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What do we mean by mean?*

Abstract. In order to be successful with a certain concept at school, it is often sufficient to master some types of tasks, without having to be aware of the different perspectives that a concept can be viewed from. It is, however, definitely not enough when the goal is to understand mathematics and become flexible and creative in solving mathematical tasks. Not only school students, but also prospective teachers should be encouraged to reflect on what particular concepts actually mean and how they can be thought of.

In this article we focus our attention on the concept of mean. In the introduction we outline the areas where our contribution adds to the results of the research carried out so far. Then we provide a brief review of how the Polish core curriculum and two very popular mathematics textbook series (one each for primary and secondary school) treat the topic of means. Next, we present results from a study conducted on a group of first-year students of mathematics and a group of teachers undertaking postgraduate studies qualifying for teaching mathematics as a second subject. The study was aimed at diagnosing pre-service teachers' substantive competences in mathematics. In this paper we show only the results obtained in two tasks concerning the arithmetic and geometric means. The paper ends with several suggestions and recommendations which can be useful for and applied by teachers of mathematics as well as mathematics teachers' educators.

1. Introduction

According to Bakker and Gravemeijer (2006)

> *the term "average" itself stems from maritime law, in connection with insurance and the fair share of profit and loss. The Oxford English Dictionary (Simpson and Weiner, 1989), under the heading of "average," states that one of the meanings of average in maritime law is "the equitable distribution of expense or loss, when of general incidence, among*

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> *all the parties interested, in proportion to their several interests." More generally, the average came to mean the distribution of the aggregate inequalities (in quantity, quality, intensity, etc.) of a series of things among all the members of the series, so as to equalize them. In its transferred use the term average thus came to signify the arithmetic mean* (Bakker and Gravemeijer, 2006, p. 153).

Already centuries ago, intuitions related to the concept of the arithmetic mean were used to make estimations and solve practical problems (Bakker, 2003). But in the times of Pythagoras there were also other two known mean values: the harmonic and geometric mean (Heath, 1981). However, as noted by Landtblom (2018), in the traditional teaching of averages, the notion of arithmetic mean has received more attention than other statistical concepts. It seems important for researchers to pay more attention also to other averages since they are taught at school.

Several authors have made a great effort to characterize in details some of the means for educational purposes. For instance, Strauss and Bichler (1988) found and highlighted seven important features of the arithmetic mean (the average in the passage below):

> *A. The average is located between the extreme values.*
>
> *B. The sum of the deviations from the average is zero.*
>
> *C. The average is influenced by values other than the average.*
>
> *D. The average does not necessarily equal one of the values that was summed.*
>
> *E. The average can be a fraction that has no counterpart in physical reality.*
>
> *F. When one calculates the average, a value of zero, if it appears, must be taken into account.*
>
> *G. The average value is representative of the values that were averaged* (Strauss and Bichler, 1988, p. 66; see also: Leon and Zawojewski, 1990).

Bakker and Gravemeijer (2006) point out to some other aspects of mean that they revealed when looking at this concept through the phenomenological lenses. They consider the following elements important:

> *the implicit use of the mean to estimate large numbers, the strategy of calculating the mean as a means to eliminate measurement errors, the change of perspective by which the mean becomes an entity in and of itself, the role of mean-as-a-measure, and the difference between the natural sciences (where there seems to be an implicit expectation of symmetrical distributions) and the social sciences (where the apparent skewed distributions create the need to describe those distributions, which creates a role for the median).* (Bakker and Gravemeijer, 2006, p. 161).

School students associate the concept of average with various notions e.g., the sum of all numbers considered, a half, the average of the smallest and largest number, the sum of numbers divided by the number of these numbers (Bakker, 2003). Mokros and Russell (1995) found five predominant approaches to solving problems involving averages. In the group they studied, the students thought of an average as mode, an algorithmic procedure, a tool for making sense of the data set, midpoint or a mathematical point of balance. As research shows students can be able to recognize the algorithm for a particular mean, calculate the mean with the use of appropriate algorithm, or even find a new mean when some number is added to the examined set of numbers, but it may be more difficult to them to find an unknown value of a number when given the other numbers and the mean of the whole set (Cai, 1995; Guimarães et al., 2010). Also the students often lack understanding of the arithmetic mean as a representative value of a set of numbers, which has been found to be the most difficult property of this mean (Strauss and Bichler, 1988).

A significant problem is that students may be able to calculate the average of a given set of numbers, and yet remain unaware of its certain qualitative features. Leavy and O'Loughlin (2006), for instance, distinguish between computational and conceptual interpretation of the arithmetic mean:

> *Computationally, the arithmetic average is the score around which deviations in one direction exactly equal deviations in another direction. Conceptually, the mean acts as a mathematical point of balance in the distribution similar to a center of gravity, and is considered representative of a data set.* (Leavy and O'Loughlin, 2006, p.55)

Knowing how to determine a given mean is not an indication of understanding of the concept (Batanero et al., 1994; Watson and Moritz, 2000). Students may experience difficulties with means at both the procedural level and the level of conceptual understanding. Teaching which is dominated by the use of algorithms (e.g., Gfeller, Niess and Lederman, 1999), can lead students to become helpless in the face of problems requiring the use of conceptual understanding of means.

Surprisingly, or perhaps not, prospective teachers happen to experience very similar difficulties in understanding mathematical concepts as the students (Jacobbe and Carvalho, 2011). Even recently (Landtblom, 2018), it has been concluded that there is still too little research available that advances our knowledge of pre- and in-service teachers' understanding of means.

There are four gaps which this paper addresses to some extent. First, we are not aware of any recent Polish educational study on mathematics, that has looked at issues related to the concept of means. Secondly, in the piece of research that we report, we show how some teachers taking postgraduate studies qualifying for teaching mathematics deal with arithmetic and geometric mean tasks, and this group of prospective teachers has been underreported in research to date. Thirdly, while among the different means, the arithmetic mean seems to attract the most attention from researchers, we also address the problem of the geometric mean and mention the harmonic mean. Last, but not least, at the end of this paper, we propose a certain universal – yet not found in contemporary Polish school textbooks – perspective on the three means, which helps to unify the way we talk and think of them.

2. The treatment of means in the Polish core curriculum and mathematics school textbooks

2.1. Means in the core curriculum

In the core curriculum for primary school (grades 4-8), in the section entitled ”Reading data and elements of descriptive statistics”, it is stated that the student is to calculate the arithmetic mean of several numbers. In the section ”Conditions and method of implementation”, which contains extended comments of the authors of the document, there is an example of a task that a primary school student should be able to solve. It reads as follows:

> *Maciek got 10 marks in mathematics. Here are 9 of them: 2, 2, 2, 3, 3, 4, 5, 5, 6. The arithmetic mean of all his 10 marks is equal to 3.6 . Determine the missing grade.*

In the core curriculum for secondary schools (4-year secondary school or 5-year technical school), the topic of averages appears in the section ”Probability and statistics”. It says that a student who pursues mathematics education at the basic level ”calculates the arithmetic mean and the weighted mean”.

It is important to notice, however, that these averages (and others not specified above) are also used by high school students many times on the occasion of other topics, not explicitly related to the average, and this is not emphasized in the curriculum. The arithmetic mean, for instance, is used to determine the coordinates of the vertex of a parabola (being a graph of a quadratic function), when the coordinates of two points of the parabola, symmetrically located with respect to its main axis, are known. Also, in an arithmetic sequence, of every three consecutive terms of the sequence, the middle term is the arithmetic mean of the neighboring terms. The weighted average is used by the students when they calculate the average of their grades, e.g. from tests, which are assigned different weights. In a geometric sequence with positive terms, out of every three consecutive terms of the sequence, the middle term is the geometric mean of the neighboring terms. Moreover, the length of the height subtended from the vertex of a right angle in a triangle is the geometric mean of the lengths of the segments into which it divides the hypotenuse.

A trapezoid is an example of a geometrical figure such that one can see four different means in it[1]:

- the length of the segment connecting the centers of the arms of a trapezoid is the *arithmetic mean* of the lengths of its bases,
- the length of the segment parallel to the trapezoid bases and dividing this trapezoid into two similar trapezoids is the *geometric mean* of the lengths of the trapezoid bases,

[1] Another example of such a figure could be the semi-circle (Hess, 1961; see also: http://www.eudoxos.pl/srednie/).

- the length of the segment parallel to the trapezoid bases and passing through the point of intersection of its diagonals is the *harmonic mean* of the lengths of the trapezoid bases,
- the length of the segment parallel to the bases of the trapezoid and dividing it into two trapezoids of equal areas is the *quadratic mean* of the lengths of the trapezoid bases.

2.2. Arithmetic mean in a popular series of Polish primary school mathematics textbooks

In the series of textbooks for primary school under review[2], the concept of arithmetic mean appears for the first time in grade 5. The students first calculate the arithmetic mean of several numbers while discussing decimal fractions, and then return to this concept and practice the skills they have acquired when discussing whole numbers. The concept is introduced as follows:

> *The arithmetic mean of two numbers is the quotient of the sum of those numbers by 2.*
>
> *The arithmetic mean of the numbers 3 and 5.5:*
>
> $$\frac{3+5,5}{2}=\frac{8,5}{2}=4,25$$
>
> *The fractional dash replaces the division sign.*
>
> *The arithmetic mean of three numbers is the quotient of the sum of those numbers by 3.*
>
> *The arithmetic mean of the numbers 6, 7.3 and 8:*
>
> $$\frac{6+7,3+8}{3}=\frac{21,3}{3}=7,1$$
>
> *The arithmetic mean of four numbers is the quotient of the sum of those numbers by 4.*
>
> *And so on* (5^{th} grade textbook, p. 166).

The task which precedes this introduction of the arithmetic mean concerns a ream of paper (500 sheets), whose thickness and weight are given. The students' task is to calculate the thickness and weight of one sheet of paper. This task requires students to consider a fair share that would take place if the thickness and weight of all the sheets were distributed equally. Immediately after the introduction of the arithmetic mean, the students calculate arithmetic means of two, three and four decimal numbers given and in a next task they calculate the average of some girl's grades. The topic of means returns when the whole numbers are discussed. Among other things, students calculate the average air temperature on the basis of temperature measurements taken on several consecutive days.

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[2] We chose the textbooks published by Gdańskie Wydawnictwo Oświatowe, as they are very (if not the most) popular among mathematics teachers and widely used at primary schools in Poland.

In the 6^{th} grade textbook, there are tasks requiring the determination of the arithmetic mean, but the topic itself is not explicitly addressed.

In grade 7, the students revisit the concept of the arithmetic mean, they know how to calculate the arithmetic mean and also how to solve a text problem related to the mean. This concept comes at the end of the school year, in the ”Statistics” section. Only in this textbook in the analysed series, a separate chapter is devoted to the concept of arithmetic mean. It begins with two tables showing the final grades in mathematics in two 7^{th} grade classes, with 20 and 25 students respectively. Then the mean of the grades in each class is calculated separately and also the mean of the grades in both classes together. The authors of the textbook draw students' attention to the fact that the mean of the two initially calculated means is not the mean sought after in the final task.

In this chapter, the authors of the textbook offer a number of tasks. Some of them require only the calculation of the arithmetic mean of a set of numbers placed in a realistic context. But there are also tasks which require more than just the application of the learned method of calculating the average. For instance: *Jurek has four grades in mathematics, we know the average of his grades, we ask by how much this average will increase if the boy gets a six* (Polish counterpart for A+); or: *Zuzanna has 12 grades on her certificate, they are only grades 4 and 5* (B's and A's respectively), *we know the average of her grades, we ask how many 5's she has.* However, in our opinion, the following three tasks deserve special attention:
Task 4, page 300

> *Ala and her 21 classmates were writing an English test. A maximum of 44 points could be scored. The average score was 28. No one wrote below the average. How many points did Ala get?*

Task 7, page 300

> *Anne keeps all her books on six shelves. There are 20 books on one shelf on average. Determine how the average number of books on a shelf will change when Anne:*
>
> *(a) moves 10 books from the highest shelf to the lowest*
>
> *b) adds 1 more book to each shelf,*
>
> *c) buys 12 more books and puts them all on one of the shelves,*
>
> *d) sells $\frac{1}{4}$ of all her books.*

Task 13, page 301

> *Each of ten glasses contains an average of s milliliters of syrup. Write an algebraic expression that represents how much syrup on average will be in each glass when:*
>
> *a) we add 40ml of syrup to each glass,*
>
> *b) we pour out half of what is in each glass,*
>
> *c) we pour 100ml of syrup into one glass,*
>
> *d) we pour another 100ml of syrup into each of three glasses,*

e) to the ten glasses we add a glass containing 150ml of syrup,

f) we add an empty glass to the ten glasses.

The first of these tasks, contains a statement that may initially cause consternation in students: ”no one wrote below the mean”. This information forces students to think. It is not enough to know the formula for the arithmetic mean of several numbers, the students need to be able to interpret the information about the mean well. Here, the quoted passage obviously indicates that, in that case, no one wrote above the average either, and therefore everyone, including Ala, got the same result.

The next two tasks can greatly contribute to the development of conceptual understanding of the arithmetic mean in students. They allow to observe and analyse how the arithmetic mean is affected by a change made to a set of numbers or its parts. The students can discover and convince themselves that:

- the average of the numbers will not change if we decrease one number and increase another number by the same amount,
- the average will increase (decrease) by k if we increase (decrease) each number by k,
- the average of n numbers will increase (decrease) by $\frac{k}{n}$ if we increase (decrease) any of the numbers by k,
- if to the set of n numbers whose sum is S we add an $n+1$th number with value k, then the average changes from $\frac{S}{n}$ to $\frac{S+k}{n+1}$ (if $k = \frac{S}{n}$ the change equals 0, since the arithmetic mean remains the same, which is yet another, interesting discovery the students can make), and so forth.

In all three examples it is not the procedure for determining the average that is important, but the conceptual understanding of it.

2.3. Four means in a popular series of Polish secondary school mathematics textbooks

Textbook definitions

In a 1^{th} grade of secondary school mathematics textbook that we chose to examine[3], a separate subsection in the chapter on algebraic expressions is devoted to the means. The authors inform the students that a mean of a group of numbers allows to characterize this group in a simple way and that it is contained between the smallest and the largest number from the set. The authors introduce four types of means:

[3]We chose the textbooks published by Oficyna Edukacyjna Pazdro, as they are very (if not the most) popular among mathematics teachers and widely used at high schools in Poland. The topic of means is discussed in the first grade. Due to the education reform and the replacement of former textbooks scheduled for several years, the textbooks for the fourth grade are not yet available. This is why we do not know whether the topic will return there when statistics is discussed and whether other means, such as quadratic, will be mentioned.

1) Arithmetic mean of numbers $a_1, a_2, \dots, a_n$ *equals* $S_a = \frac{1}{n}(a_1 + a_2 + \dots + a_n)$.

2) Geometric mean of positive numbers $a_1, a_2, \dots, a_n$ *equals* $S_g = \sqrt{a_1 \cdot a_2 \cdot \dots \cdot a_n}$.

The following relationship holds between the geometric mean and the arithmetic mean: if $a_1, a_2, \dots, a_n$ *are positive numbers then* $S_g \leq S_a$*, with equality occurring only if* $a_1 = a_2 = \dots = a_n$.

3) The weighted mean of the numbers $a_1, a_2, \dots, a_n$ *with positive weights* $w_1, w_2, \dots, w_n$ *is equal to*

$$S_w = \frac{a_1 w_1 + a_2 w_2 + \dots + a_n w_n}{w_1 + w_2 + w_3}$$

4) The harmonic mean of positive numbers $a_1, a_2, \dots, a_n$ *is equal to*

$$S_h = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_n}}$$

Between the geometric mean and the harmonic mean of the n positive numbers $a_1, a_2, \dots, a_n$*, the following relation holds:* $S_g \geq S_h$*, with equality occurring only when* $a_1 = a_2 = \dots = a_n$ (OE Pazdro, p. 132).

It is worth noting that:

- In the case of the arithmetic mean, the authors do not give any conditions regarding the numbers for which the mean is determined. The learner may easily guess that any real values of numbers are allowed.

- Similarly, in case of the weighted mean no conditions are given concerning the numbers themselves, but it is emphasized that the weights take positive values. Here it is quite easy to conclude that negative weights would not make much sense. But what if we allow 0 as a weight? What would happen if a number received a weight of 0? Such number would not be taken into account (would not matter), so allowing for a zero weight would also make no sense[4].

- Geometric and harmonic means are calculated only for positive numbers. In the case of the harmonic mean, it is obvious that none of the numbers can be zero, for when calculating this mean we use inverses of the numbers. However, why is zero not allowed in the case of the geometric mean? The root of zero is well defined after all. The authors do not comment on this in any way. Moreover, the literature on this matter is inconsistent, since other authors allow the numbers whose geometric mean is calculated to have zero values (e.g., Cewe, Nahorska and Pancer, 2003).

[4]However, a clear distinction must be made between a situation when the weight would be 0 and a situation when one of the numbers in the set is 0. In the latter, of course, the number 0 will be counted in the calculation of the average.

- The exclusion of negative numbers in the case of both last mentioned means is also left without a comment. Here, in turn, while it is easy to see that not in every case it would be possible to determine the geometric mean in the set of real numbers, one does not immediately see the reason why it would not be possible to calculate the harmonic mean. If we allowed negative numbers, for example, for the set of numbers -3,-2, 2, and 3 we would get zero in the denominator of the fractional expression for the harmonic mean. Is this the reason why negative numbers were eliminated? The authors say nothing about this.

- Nowhere does the textbook also say that if we have one number, it is its own arithmetic, geometric and harmonic mean at the same time. This trivial case, perhaps in the authors' opinion not even worth considering, allows for, among other things, a discussion of the degree of the arithmetic root $\sqrt[n]{a}$. Usually, textbook authors take the view that n is a natural number at least equal 2. Meanwhile, Chronowski (1999) also allows for a root of degree one stating that it is simply the same number (the root "does nothing"). Chronowski's approach makes the notion of a geometric mean also meaningfully applicable to a single number.

It should also be added that the quadratic mean, which students are going to use (perhaps unknowingly) in the future to determine the standard deviation, is not considered in the 1^{st} grade textbook. For the sake of completeness, let us add that the quadratic mean is defined[5] as follows (e.g., Cewe, Nahorska and Pancer, 2003):

Quadratic mean	For $a, b > 0$ $S_q = \sqrt{\frac{a^2+b^2}{2}}$	For $a_1, \ldots, a_n > 0$ $S_q = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$ (the root of the arithmetic mean of the squares of numbers $a_1, \ldots, a_n$)

Textbook examples

In the first example provided by the authors, the results of temperature measurements from several consecutive days are given and it is shown how the average temperature for the analyzed period was determined.

The second example states that in a certain workplace, which employed 19 workers, the average salary was 2650 PLN. After paying a new employee, the average salary increased by 2%. It is shown how to calculate the salary of a new employee. Both of these examples deal with the arithmetic mean.

In the third example, the following task is given:

> *In a clothing store, the price of a certain blouse increased first by 50%, then by 20%. At the end of the season the price of the blouse has been reduced by 30%. What was the average percentage change in the price of the blouse?* (OE Pazdro, p. 133)

[5] Again a question arises, why do we exclude 0 from the set of the numbers whose quadratic mean is to be found?

In the solution given by the authors, the initial price of the blouse is denoted as a. The final price was obviously $0,7 \cdot 1,2 \cdot 1,5a$. The authors state:

> *Let p denote the average factor of change in the price of the blouse. Since there were three changes in the price of the blouse, we get the equation*
>
> $$p^3 a = 0,7 \cdot 1,2 \cdot 1,5a \ \ (\dots)$$
>
> $$p = 1,080082\dots$$
>
> *We can say that at each price change, the blouse became more expensive by about 8% on average.*

Example four shows how to calculate the weighted average of two students' grades. The example illustrates that although two students may have the same grades, the weighted averages of their grades may differ if they earned different grades for activities to which the teacher assigned different weights.

The last example contrasts two situations that are often confused by the students. The task shown in this example is the following:

> *A car drove a distance from city A to city B. At what average speed did the car travel if:*
>
> *(a) it traveled half of the time with speed v_1 and the other half of the time with speed v_2,*
>
> *b) it travelled half of the distance with speed v_1 and the other half of the distance with speed v_2.*

In the first case, the average speed of the car over the entire distance is equal to the arithmetic mean of speeds v_1 and v_2. The main role of this example, however, is to show the students that when an object travels distance s at speed v_1 and then the same distance at speed v_2, its average speed over the entire route is equal to the harmonic mean of speeds v_1 and v_2, and not the arithmetic mean as one may mistakenly guess.

Students are often familiar with the formulas for different means. That is just part of superficial procedural knowledge (Star 2005) on means. A deeper understanding of the means allows, for example, not to calculate them from the formulas when they can be determined otherwise. Unfortunately in the examined high school textbook there are no tasks analogous to the examples we highlighted in the primary school textbook (*How would the mean change if we...*) that would stimulate out-of-the-box thinking.

Of the means students learn in high school, the geometric and harmonic means seem to be the least intuitive. While each can be calculated for a given set of positive numbers, there is no clear message from the authors conveying the meaning of the values obtained as means. In a task where we are not told which mean we should use (like in the third and the last textbook examples), those who know the meaning of each mean, should resist being deceived. The geometric mean seems to be particularly interesting in this respect. Even just formulating the task clearly can be a challenge in the case of this mean. On the other hand, in typical, unambiguously formulated tasks where the students are expected to apply the formula for the harmonic mean, it is usually possible to solve the task without using formulas, and even without being aware that one is calculating one of the means.

3. Investigating the substantive competences of pre-service teachers of mathematics – tasks on arithmetic and geometric mean

3.1. The study and its objectives

This study has been conducted as part of a larger project supported by the National Science Center in Poland. The project's main objective was to diagnose the substantive and didactical competencies of pre-service teachers of mathematics, mainly those participating in postgraduate qualifying courses. The pilot study however has been conducted on a number of undergraduate students.

With respect to what we present in this paper, we were first interested in whether the study participants could correctly solve the tasks involving arithmetic and geometric means. As regards the task on arithmetic mean, we wanted to find out whether the respondents would calculate the average from a formula. We are aware that just because someone did not write down the calculations, it does not mean that they did not perform them in their memory. On the other hand, however, any traces of calculations found in the respondents written answers would be a clear evidence of a procedural approach. In the case of the second task, we wanted to check whether the respondents recognize in it, a situation in which the geometric mean, not the arithmetic one, should be calculated. This type of task appears in a popular secondary school mathematics textbook, so teachers working at this level of education are likely to be confronted with this topic in their near future.

3.2. Participants

The study involved 59 students entering undergraduate programs in mathematics (hereafter referred to as 'students') and 14 teachers (hereafter called 'teachers') who entered postgraduate programs qualifying them to teach mathematics as another subject. Among the students, 8 people had taken the high school leaving exam only at the basic level, 48 people at the basic and extended level, and 3 people did not provide information on this matter.

Among the teachers surveyed, 1 person did not complete a short questionnaire asking for information about age, former education and previous work experience. Of the remaining 13 persons, there were 3 who indicated education in early childhood and pre-school education, another 3 in accounting and finance, and also 3 in philology (German or Polish). Each of the following disciplines was mentioned by two teachers: management, physics, chemistry or computer science. Single persons also indicated theology, logopedics, physical education and sociotherapy.

Three persons declared that they had no work experience as teachers. Two others had only experience of working in a day-care center, kindergarten or nursery, or as a supporting teacher or physical education teacher. One person declared to teach physics at a university, another to give private lessons in mathematics, including preparation for the international high school final exams. One person, who had completed a PhD in chemistry, had been teaching physics and computer science at secondary school in addition to chemistry. The others indicated that they were employed in primary or secondary schools accordingly to their educational background.

What the students and teachers have in common is that both groups began their studies to obtain preparation for teaching mathematics. An important difference lies in the fact that while the students have 5 years to acquire relevant knowledge and skills, the teachers are expected to achieve the same competencies in three semesters, i.e. 1.5 year. The formal qualifications that both groups are going to obtain at the end will be the same.

3.3. Organization of the study

A group of students were recruited at the university where one of the authors works. Undoubtedly, the new situation and the feeling of obligation influenced strong commitment in completing the tasks and reliably handing over to the test organizers all the notes taken while solving the tasks.

The students were informed that the test was only for diagnostic purposes and was not subject to assessment. Students were asked to solve the tasks on the day the academic year began, when they had not yet taken any classes. When solving the tasks, they could only rely on the knowledge they had acquired in their previous education. The study conducted on this group was a pilot study and was aimed at final verification of the tasks to be used in the study of the second group. The tasks were carefully analyzed and selected beforehand, and the test sets received positive reviews from four experts in mathematics education. The pilot study did not reveal any problems with understanding the content of instructions, and it did not result in any changes to the test. Therefore, the results obtained during this study provide a certain point of reference for subsequent groups tested with the same set of tasks.

Due to pandemic restrictions, students were divided into 6 groups which attended the test in a hybrid mode. Three consecutive blocks of meetings were scheduled on the day of the study. Within each block, the tasks were solved by two groups, with one group solving the tasks in a room at the university building and the other group completing the test remotely at the same time. A total of 31 people completed the tasks stationary, and 28 remotely. Those who took the test remotely completed the closed and gap tasks via Google Form and were asked to send photos or scans of their solutions to these tasks together with solutions to open tasks that were also given to the students.

Because of the pandemic restrictions, far less teachers participated in the study than was originally expected. At the planning stage of the study, face-to-face meetings with teachers were intended to take place during assemblies at the universities where the studies are conducted. Due to the pandemic, the stationary classes were replaced by remote meetings. In order to reach the teachers, we attempted to contact first the different universities and centers conducting postgraduate studies qualifying for teaching mathematics as a second subject. Some of the centers did not respond to our requests, some stated that they did not currently offer the courses we were interested in, some declined to help us contact students and a few centers promised to forward the message we had prepared for their students to invite them to participate in the study. However, even when the invitation reached a particular teacher, he or she could make the decision voluntarily. We suspect that many teachers did not receive our message, and of those who did, a significant number were unwilling to participate. Ultimately, the results we were able to collect were obtained in two periods XI–XII 2020 and X–XI 2021.

For the convenience of teachers solving the test remotely, the tasks they were asked to solve were divided into two blocks: a block of closed tasks, gap tasks and true-false tasks, which could be answered via Google Forms, and a separate block of open tasks, where the solution had to be written down in full on a piece of paper and sent in the form of a scan or a photo. Students had both of these blocks combined into one form.

3.4. The tasks

In the test, which was completed by both students and teachers, there were a total of 45 tasks, including 6 true-false tasks, 30 closed or gap tasks and 9 open tasks. In addition, a group of teachers received 5 open tasks related to the teaching competences at a separate time, which is not the subject of our attention here. In this article, we report the results obtained in the two tasks on the averages that were included in the test. Both tasks were the gap tasks.

Task 1

The content of the first task we would like to present here was as follows:

The arithmetic mean of the numbers x and y is equal to 6. The arithmetic mean of the numbers $x + 10$ and $y + 10$ is equal to ...

This task is similar to some of the examples from the 7^{th} grade textbook that we referred to earlier in the text. If the arithmetic mean of two numbers x and y equals 6, it means that each of these numbers could be replaced by 6 and the sum of the two numbers would remain the same. If we increase each number by 10, it means that now each of the newly obtained number could be replaced by 16, which is the arithmetic mean to be found. No calculation is needed for solving this task.

Task 2

The second task was the following:

In a newly built housing complex, the number of inhabitants increased in two consecutive years by 69% and 21% respectively. The average annual percentage increase in the number of residents of the estate was .... %

The task is similar to one of the examples presented in the high school mathematics textbook. By asking about the average annual percentage increase we are asking what the percentage increase should be if we wanted it to be equal in each year and leading to the same result as the two changes mentioned in the task. This however, may not be so obvious to the students, as it is even nowhere explained in the textbook in such a manner. By asking respondents to solve such a task, we wanted to observe whether they would determine the geometric mean or interpret the task as a question about the arithmetic mean of the percentage changes. As the formulation of the task itself may be ambiguous for the recipients (however, we decided to keep the wording close to the one from the textbook), we were curious if anyone would comment on it in their answer.

It is worth mentioning here that the solution involving the geometric mean does not require the use of a calculator if one performs the calculations cleverly, which also proves one's skills, a certain mathematical proficiency, as well as a developed number sense. Denoting by a the initial number of inhabitants of the housing estate, one should find such a number p, that:

$$p^2 \cdot a = 1,21 \cdot 1,69a.$$

Hence:

$$p^2 = 1,1^2 \cdot 1,3^2$$
$$p^2 = (1,1 \cdot 1,3)^2$$
$$p^2 = 1,43^2$$
$$p = 1,43.$$

Thus the answer is 43%.

3.5. Results

Task 1

The results obtained in Task 1 are shown in the table below:

Students (N=59)		**Teachers (N=14)**	
Answer	**Number of answers**	**Answer**	**Number of answers**
16	46	16	12
8	4	60	1
11	2	16 and 36	1
Other	4		
No answer	3		

Students' solutions

We found written records of the solutions to this task in the papers of 25 out of 31 students solving the tasks stationary and in 19 out of 28 scans submitted by the students writing the test remotely. This means that 44 of the 59 students writing the test needed a record of the performed calculation. These students did not appeal to a conceptual understanding of the arithmetic mean, but performed calculation using an algorithm learned back in primary school. This observation alone shows that the arithmetic mean, which is perhaps the mean most familiar to the students, is treated by the respondents on a largely procedural basis. As for the other 15 students, we do not know how they proceeded. The calculations, even if done in this task, are not tedious and they could easily be done in mind.

It came as no surprise to us that most students answered the question posed in the task correctly. The most frequent solution was the following:

Task 1, example 1

$$\frac{x+y}{2} = 6$$

$$x + y = 12$$
$$\frac{x + 10 + y + 10}{2} = \frac{12 + 20}{2} = 16$$

Two students solved the task without taking this second step:

Task 1, example 2

$$\frac{x + y}{2} = 6$$
$$\frac{x + 10 + y + 10}{2} = \frac{x + y}{2} + \frac{20}{2} = 6 + 10 = 16$$

In several works, the arithmetic mean was determined as follows:

Task 1, example 3

$$\frac{x + y}{2} = 6$$
$$x + y = 12$$
$$x = 12 - y \quad (\text{or } y = 12 - x)$$
$$\frac{x + 10 + y + 10}{2} = \frac{12 - y + 10 + y + 10}{2} = \frac{32}{2} = 16$$

Four students obtained number 8 as their final result. We have a record of such a solution from three of them. In each case, the error was the same: the student treated $x + 10 + y + 10$ as the sum of four numbers and performed the following division:

Task 1, example 4

$$\frac{x + 10 + y + 10}{4} = 8.$$

It is likely that the answers indicating number 11 resulted from calculating the sum of 6 (half of the sum of x and y) and 5 (half of the number 10), but we did not find a record of such a solution in any paper with this result.

The other answers given by the students were: 17, 2, 22 and 5,5. The result 17 was obtained because 32 was incorrectly divided by 2 (the method was valid, only there was a calculation error at the end). The result 22 was obtained in this way:

Task 1, example 5

$$\frac{x + y}{2} = 6$$
$$x + y = 12$$
$$\frac{x + 10 + y + 10}{2}$$
$$12 + 10$$

The author of this solution either assumed by mistake that $\frac{x+y}{2} = 12$ or she mistakenly simplified the arithmetic part of the expression $\frac{x+10+y+10}{2}$.

The outcome of 5,5 was obtained due to a calculation error combined with a wrong method adapted by one student:

Task 1, example 6

$$\frac{x + 10 + y + 10}{4} = \frac{12 + 10}{4} = 5,5.$$

We do not know where the result 2 comes from. It appears in the work of a student, who filled in the form, but did not send a scan of his written solution to this task (maybe he did not write anything down, or maybe he did not make a scan of a sheet of paper with the solution to this task). It is possible that the student wanted to give a different result, e.g. 22, but entered the data incorrectly in the form. It is difficult for us to comment on this.

Teachers' solutions

It is also difficult to analyse the results submitted by the teachers. Although they were asked to send scans showing everything they wrote or drew while solving the closed and gap tasks, we received such a scan showing the calculations made in the task in question only from one teacher. His solution was the same as given in example 1. We can only assume that other correct answers in this group were also obtained via methods similar to those used by the students.

Task 2

Students (N=59)		**Teachers (N=14)**	
Answer	**Number of answers**	**Answer**	**Number of answers**
No answer	28	No answer	2
45%	20	45%	6
43%	1	52%	3
Other	10	43%	1
		Other	2

Among the "other" results, the following appeared:

- In the group of students: two people indicated a result 52.245%, of the remaining results, each was indicated by one person: 35,49%; 48,5%; 58%; 44%; 40%; 52%; 50% and 22%;
- In the group of teachers: one person wrote 52,245%, the other wrote 12%.

Students' solutions

In this task, any written responses appeared in the works of 19 out of 31 students writing the test stationary (with two works containing only the numbers given in the task), and in the works of 8 students working remotely.

Eight students' solutions included the calculation of the arithmetic mean. In six of them, the calculation was as follows (with or without the percentage sign):

Task 2, example 1

$$\frac{69\% + 21\%}{2} = 45\%,$$

and two people, except of using the wrong method, also made another error when inserting the numbers:

Task 2, example 2

$$\frac{69\% + 28\%}{2} = \frac{97\%}{2} = 48,5\%$$

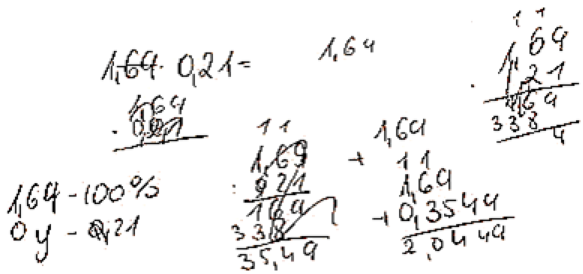
or in calculations:

Task 2, example 3

$$\frac{69+21}{2}=\frac{80}{2}=40.$$

There were six students who, despite starting with calculations, most of which were very tedious, eventually gave an answer of 45%, most likely arriving at this result by ultimately abandoning complicated counting and finding the arithmetic mean in memory. It can be assumed that the ”ugly” calculations led students to consider them impossible and wrong. Here are two examples of the situation just described:

Task 2, example 4



Task 2, example 5

$$169\% - \mathrm{I}$$

$$1,21 \cdot 169\% \approx 204,5\%$$

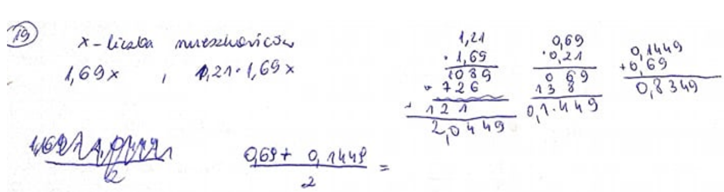
$$204,5+169=373,5$$

$$373,5:2=$$

In both examples it can be seen that the intention of the initial calculations was good. In Example 4, the student wanted to calculate 21% of the number 1,69 and then add the result to that number. In Example 5, the student immediately calculated 121% of 169%. While in the case of the author of the solution in example 4 we do not know what he intended to do next, example 5 shows not only an unjustified addition, but also a plan to calculate the arithmetic mean at the end.

Four students started with some calculations, but did not come up with any answer. This was the case, for example, in the works of students whose solutions were the following:

Task 2, example 6



The calculations done by the student in *Example 6* were as follows:

$$1,69 \cdot 1,21 = 2,0449$$

$$0,21 \cdot 0,69 = 0,1449$$

$$0,69 + 0,1449 = 0,8349$$

Finally, the student wanted to determine the arithmetic mean of 69% and 21% of 69%:

$$\frac{0,69 + 0,21 \cdot 0,69}{2}.$$

Perhaps what the student was trying to do, but got lost during partial computations, was what could be done if we knew that a certain number x was increased by 69% and then by 21% and the question was how much (by what value, not by what percentage) the number increased on average.

x – initial value of a number

$1,69x$ – value of the number after the first change;

$0,69x$ – the rate of first increase

$1,21 \cdot 1,69x$ – value of the number after the first change;

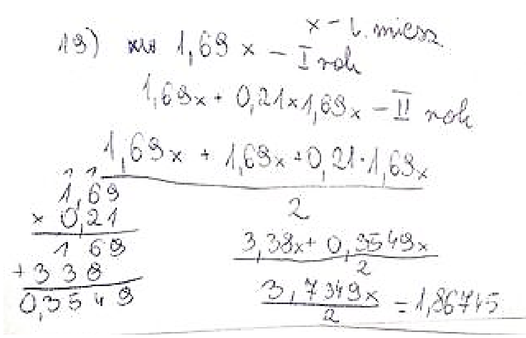
$0,21 \cdot 1,69x$ – the rate of second increase

Now, the arithmetic mean

$$\frac{0,69x + 0,21 \cdot 1,69x}{2}$$

would inform us what the average increase was if we were interested in the size (value) of the increment and not the percentage change in the value of the number. For instance, if our initial number was 40, and the number would be increased by 50%, this would result in 60. We would then get 90 by increasing 60 again by 50%. We could say that the average change in the value of the number was $(20 + 30) : 2 = 25$. Perhaps the reasoning of the author of example 6 was similar to the above. This would explain how the student interpreted the question posed in the task.

Task 2, example 7



The author of *Example 7* in her written solution was also calculating the arithmetic mean. What was calculated was the average of the number of inhabitants after the first and after the second year. The result obtained by the student would indicate that on average in each of these two years, about 86% more people lived in the estate than at the beginning.

The following solution has ultimately led the student to give 35,49% as an answer:

Task 2, example 8

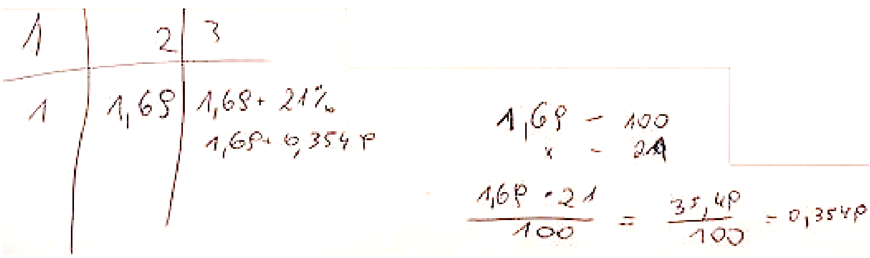
$$0,21 \cdot 1,69 = 0,3549$$

$$1,69 + 0,3549$$

The result of 35,49% actually says that the increase of 21% in the following year compared to the previous year was 35,49% of the initial population.

A final answer of 44% came from a student who in his work also got the number 0,3549:

Task 2, example 9



A student whose solution is shown below, wrote 58% as an answer:

Task 2, example 10

$$x_1 = year\ 1 \qquad x_1 = 69\%y + y$$

$$x_2 = year\ 2 \qquad x_2 = \frac{((69\%y + y) + 21\%y)}{2}$$

A result of 52,245% was given by two students. In both cases this was due to the following calculation:

Task 2, example 11

$$1,69 \cdot 1,21 = 2,0449$$

$$2,0449 - 1 = 1,0449$$

$$1,0449 : 2 = 0,522245$$

One person gave a result of 50% and this was achieved as follows:

Task 2, example 12

$$1 - x$$

$$2 - 1,69x$$

$$3 \sim 2x$$

$$\frac{4,69x}{3} \approx 1,5x.$$

A good answer was given only by one student. In the scan she sent, we can see the following written entry:

Task 2, example 13

$$x \to 1,69x \to 1,69 \cdot 1,21x = 2,0449x$$

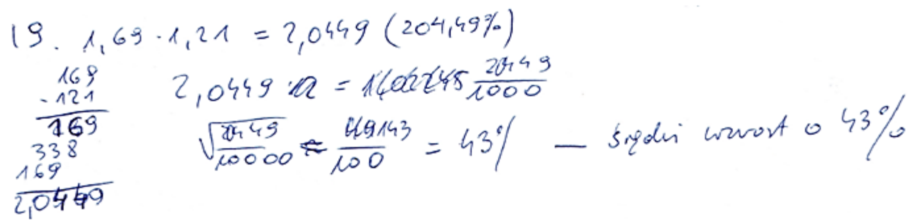
We can assume that the student calculated the root of 2,0449, obtained 1,43, and interpreted this result as an average annual percentage increase of 43%.

The lack of a written solution for this task could indicate either that a student had no clue how to tackle the task or that (s)he believed the task was trivial and all that needed to be done was to find the arithmetic mean of two numbers, which required only simple mental arithmetic. The lack of a written solution co-occurred in the group of students with the lack of an answer as many as 22 times, while the lack of a written solution and an answer of 45% occurred 11 times. This shows that there were twice as many students who were helpless with this task as those who were confident that the result was an arithmetic mean which could be calculated in memory. None of the students expressed doubts about the wording of the task, but it is possible that some of those who did not give an answer were confused about what mean was meant.

Teachers' solutions

Only one of the teachers sent a scan of a written solution to this task. It was the teacher who gave a correct answer. His solution was the following:

Task 2, example 14



19. $1,69 \cdot 1,21 = 2,0449$ (204,49%)

$2,0449$ ~~…~~ $= \frac{20449}{10000}$

$\sqrt{\frac{20449}{10000}} \approx \frac{143}{100} = 43\%$ — średni wzrost o 43%

169 · 121 = 169, 338, 169, 2,0449

The teacher was aware that the mean being sought was the geometric mean. His result is correct, only the calculations may not have been so complicated.

In the case of 6 teachers who gave 45% as an answer we may assume that the arithmetic mean was calculated in memory. We also suspect that the result of 52% was obtained by rounding up the result of 52,245%, that was given by two students and another teacher. It is possible that the solutions of these teachers were similar to the one given in example 11. It is difficult for us to justify the result of 12% given by one of the teachers. We also do not know whether the teachers who did not answer to this task wrote down any calculations. We hypothesize that the teachers were reluctant to share their notes with us, which they may have considered to be a rough draft, not suitable for showing to anyone.

4. Limitations of the study

Before we move on to summarize the findings, we would like to point out that the study we conducted had some limitations that need to be taken into account when drawing conclusions. The number of participants in the study was not big enough to give us the basis for generalizing the results to the whole population of students or teachers undertaking postgraduate courses qualifying for teaching mathematics. The under-representation of the latter group in the study was due to the pandemic driven measures strongly affecting the conduct of the study. Especially the restrictions imposed by the Polish government have led to changing the form of teaching on postgraduate programs from stationary to remote. The lack of opportunity for direct, face-to-face contact with the study participants precluded additional, follow-up questions that would have helped to establish, for example, how the solvers who wrote only their answer thought about the task when taking the test.

However, the revealed difficulties of the respondents should be taken into account in the training of future teachers. The results we obtained allow us to state with certainty that in the group of pre-service teachers there are persons who may have problems in solving some tasks concerning means. Moreover, it should not be presumed that, due to many years of accumulating mathematical experience at different stages of mathematical education, candidates for the teaching profession have developed conceptual understanding of means.

5. Conclusions and recommendations

As the organizers of the study, we were interested in not only knowing the results obtained by prospective mathematics teachers, but also in seeing how they arrived at these results. However, it is surprising that at least 45 of 73 respondents needed to write down the solution to the first task. This means that, for these people, handling the arithmetic mean is strictly related to performing some calculations and applying a well-known algorithm.

It is even more disturbing that as many as 30 out of 73 respondents from both groups did not give any answer in task two, and only 2 respondents were able to see the geometric mean there and give a correct answer. Our opinion is that the geometric mean is given too little attention at school, and its inclusion in the textbook, which has been presented, does not sufficiently explain when this mean should (could) be used and why. The geometric mean seems to be not intuitive enough. The current way of introducing this mean does not appeal to the imagination of the students, does not allow them to get a sense of what the nature of this mean is and what role it plays for a given set of numbers. Special attention may also need to be paid to the wording of the tasks themselves, so that the problem for the learners is not to figure out what they are actually being asked for, but to make an intellectual effort to solve the problem.

In the light of the obtained results, we strongly suggest that during the classes on didactics of mathematics a special attention be paid to the tasks 4, 7 and 13 from pages 300–301 of the 7th grade textbook referred to in this paper. Mathematics educators should also encourage prospective teachers to formulate analogous tasks concerning the geometric and harmonic mean, and thus investigate their nature. One might also be tempted to explore the relationship between the means and other concepts, e.g., it is relatively easy to show (and may be a big surprise to prospective teachers) that the logarithm of the geometric mean of a set of positive numbers is the arithmetic mean of the logarithms of the individual numbers.

Lastly, we want to suggest a way of introducing arithmetic, geometric and harmonic means that makes it possible to think and talk about them in a unified way.

Arithmetic mean

The arithmetic mean answers the question: if, in a given set of numbers, all the numbers had the same value, what would that value have to be for the sum of all the numbers to be the same?

For example: for the numbers 3 and 5, the arithmetic mean is 4, because if we replace each of the numbers 3 and 5 with 4, the sum of the two numbers remains the same: $3 + 5 = 4 + 4$.

Method of determination: (for a set of n numbers) we add all the n numbers in the set and then divide the resulting sum by n, the number of components.

Geometric mean

The geometric mean answers the question: if, in a given set of n positive numbers, all the numbers had the same value, what would that value have to be in order for the product of all the numbers to be the same?

For example: for the numbers 2 and 8, the geometric mean is 4, because if we replace each of the numbers 2 and 8 with 4, the product of the two numbers is the same: $2 \cdot 8 = 4 \cdot 4$.

Method of determination: (for a set of n positive numbers) multiply all n numbers from the set, then calculate the nth degree root of the resulting product.

Harmonic mean

The harmonic mean answers the question: if the reciprocals of each positive number in a given set were replaced by the reciprocal of one, the same number, what would that number have to be to give the same sum of reciprocals?

For example: for the numbers 3 and 5, the harmonic mean is $\frac{15}{4}$, because if we replace the inverse of each of the numbers 3 and 5 with $\frac{4}{15}$, the sum of the two inverses in each case will be the same: $\frac{1}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15} = \frac{4}{15} + \frac{4}{15}$. So, for the positive numbers a and b, we seek to find such a positive number c that:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{c}$$

Method of determination: (for a set of n positive numbers) the number n is divided by the sum of the inverses of each number.

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