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Conic curves in jungle river metric *

Abstract. In this article, we refer to different way of measuring distances than commonly used Euclidean metric. It is shown how some objects of two-dimensional geometry look like in the jungle river metric. Some examples of the objects are presented in pictures made in the free available *Geogebra* program, the conditions of occurrence of this objects are exactly described. Reading the text, one may learn that a sphere need not be round and a parabola may not be a bounded curve. Some task for the reader are included.

1. Introduction

In plane geometry, many objects and concepts can be defined by distance: collinearity of points, circle, spheres, segments, lines, conic sections: ellipse, hyperbola and parabola.

The distance we use every day is the Euclidean metric, so we are used to the shape of Euclidean (regular) line segments, lines, circles, spheres, ellipses, hyperbolas and parabolas. In this paper we will use a different way to measure distances.

Let us start from the metric and metric space definitions.

1.1. Definition

The metric (Engelking, 2007) on the set $\mathbb{R}^2$ is the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that satisfies the following conditions:

$$d(x, y) = 0 \Leftrightarrow x = y,$$

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$$\forall x, y \in \mathbb{R} : d(x, y) = d(y, x),$$

$$\forall x, y, z \in \mathbb{R} : d(x, z) \leqslant d(x, y) + d(y, z).$$

In the article, we will use a two-dimensional metric space with a jungle river metric.

Imagine that you are in a very dense forest with a river flowing through it with a very rushing stream. To get from one place to another on the same side of the river, we can only walk along the paths perpendicular to the waterhole which have been trampled by wild animals or along the bank of the river, we have to cross the forest along well-trodden paths and in the meantime to sail boat along the river.

Now we may to introduce a proper definition of the river metric on the plane.

1.2. Definition

Let us consider $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$.

The jungle river metric (Engelking, 2007) is the distance defined by the formula:

$$d(x, y) = \begin{cases} |x_2 - y_2|, & \text{when } x_1 = y_1, \\ |x_1 - y_1| + |x_2| + |y_2|, & \text{when } x_1 \neq y_1. \end{cases}$$

The above function is a metric – proof in (Jedrzejewski, Wilczyński, 1999).

By river we will mean to the axis Ox.

1.3. Remark

Let now introduce the definitions of selected objects with their exemplary visions (made in a free available program *Geogebra*) and conditions of occurrence of some peculiar examples.

1.4. Definition

Let $(\mathbb{R}^2, d)$ be a metric space, $x \in \mathbb{R},\ r > 0$.

The sphere (Engelking, 2007) with the center at the point x and of radius r is the set:

$$S(x, r) = \{y \in \mathbb{R} : d(x, y) = r\}.$$

The open ball (Engelking, 2007) with the center at the point x and of radius r is the set:

$$K(x, r) = \{y \in \mathbb{R} : d(x, y) < r\}.$$

A closed sphere is the sum of the sphere and the open ball (with respectively the same center and of the same radius).

1.5. Example

The pictures below show examples of two-dimensional spheres $S(x,r)$ and the open ball $K(x,r)$ in jungle river metric:

a) Spheres:

i) (Fig. 1) – two points equidistant from the center of the sphere on a perpendicular line to the axis Ox through this center when the radius is less than the distance from the center of the sphere to the river,

ii) (Fig. 2) – the boundary of a square with the “shot” vertex lying on a line perpendicular to the river through the center of the sphere when the radius is greater than the center distance spheres from the river;

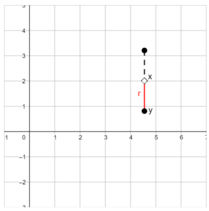
Figure 1

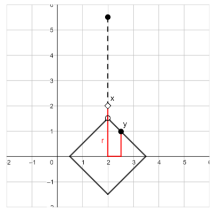
Figure 2

b) Open balls:

i) (Fig. 3) – segment perpendicular to the river (without ends), when the radius is less than distance from the center of the sphere to the river,

ii) (Fig. 4) – the inside of a square with a "tail" segment when the radius is greater than the center distance ball from the river.

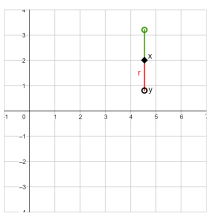
Figure 3

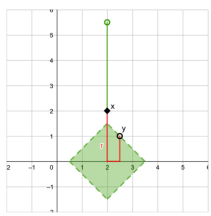
Figure 4

1.6. Definition

The metric segment (Lisiewicz, 1993) with ends A and B ($A \neq B$) in the metric space $(\mathbb{R}^2, d)$ we will meant the set:

$$\overline{AB} = \{C \in \mathbb{R} : d(A,C) + d(C,B) = d(A,B)\}.$$

The metric line L (Engelking, 2007) determined by points A and B ($A, B \in \mathbb{R}$, $A \neq B$) in metric space $(\mathbb{R}^2, d)$ is set of points:

$$L(A,B) = \big\{C \in \mathbb{R} : d(A,C) + d(C,B) = d(A,B) \ \vee \ d(A,B) + d(B,C) = d(A,C) \\ \vee \ d(B,A) + d(A,C) = d(B,C)\big\}.$$

1.7. Remark

For the well-known Euclidean metric, the metric segments and the metric lines are ordinary lines and lines known to us from geometry. This is due to the fact that in the space of $\mathbb{R}^2$ the segments and lines in the Euclidean metric are geometric concepts and in other metrics they are concepts related to a given metric.

1.8. Example

The pictures below show some examples of metric segments and metric lines in jungle river metric.

a) Segments (Fig. 5):

i) Euclidean segments parallel to the axis of the system, when the ends of the segment lie on the river or on a straight line parallel to the axis Oy (marked in blue and black),

ii) when the segment ends on the one side of the river, we get a section $\overline{EF}$ (marked in red),

iii) when the ends of the segment lies on opposite sides of the river on a straight line not parallel to the axis Oy, we get the segment $\overline{GH}$ (marked in green);

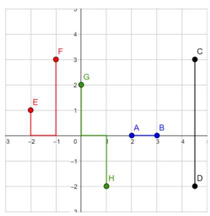

Figure 5

b) Lines (Fig. 6–9):

i) we get all the plane $\mathbb{R}^2$, when one of the points marking the straight line lies on the river and the other lies on a straight line perpendicular to the river passing through the first point (marked in orange),

ii) we get the sum of two closed Euclidean semi-planes and a Euclidean segment with ends marking the straight line in the jungle river metric, when these points lie on the river (marked in blue),

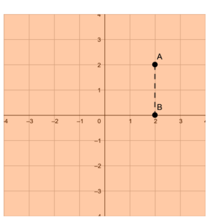

Figure 6

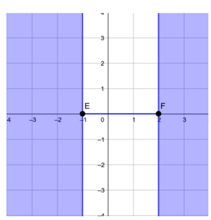

Figure 7

iii) we get the sum of the Euclidean half-plane and the half-line perpendicular to the river with it is beginning on the river and the segment that connects them on the river, when the points marking the straight line do not lie on one straight line perpendicular to the river and one of them lies on the river (marked in purple),

iiii) we get Euclidean line, when the points marking the line lie on a straight line perpendicular to the river (marked in black),

iiiii) we get the sum of two Euclidean half-lines and the segment lying on the river, when the points marking the line lie outside the river and do not lie on a straight line parallel to the axis Oy (marked in red and green).

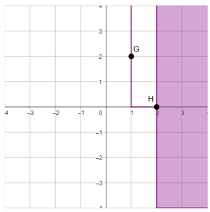

Figure 8

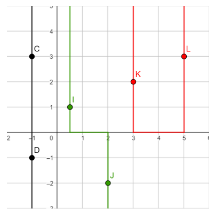

Figure 9

1.9. Example

Let try to find the segments in the jungle river metric.
Let us assume that $A = (x_A, y_A), B = (x_B, y_B)$ and $C = (x, y)$.
By definition of the jungle river metric and the metric segment:

a) for $x \neq x_A \neq x_B \neq x$, we get:

$$d(A, B) = |x_A - x_B| + |y_A| + |y_B|,$$

$$d(A, C) = |x_A - x| + |y_A| + |y|,$$

$$d(C, B) = |x - x_B| + |y| + |y_B|.$$

$$|x_A - x| + |y_A| + |y| + |x - x_B| + |y| + |y_B| = |x_A - x_B| + |y_A| + |y_B|,$$

$$|x_A - x| + |x - x_B| + 2|y| = |x_A - x_B|.$$

The above equation will have a solution when:

$$|x_A - x_B| \geqslant |x_A - x| + |x - x_B|.$$

Solving this inequality, we have:

i) when $x_B > x_A$, we get $x_B \geqslant x \geqslant x_A$,

ii) when $x_B < x_A$, we get $x_B \leqslant x \leqslant x_A$,

iii) when $x_B = x_A$, we get $x = x_A$.

For each of these cases, we get $y = 0$.

b) for $x_A = x \neq x_B$ (case $x_B = x \neq x_A$ is symmetrical), we get:

$$d(A,B) = |y_A| + |y_B| + |x_A - x_B|,$$

$$d(A,C) = |y_A - y|,$$

$$d(C,B) = |y| + |y_B| + |x_A - x_B|.$$

$$|y_A - y| + |y| + |y_B| + |x_A - x_B| = |y_A| + |y_B| + |x_A - x_B|,$$

$$|y_A - y| + |y| = |y_A|.$$

Solving the above equation:

i) when $y_A > 0$, we get $y_A \geqslant y \geqslant 0$,

ii) when $y_A < 0$, we get $y_A \leqslant y \leqslant 0$,

iii) when $y_A = 0$, we get $y = 0$.

c) for $x_A = x = x_B$, we get:

$$d(A,B) = |y_A - y_B|,$$

$$d(A,C) = |y_A - y|,$$

$$d(C,B) = |y - y_B|.$$

$$|y_A - y| + |y - y_B| = |y_A - y_B|.$$

Solving the above equation, we have:

i) when $y_B > y_A$, we get $y_B \geqslant y \geqslant y_A$,

ii) when $y_B < y_A$, we get $y_B \leqslant y \leqslant y_A$,

iii) when $y_B = y_A$, we get $y = y_A$.

By changing the coordinates of points A and B, we can obtain the segments in the Fig. 7.

1.10. Definition

Let $(\mathbb{R}^2, d)$ be a metric space, $x \in A,\ \varepsilon \geqslant 0$.

The set:

$$A_\varepsilon = \{y \in \mathbb{R} : d(x, y) \leqslant \varepsilon\}.$$

is called the $\varepsilon -$ *envelope* of the set A.
Of course $A_0 = A$.

The set:

$$A_\varepsilon = \{y \in \mathbb{R} : d(x, y) = \varepsilon\}.$$

ic called the $\varepsilon -$ *border* of the set A (Engelking, 2007).

1.11. Example

The pictures below show examples of ε - envelope and ε - border for Euclidean line in the jungle river metric:

a) (Fig. 10) – ε - envelope;

b) (Fig. 11) – ε - border.

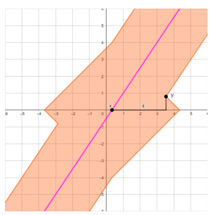
Figure 10

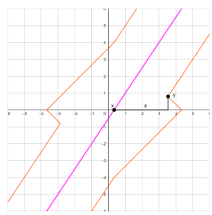
Figure 11

1.12. Definition

Let $(\mathbb{R}^2, d)$ be a metric space and $\varnothing \neq A \subset \mathbb{R}$.
The distance of point $x \in \mathbb{R}$ of the set A (Engelking, 2007) is called the number defined as below:

$$\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y).$$

1.13. Remark

1. For $x \in A$ $\operatorname{dist}(x, A) = 0$,

2. In practice for $x \notin A$ (where A is closed set) the formula is used:

$$\operatorname{dist}(x, A) = \max\{r : K(x, r) \cap A = \varnothing\}.$$

1.14. Example

The pictures below show some examples of the distance of the point of the set in the jungle river metric.
In all the cases, the distance of the point x from the set A is realized by the radius of a sphere in the jungle river metric connecting the center of the sphere to the "tangent point" y of the sphere with the set A. The point x and the set A are marked in black, the radius of the sphere that realizes the distance is marked in red, and the "tangent point" y in orange:

a) (Fig. 12)– when the point x and the "tangent point" y lie on one straight line perpendicular to the river;

b) (Fig. 13)– when the point x and the "tangent point" y do not lie on one straight line perpendicular to the river.

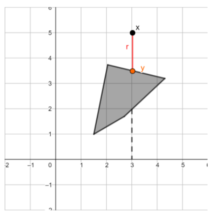
Figure 12

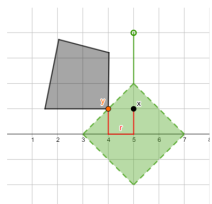
Figure 13

2. Ellipse

2.1. Definition

The ellipse (Leja, 1973) in metric space $(\mathbb{R}^2, d)$ is the set composed of all points P, for which distances from fixed points F_1 and F_2 (focuses) satisfied the condition:

$$d(P, F_1) + d(P, F_2) = 2a,$$

where $2a$ is a positive constant greater than the distance between the focuses.

2.2. Remark

Let us suppose that the focuses of the ellipse are points $F_1 = (x_{F_1}, y_{F_1})$, $F_2 = (x_{F_2}, y_{F_2})$ and the point $P = (x, y)$ is a point belonging to this ellipse.

The ellipse in a Euclidean metric is a set of points satisfying the equation:

$$\sqrt{(x_{F_1} - x)^2 - (y_{F_1} - y)^2} + \sqrt{(x_{F_2} - x)^2 - (y_{F_2} - y)^2} = 2a.$$

If the focuses coincides we get a sphere in the metric with the center at the point $F_1 = F_2$ and of radius a.

Using the classic definitions of an ellipse in the metric space $(\mathbb{R}^2, d)$ and changing the Euclidean distance to the distance related to the jungle river metric, we get the following conditions describing the belonging of the point P to the ellipse depending on the position of the point P and the focuses:

a) if points P, F_1, F_2 lie on one line perpendicular to the river $(x = x_{F_1} = x_{F_2})$:

$$|y - y_{F_1}| + |y - y_{F_2}| = 2a;$$

b) if points F_1, P lie on a line perpendicular to the river and F_2, P do not lie on this line $(x_{F_1} = x \neq x_{F_2})$:

$$|y - y_{F_1}| + |y| + |x - x_{F_2}| + |y_{F_2}| = 2a;$$

c) if points F_2 and P lie on a line perpendicular to the river and F_1 and P do not lie on this line $(x_{F_2} = x \neq x_{F_1})$:

$$|y - y_{F_2}| + |y| + |x - x_{F_1}| + |y_{F_1}| = 2a;$$

d) if no two of the points P, F_1, F_2 do not lie on a straight line perpendicular to the river $(x \neq x_{F_1} \neq x_{F_2} \neq x)$:

$$2|y| + |x - x_{F_1}| + |x - x_{F_2}| + |y_{F_1}| + |y_{F_2}| = 2a.$$

2.3. Example

The distance realises the distance $2a$ marked in red.

Depending on the location of the focuses, we get different shapes of the ellipses:

a) the boundary of a hexagon when the focuses lie on the river (as in Fig. 14);

b) the boundary of a hexagon with two points “shot” in the direction of the axis Oy of the distance of the focuses from the river (x coordinates of these points coincide with x coordinates of the focuses of the ellipse):

   i) when the focuses lie on different sides of the river (as in Fig. 15),

   ii) when the focuses lie on the same side of the river (as in Fig. 16);

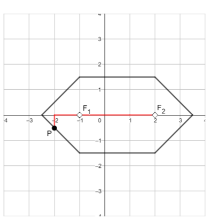

Figure 14

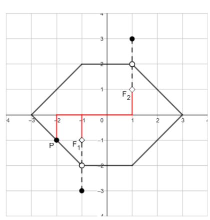

Figure 15

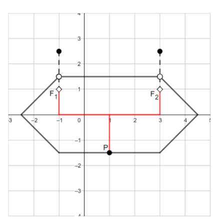

Figure 16

c) a set composed of two points distant from the focuses by a value $z = a - \frac{d(F_1,F_2)}{2}$ on a line perpendicular to the river, when the focuses lie on the line perpendicular to the river and the distance of the focuses from the river is greater than the value $2a$ (as in Fig. 17);

d) the boundary of a square with the "shot" point, when the focuses lie on one straight line perpendicular to the river and the distance of the focuses from the river is less than the value $2a$ (as in Fig. 18).

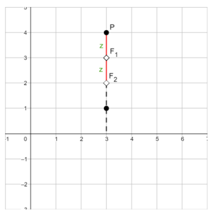
Figure 17

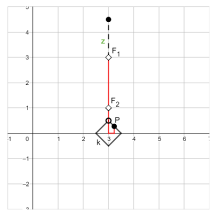
Figure 18

3. Hyperbola

3.1. Definition

Let $(\mathbb{R}^2, d)$ be a metric space and let F_1 and F_2 be fixed points of $\mathbb{R}$. *The hyperbola* (Leja, 1973) with focuses F_1 and F_2 we will meant the set of all such points P for which the distances $d(P, F_1)$ and $d(P, F_2)$ satisfied the condition:

$$\left|d(P, F_1) - d(P, F_2)\right| = 2a,$$

where $2a$ is a positive constant smaller than the distance between the focuses.

3.2. Remark

Let us suppose that the focuses of the hyperbola are points: $F_1 = (x_{F_1}, y_{F_1})$, $F_2 = (x_{F_2}, y_{F_2})$ and point $P = (x, y)$ belong to this hyperbola.

The hyperbola in a Euclidean metric is a set of points satisfying the equation:

$$\left|\sqrt{(x_{F_1} - x)^2 - (y_{F_1} - y)^2} - \sqrt{(x_{F_2} - x)^2 - (y_{F_2} - y)^2}\right| = 2a.$$

Using the classic definitions of a hyperbola in the metric space $(\mathbb{R}^2, d)$ and replacing the Euclidean distance by the distance related to the jungle river metric, we get the following conditions describing the belonging of the point P to the hyperbola depending on the position of the point P and the focuses:

a) if points P, F_1, F_2 lie on the same line perpendicular to the river $(x = x_{F_1} = x_{F_2})$:

$$\big||y - y_{F_1}| - |y - y_{F_2}|\big| = 2a;$$

b) if points F_1, P lie on a line perpendicular to the river and F_2, P do not lie on this line ($x_{F_1} = x \neq x_{F_2}$):

$$\big||y - y_{F_1}| - |y| - |x - x_{F_2}| - |y_{F_2}|\big| = 2a;$$

c) if points F_2, P lie on a line perpendicular to the river and F_1, P do not lie on this line ($x_{F_1} \neq x = x_{F_2}$):

$$\big||y - y_{F_2}| - |y| - |x - x_{F_1}| - |y_{F_1}|\big| = 2a;$$

d) if no two of the points P, F_1, F_2 do not lie on a straight line perpendicular to the river ($x \neq x_{F_1} \neq x_{F_2} \neq x$):

$$\big||x - x_{F_1}| + |y_{F_1}| - |x - x_{F_2}| - |y_{F_2}|\big| = 2a.$$

3.3. Example

In the pictures below the segments which realize the distances $d(P, F_1)$ and $d(P, F_2)$ are marked in green.

The shape of a hyperbola in the jungle river metric depends on the location of its focuses in regard to the river:

a) when the focuses lie on the same distance from the river, the hyperbola consists of:

   i) two half-planes without two semi-lines perpendicular to the river through focuses when $|x_{F_1} - x_{F_2}| = 2a$ (as in Fig. 19),

   ii) two points equidistant from the focuses on lines perpendicular to the river through the focuses when $|x_{F_1} - x_{F_2}| < 2a$ (as in Fig. 20),

   iii) two Euclidean lines equally distant from the focuses when $|x_{F_1} - x_{F_2}| > 2a$ (as in pic.21);

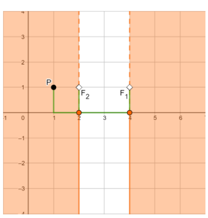

Figure 19

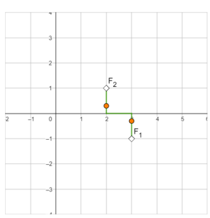

Figure 20

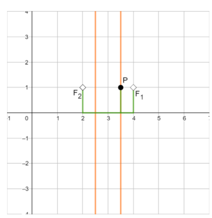

Figure 21

b) when the focuses lie at different distances from the river, the hyperbola is composed of:

   i) a half-plane without a semi-line perpendicular to the river through the focus and point is lying on such line when

$$|y_{F_1}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| = 2a$$

   (as in Fig. 22–23),

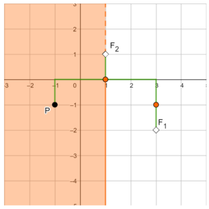

Figure 22

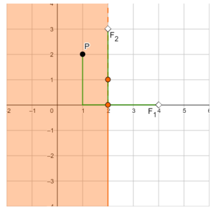

Figure 23

ii) two points are lying on semi-lines perpendicular to the river through the focuses,

$$|y_{F_1}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| < 2a \text{ and } |y_{F_2}| + |x_{F_1} - x_{F_2}| - |y_{F_1}| < 2a$$

(as in Fig. 24),

iii) Euclidean line perpendicular to the river and the point when

$$|y_{F_1}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| < 2a \text{ or } |y_{F_2}| + |x_{F_1} - x_{F_2}| - |y_{F_1}| > 2a$$

or

$$|y_{F_2}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| > 2a \text{ or } |y_{F_1}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| < 2a$$

(as in Fig. 25),

iiii) the Euclidean lines prependicular to the river when

$$|y_{F_1}| + |x_{F_1} - x_{F_2}| - |y_{F_2}| > 2a \text{ and } |y_{F_2}| + |x_{F_1} - x_{F_2}| - |y_{F_1}| > 2a$$

(as in Fig. 26).

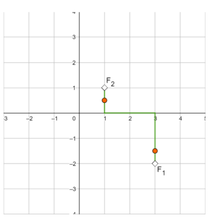

Figure 24

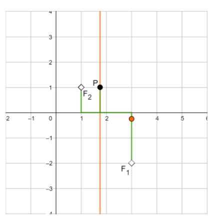

Figure 25

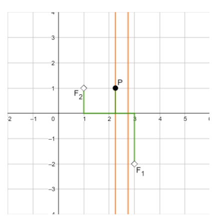

Figure 26

4. Parabola

4.1. Definition

The parabola (Leja, 1973) in metric space $(\mathbb{R}^2, d)$ is the set composed of all points P, for which distances from a fixed point F (focus) and from an established line l (directrix) not through F are equal:

$$d(P, F) = d(P, l).$$

4.2. Remark

In the most metrics, the metric line differs from the Euclidean line, so we will consider pseudo-parabolas (which include points with the same distance from the focal point and the Euclidean line) and metric parabolas in the strict sense, where the role of the directrix is played by the line given by the metric.

In order to receive the chart of the parabola, we find the points belonging to it as common points of the ε - envelope with directrix and the sphere $S(F,\varepsilon)$ for possible nonnegative values ε with the help of an applet created in *Geogebra*.

4.3. Example

Pseudo-parabola.

Let directrix be an Euclidean line with the equation $l: Ax + By + C = 0$.

Let the focus of the hyperbola is point: $F = (x_F, y_F)$ and $P = (x, y)$ is a point belonging to a parabola.

The parabola in a Euclidean metric is a set of points satisfying the equation:

$$\sqrt{(x_F - x)^2 - (y_F - y)^2} = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}.$$

Parabola in the jungle river metric consists of:

a) two half-lines perpendicular to the river connected by two regular segments when the directrix is a line parallel to the river (as in Fig. 27);

b) a half-plane when the directrix is a line parallel to the bisector of the angle between the axes of the coordinate system (as in Fig. 28);

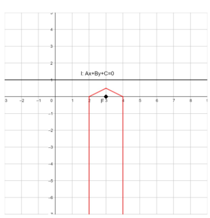

Figure 27

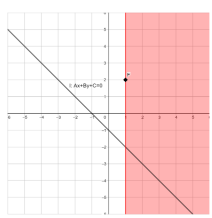

Figure 28

c) one point when the value of $|y_F|$ is greater than $\left|\frac{C}{A} - x_F\right|$ (as in Fig. 29);

d) a line perpendicular to the river when the value of $|y_F|$ is less than $\left|\frac{C}{A} - x_F\right|$ (as in Fig. 30).

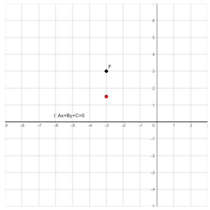

Figure 29

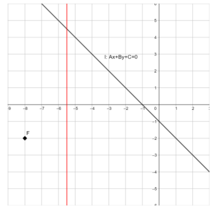

Figure 30

4.4. Example

The metric parabola in the strict sense.

Let the directrix will be a line L in the jungle river metric through the points A and B.

In order to receive the chart of the parabola, we find the points belonging to it as common points of the ε - envelope with directrix and the sphere $S(F, \varepsilon)$ for possible nonnegative values ε with the help of an applet created in *Geogebra*.

The form of the condition describing the affiliation of the point P to the parabola in the jungle river metric depends on the position of the point P and the focus:

a) if P and F lie on one straight line perpendicular to the river $(x = x_F)$:

$$|y - y_F| = \mathrm{dist}(P, L);$$

b) if P and F do not lie on one straight line perpendicular to the river $(x \neq x_F)$:

$$|y| + |x - x_F| + |y_F| = \mathrm{dist}(P, L).$$

4.5. Example

Depending on the position of the focus relative to the directrix, the parabola is:

a) a half-plane when the focus lies on a line parallel to the bisector of the angle between the axes through the focus and the nearest point on the directrix (as in Fig. 31);

b) in a single-point set, when the distance of the focus from the river is greater than the distance between the projection of the focus into the river and the nearest point marking the line (as in Fig. 32);

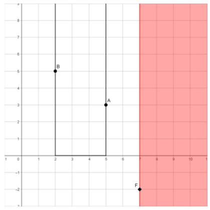

Figure 31

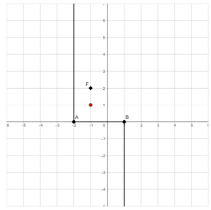

Figure 32

c) a line, when the distance of the focus from the river is less than the distance between the projection of the focus on the river and the nearest point marking the line (as in Fig. 33).

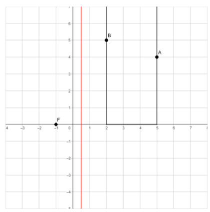

Figure 33

4.6. Remark

The objects that we get in the jungle river metric differ from their Euclidean counterparts. Spheres and ellipses can contain “shot” points, spheres can have “tails”, hyperbolas and parabolas can be semi-planes, straight lines, or one-point/two-point sets. This is due to the fact that in this metric, the method of calculating the distance changes depending on the mutual position of the points in relation to the river.

In each of the considered objects there are Euclidean segments or lines.

5. Example for readers

5.1. Definition

Let $x = (x_1, x_2),\ y = (y_1, y_2)$ denote any points from $\mathbb{R}^2$.
The taxicab metric d_1 (Lisiewicz, 1993) is the distance defined by the formula:

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

5.2. Example

In the taxicab metric, metric lines may look like in the pictures (Fig. 34–36):

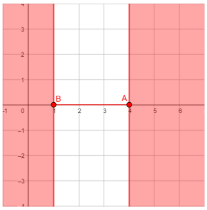

Figure 34

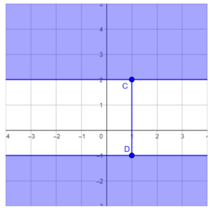

Figure 35

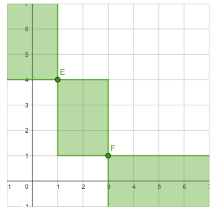

Figure 36

5.3. Example

For a set A its ε - envelope and ε - border may be as in pictures (Fig. 37–38):

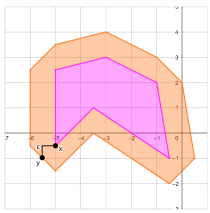

Figure 37

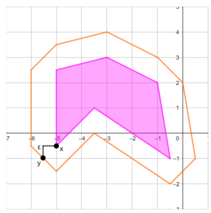

Figure 38

And what will the parabolas look like in this metric? Will then be closed or open curves?

Some examples you can see below (Fig. 39–42):

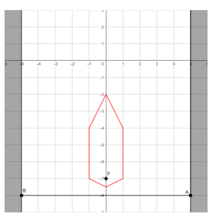

Figure 39

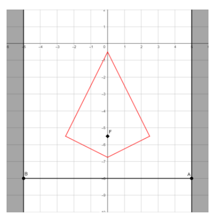

Figure 40

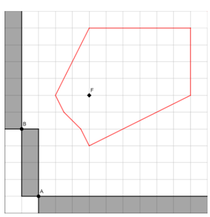

Figure 41

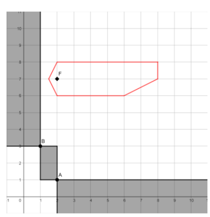

Figure 42

We invite you to search for other shapes of parabolas and some shapes of hyperbolas.

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