draw the gradient triangle — Read the slope — **Equation**

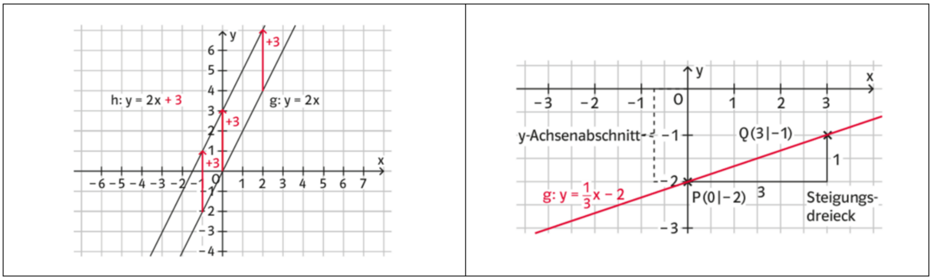

Figure 2: Typical German procedures for changes between graph and equation (Freudigmann et al., 2016, p. 71, 72)

Equation — Calculate the coordinates of two points — Mark the points into the coordinate system — **Graph**

Graph — Read the coordinates of two points — Set a system of linear equations — Solve the system — **Equation**

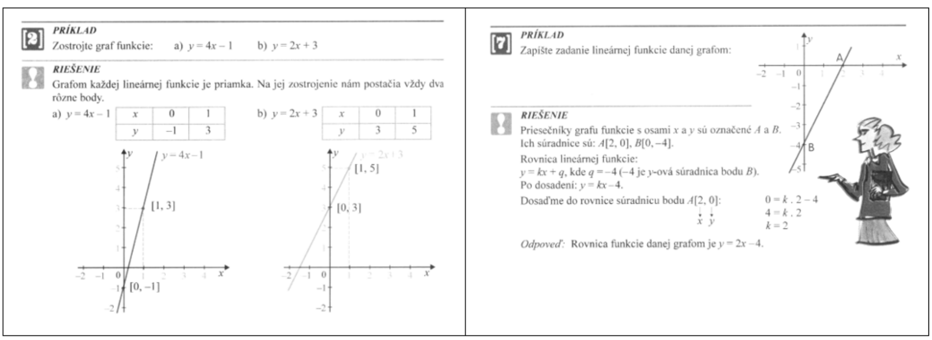

Figure 3: Typical Slovak procedures for changes between graph and equation (Šedivý, 2001, p. 32, 35)

Moreover, Birgin (2012) reports from the common error among students to interpret the slope as the tangent of the smaller angle between the graph and the x-axis instead of the tangent of the graph and the positive x-axis. This, of course, works for positive slope, however negative slope cannot be obtained in this way (see Figure 4). The last-mentioned error does not relate to the German nor Slovak procedure, and hence, is not to be expected among German or Slovak students. However, it is common among students whose teachers introduce the topic using trigonometry definition.

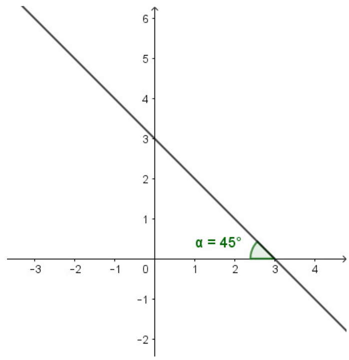

Figure 4: Common error of Turkish students concerning the slope

Moreover, there are also further aspects of functions causing students' difficulties and errors. For instance, Zaslavsky, Sela, Leron (2002) pinpointed the differences in the slope understanding between students using a visual respectively an analytical approach. As shown, an analytical approach helped students to understand slope as the property of the corresponding function, whereas a visual approach restricted students' understanding of invariance of the slope when the scale of the axis was disproportionally changed.

The precedent description of the KCS facet related to students' common errors (Hill et al., 2008) implies that, teachers' corresponding KCS should be considered in a country-specific way, i. e., it should, for instance, account for differences between typically used procedures and subsequently expected common errors. This study focuses on determining such country-related specificities of teachers' knowledge. In particular it evaluates if the nationally preferred procedures (indicated by common German and Slovak textbooks) are also reflected in teachers' expectations about common student errors that might be a result of (incorrectly) using these different procedures. The corresponding research question is:

Do the differences in how the changes between graph and equation are taught in Germany and in Slovakia also show up in teachers' *Knowledge of Content and Students* (KCS) related to corresponding common student errors?

3. Method

In this study, data from 105 in-service teachers were analyzed. The teachers came from Germany (N=49) and Slovakia (N=56) (see table 1 for details) and participated at different professional development courses, where the data for the study were collected. Their participation was not obligatory. The participants were acknowledged about the scientific goals of the data collection, which was conducted anonymously. Analysis of their answers and the subsequent discussion was a crucial part of the course, therefore, they were motivated to think about the answers deeply.

	German teachers (N=49)	Slovak teachers (N=56)
Sex (M; W)	13; 36	7; 49
Age (years)	**25** – **63** (M = 38.6, SD = 9.4)	**25** – **67** (M = 43.1, SD = 8.9)
Teacher experience (years)	**2** – **38** (M = 12.2, SD = 9.9)	**1** – **37** (M = 16.8, SD = 9.4)
Linear function taught (times)	**0** – **20** (M = 4.4, SD = 4.4)	**0** – **50** (M = 7.8, SD = 10.0)

Table 1: Characteristics of the teacher sample

The data about the teachers' KCS was collected via a test consisting of five open-ended tasks with several sub-items. Teachers had about 45 minutes to work on the test. In this paper we only report the results of the tasks 1a) and 2a). Here, the teachers are asked to write which typical student mistakes and learning difficulties they expected when changing from graph to equation (task 1a: $y = \frac{2}{3}x - 2$) and when changing from equation to graph (task 2a: $y = 5x - 2$). These two representational changes constitute standard tasks in both countries.

The data was coded with the help of a codebook developed by the German author and collaboratively adapted to the Slovak sample. Due to language barrier, the German researchers were not able to code the Slovak data or vice versa. Therefore, the Slovak authors were trained to code the data in the same way as the German researchers. The coding of the data was conducted independently by two German respectively two Slovak researchers. The inter-rater reliability for these tasks amounted to $1.00 \geq \kappa \geq 0.78$ in the German sample and to $1.00 \geq \kappa \geq 0.79$ in the Slovak sample.

Based on the coding, the researchers identified error codes typical for the German respectively Slovak procedures. As pinpointed above, this differentiation had its foundation in the common procedures typically used in these countries. In order to focus on the differences, we here only state the codes that could be clearly assigned to the German, or the Slovak procedure and we do not report the codes that were applicable for both procedures.

The codes typical for the German procedures for changes between graph and equation, explained via exemplary teacher answers, are the following:

FA Focusing the x-axis interception instead of the slope; e.g.: *"As the graph intersects the x-axis at (2,0), the student might think that 2 is the slope, because he also uses this strategy for the y-intercept."*

SL General problems with the slope; e.g.: "*The student isn't able to determine the slope as it is more difficult to read from the graph than the y-intercept."*

SE Sign errors; e.g.: "*The graph starts on the negative y-axis, however, the student omits the negative sign when reading the y-intercept".*

ND Inverting the numerator and the denominator of the slope; e.g.: *"Numerator and denominator are often incorrectly placed in the fraction provided by the gradient triangle."*

CP Confusion of both parameters; e.g.: *"The students know the procedure how to read the two relevant parameters of a linear function, namely slope and y-intercept, from a graph. But they confuse the placing of these parameters in the equation."*

SC Mistakes in reading the scale; e.g.: "*Refer to 1cm as 1 box of the coordinate system and therefore reading a wrong value for the y-intercept.*"

00 Ignoring the y-intercept; e.g.: *"Students correctly draw the slope via the gradient triangle, but they omit the y-intercept and start their graph from the origin."*

The codes typical for the corresponding Slovak procedures are:

CO Problems with reading the point coordinates from the graph; e.g.: "*Students usually confuse the coordinate axes, most of all, they have troubles with the points lying on the axes.*"

GF Not recalling the general form of the linear equation; e.g.: "*They do not remember the form of the equation, they underestimate it. They do not know where to start the procedure.*"

IC Problems with inserting the coordinates into the linear equation; e.g.: "*Students sometimes confuse coefficient "a" and the variable "x" when they insert coordinates into the general form* ($y = ax + b$)*.*"

LES Mistakes in solving a system of linear equations; e.g.: "*Student sets the system of linear equations, however he makes a mistake when solving it. He gets the decreasing function and even though he is satisfied with the result.*"

WT Working with wrong values, e.g., due to numerical errors; e.g.:

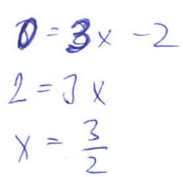

Figure 5: Example of a common student numerical error expected by a Slovak teacher

XY Switching the x and y coordinates; e.g.: "*They switch coordinate x and y when drawing the point into the coordinate system.*"

MD General problems when marking the points into the coordinate system; e.g.:

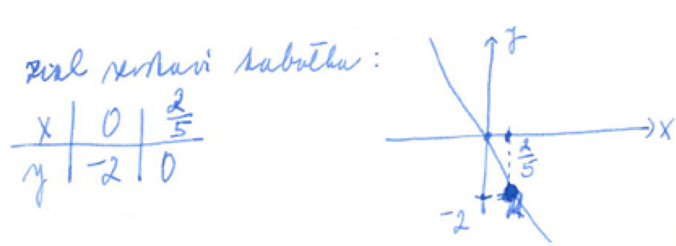

Figure 6: Example of problem with marking points in the coordinate system expected by a teacher

4. Results

For both tasks, we can see that the codes identified by the researchers as typical for the German procedures were more prevalent among German teachers (except the code FA for the change from graph into equation). Reversely, the typical codes for the Slovak procedures were more prevalent among Slovak teachers. Table 2 and 3 indicate how frequent German and Slovak teachers mentioned the errors coded as described above. The most prevalent codes for German respectively Slovak teachers are printed in bold.

	German procedure						Slovak procedure			
	FA	SL	SE	ND	CP	SC	CO	GF	IC	LES
German teachers	10.2	**49.0**	14.3	36.7	26.5	4.1	0	0	0	0
Slovak teachers	10.7	8.9	5.4	3.4	0	0	**50.0**	17.9	19.6	26.8

Table 2: Representational change – graph into equation (task 1a)

	German procedure							Slovak procedure		
	FA	SL	SE	ND	CP	SC	00	WT	XY	MD
German teachers	20.4	**36.7**	14.3	26.5	20.4	4.1	4.1	4.1	0	0
Slovak teachers	10.7	1.8	0	0	3.6	0	0	**35.7**	19.6	21.4

Table 3: Representational change – equation into graph (task 2a)

Table 2 refers to task 1a, where a change from graph into equation is required, whereas table 3 relates to the inverse representational change. In the German sample, problems of reading / drawing the slope were expected most frequently. Only the Slovak code WT (working with wrong values) was indicated by a few German teachers. In the Slovak sample, most teachers expected errors due to incorrectly reading of or working with the coordinates. The Slovak teachers rarely indicated the German codes.

5. Discussion

Based on our descriptive results, we can positively answer our research question: Despite some slight exceptions, the differences in how the changes between graph and equation are commonly taught in Germany (Freudigmann et al., 2016, p. 71, 72) and in Slovakia (Šedivý, 2001, p. 32, 35), also show up in teachers' KCS related to common student errors. To our best knowledge, there is no international research which would deal with a similar research question and therefore, we cannot compare our results to other transnational studies.

However, beyond this finding, we want to discuss the most noticeable from those slight exceptions. We can see that the code FA (Focusing the x-axis interception instead of the slope) was as prevalent among Slovak teachers as among German teachers concerning the representational change from graph into equation (see Table 2). Moreover, also concerning the representational change from equation into graph (see Table 3), approximately every 10^{th} Slovak teacher expected errors of this code. There are several possible explanations. First, several high school

Slovak teachers present to their students except the above presented "Slovak" procedure also the first step of the "German" procedure (determining the y-intercept) when changing from graph to equation. This could be sufficient condition for students to overgeneralize and read the slope from the x-intercept. Secondly, some teachers at Slovak high schools present to their students the German procedure in order to introduce the concept of slope to them. Therefore, the teachers that mentioned this error could do so in the same sense as expected by the German teachers. Another explanation could be, that the authors could have misunderstood the corresponding teacher writing as the coding of the teacher answers was – of course – an interpretative process. Moreover, the authors will monitor and validate in subsequent research steps within larger samples of German and Slovak teachers if the code FA was justifiably identified as typical German code.

Our results pinpoint the importance of considering teachers' knowledge, for instance about learning difficulties and misconceptions, in a country-specific way and to be careful when comparing such knowledge of teachers from different countries. Common concepts or procedures required by the national curriculum or taught by the corresponding teachers should be taken into account. Such information is necessary to provide and to emphasize the highly relevant context of empirical studies on teacher knowledge. Otherwise, researchers from countries with different teaching traditions could misunderstand and potentially misuse the results of the research. International groups of researchers (e.g. CERME9, CERME11) also highlighted the importance of a clear context description for empirical studies.

Nevertheless, international cooperation has the potential to document country-specific approaches to teach certain (mathematical) concepts or procedures. The first step is to identify more or less pedagogically adequate ones and to include such country specifics also into models of teacher knowledge. This might help to assess and compare teachers' knowledge also in an international perspective. Further research steps of this project will focus on classifying (adequate) teachers' knowledge about typical errors related to elementary functions and ways how to overcome them in a country-specific perspective. Moreover, the authors start the cooperation with further partners from several EU countries on the project focused on enhancing functional thinking from primary to upper secondary school that brings together their home countries' different teaching procedures; this will enable to identify country-specific and transnationally valid learning and teaching approaches related to functions.

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Veronika Hubeňáková
Pavol Jozef Šafárik University in Košice
e-mail: veronika.hubenakova@upjs.sk

Ute Sproesser
Ludwigsburg University of Education
e-mail: ute.sproesser@ph-ludwigsburg.de

Ingrid Semanišinová
Pavol Jozef Šafárik University in Košice
e-mail: ingrid.semanisinova@upjs.sk