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Central Conics Left in Place *

Abstract. A method is presented for creating a problem, solving it, and confirming that the solution is correct. The problem is to analyze a quadratic equation in two variables, in order to identify and draw, in its place – that is, without rotating the coordinate system – the central conic section that the equation defines. One creates the problem by choosing conjugate diameters for the conic that are not orthogonal; the solution requires finding orthogonal conjugate diameters, namely the axes. One can do this by the method of Apollonius, which is to intersect the conic with a concentric circle; the resulting four points are the vertices of a rectangle, whose sides are parallel to the axes. Along the way, by completing squares, one has found another pair of conjugate diameters, one of these being parallel to one of the axes of coordinates (or indeed to any line that one chooses). By sketching all three pairs of diameters, with their endpoints, one can see by inspection whether one's computations are likely to have been correct. Those computations in turn serve to confirm the general formulas that are found, in terms of the coefficients of the equation, for the endpoints of the axes of an arbitrary central conic.

1. Introduction

Perhaps in any subject, but certainly in mathematics, one must learn to evaluate one's own work. It is also good to be able to come up with one's own exercises.

For example, in linear algebra, one may be taught an algorithm for finding the inverses of matrices. One may also be shown a proof that the algorithm works. One ought to learn also to confirm one's result with any particular invertible matrix A, as by checking that indeed $A \cdot A^{-1} = I$. One does well to learn also to create new invertible matrices by applying elementary row and column operations to I; for then one can set oneself the problem of finding the inverses of those matrices without considering where they came from.

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We present here a similar example, developed as a project for students of analytic geometry under the restrictions of the Covid-19 pandemic. The project needs only high-school algebra, but getting the computations correct is a challenge. The project asks us

1) to create our own problem,
2) to solve it,
3) to confirm our solution.

That third step shows that the computations are actually meaningful.

1.1. Creation

We create our problem in the following steps.

A. In a rectangular coordinate system, we select points (a, b) and (c, d) with integral coordinates so that
$$ad - bc \neq 0 \tag{1}$$
and also
$$ac + bd \neq 0. \tag{2}$$
Thus, considered as vectors, (a, b) and (c, d) are linearly independent, but not orthogonal.

B. We select a point (e, f), different from the origin, but also with integral coordinates.

C. By choosing a value for ± 1, we form an equation
$$\big(d(x-e) - c(y-f)\big)^2 \pm \big(b(x-e) - a(y-f)\big)^2 - (ad-bc)^2 = 0. \tag{3}$$

D. We multiply out (3) to obtain
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{4}$$

Our problem now is to understand the curve defined by (4). Let us say precisely what this means.

1.2. Solution

Our curve is a central conic, namely an hyperbola or ellipse, depending on whether we have chosen the coefficient ± 1 on the second square in (3) to be -1 or 1. The conic is non-degenerate, by (1). The conic is not a circle, by (2); this is true, even though the vectors may have the same magnitude, as they do in Figure 1. To keep the problem challenging, we may make the additional requirements
$$|a| = |c| \Rightarrow |b| \neq |d|, \qquad |a| = |d| \Rightarrow |b| \neq |c|,$$
which (along with the non-orthogonality) ensure that (a, b) and (c, d) make different angles with the coordinate axes, if the vectors are equal in magnitude.

We could also just require the vectors (a, b) and (c, d) to differ in magnitude. In any case, the center of our conic is (e, f), and one thing to do will be to recover this from (4).

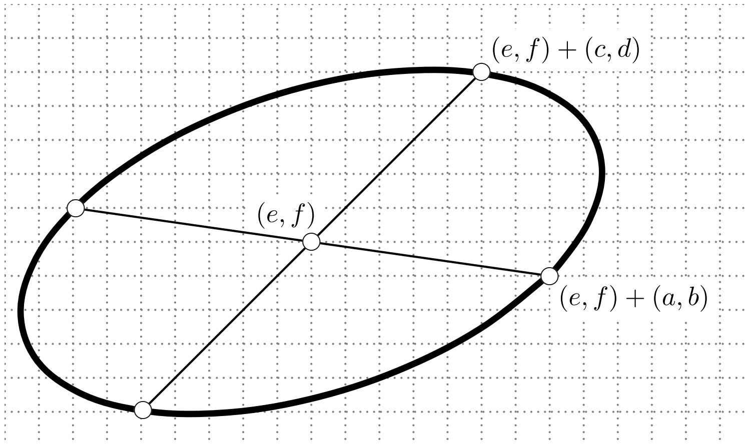

Figure 1: Ellipse defined by the positive case of (3)

In the terminology of Apollonius (Apollonius of Perga, 1998, p. 4), our conic has *conjugate diameters* in the directions of (a, b) and (c, d) respectively. Such diameters are drawn in Figure 1, and we shall talk more about them in § 2. We cannot recover the original conjugate diameters from (4), because, as Apollonius shows, *every* line through the center of the conic is one of a pair of conjugate diameters. Our equation (4) singles out no particular such line.

Our original choice of conjugate diameters can be inferred from (3), but we erase this information when we obtain (4). This new equation still defines the same conic that (3) does. As Apollonius also shows, because our conic is not a circle, it has a *unique* pair of conjugate diameters that are mutually orthogonal. These are the *axes* of the conic, and we want to find these too.

In the most precise sense, conjugate diameters are not just lines, but line segments. The conjugate diameters that we start with have endpoints $(e, f) \pm (a, b)$ and $(e, f) \pm (c, d)$ respectively. The endpoints of both diameters lie on the conic, if it is an ellipse. If it is an hyperbola, as in Figure 2, then the endpoints of only one diameter lie on it: Apollonius calls this the *transverse* diameter, while the other is the *upright* diameter. The endpoints of the latter lie on the *conjugate* hyperbola, whose equation we can obtain from (3) by interchanging (a, b) and (c, d) or by changing the sign on $(ad - bc)^2$.

A complete solution of our problem will give us, from (4) alone,

- the center (e, f) of our conic,
- the directions of its axes,
- the endpoints of the axes,
- the specification of whether the conic is an ellipse or hyperbola, and
- in the case of the hyperbola, the specification of which axis is transverse.

Solving our problem thus means obtaining the point (e, f), along with *orthogonal* vectors (α, β) and (γ, δ) for which the curve given by (4) is given also by

$$\big(\delta(x - e) - \gamma(y - f)\big)^2 \pm \big(\beta(x - e) - \alpha(y - f)\big)^2 = (\alpha\delta - \beta\gamma)^2. \tag{5}$$

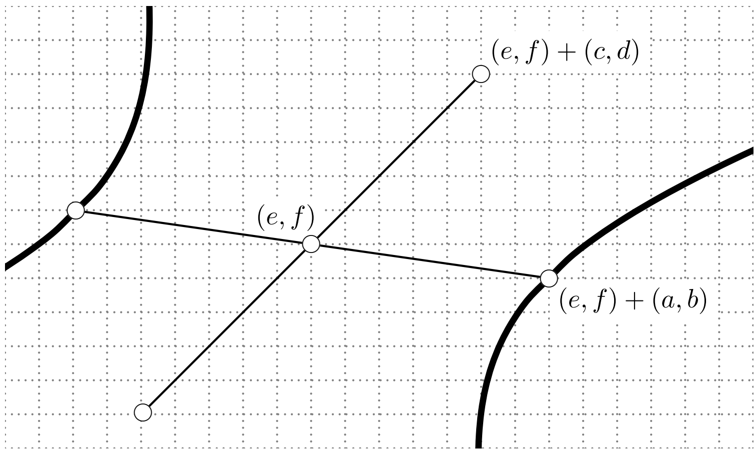

Figure 2: Hyperbola defined by the negative case of (3)

If ± 1 is positive, the conic is an ellipse; if negative, an hyperbola, and the transverse axis is in the direction of (α, β).

We shall work out numerical examples in §§ 2 and 3, and a general solution, with formulas for α, β, γ, and δ, in § 4. The specific examples will help to confirm that the general formulas are correct.

In outline, our method of solution is as follows.

1 (completion of squares) Defining

$$\Delta = 4AC - B^2 \tag{6}$$

and multiplying (4) by $4A\Delta$, we apply the high-school technique of completing squares to obtain

$$\Delta(2Ax + By + D)^2 + (\Delta y + 2AE - BD)^2 + 4A\Delta F' = 0 \tag{7}$$

for some F'. Thus our curve has conjugate diameters, one of them horizontal, given respectively by

$$\Delta y + 2AE = BD, \qquad 2Ax + By + D = 0. \tag{8}$$

Their intersection is the center, (e, f).

2 (intersection with circle) The horizontal diameter of our conic is also the diameter of a circle. As Apollonius shows (Apollonius of Perga, 1998, II.47, pp. 156–7), that circle (or any circle, concentric with the conic, that intersects the conic at all) intersects the conic in the four vertices of a rectangle, whose sides are parallel to the axes of the conic. We find those vertices using another high-school technique, namely that of "solving any pair of simultaneous second-degree equations of which one is homogeneous" (Weeks, Adkins, 1971, pp. 394–5).

3 (identification of axes) The sides of our rectangle will be parallel to the axes of the curve. We compute what those axes are and ultimately obtain (5).

1.3. Confirmation

If we are working with actual numbers, then each step of the solution gives us information that we can plot on graph paper:

1) center and a pair of conjugate diameters of our curve, as given by (8);
2) four points of the curve, where a concentric circle intersects it;
3) the axes of the curve.

Also, though we are working with (4), we have not forgotten (3), which also gives us a center and a pair of conjugate diameters to plot. If we have made a mistake along the way, it ought to be obvious in the graphs, if not in the directions of the conjugate diameters, then in their lengths. That was my experience, at least, in working out examples by hand.

Another method for checking our work is actually multiplying out what we have for (5), to see that we get (4). We can still make mistakes here though, especially since we know the answer that we want to get.

The coordinates of the axes of our conic will involve radicals within radicals; thus we may use an electronic calculator to approximate the coordinates. To check my students' work, in LaTeX (using also the packages `calc`, `ifthen`, `pstricks`, `pst-eucl`, and `pst-plot`) I wrote a command that, given (a, b), (c, d) and (e, f), draws the graph of (3) while printing out (α, β) and (γ, δ) in (5) exactly, as in the formulas to be found in § 4. It would be a good thing if students could come up with such a program for themselves, if they did not want to do computations by hand with specific numbers as in §§ 2 and 3.

Instead, a student may do those computations, starting with (3), multiplying out to get (4), completing squares to get (7), and finally coming up with an equation (5). The student may ask, "Is this right?" We can respond, "See for yourself, by sketching the curves defined by what you have found for (7) and (5); they should be the same as for (3)."

Teaching a different subject, Robert Pirsig faced a similar challenge. He had to answer students who asked, while turning in an assignment, "Is this what you want?" Pirsig described the situation in a letter (Pirsig, 1961); in *Zen and the Art of Motorcycle Maintenance,* he quoted a passage from the letter that concludes, "All those [instructors of English composition] who feel that quality of writing does exist . . . can benefit by the the following method of teaching pure quality in writing without defining it" (Pirsig, 1974, ch. 18, p. 191). The method involves having students assess one another's work.

What Pirsig wanted, as an English teacher, was for students to learn to write what *they* wanted. In the end, this would somehow be what everybody else wanted, which was *quality.*

In mathematics, an essential part of quality is *truth,* or *correctness* if you prefer. We take this to be universal. Everybody who takes an interest should be able to confirm whether

- a given proof does yield a purported theorem,
- calculated numbers do satisfy the conditions that they are intended to satisfy.

The project worked out here is supposed to help teach that lesson.

1.4. Analysis

Our project involves a procedure, which we sketched above, and which we shall spell out in detail in the later sections, for finding center and axes of a central conic defined by an equation (4), derived from (3) under conditions (1) and (2). In principle, as we have said, the procedure needs only high-school knowledge:

- completion of squares,
- solution of easy cases of simultaneous quadratic equations,
- the rectangular coordinatization of the Euclidean plane.

Strictly speaking, we need not know the theory of conjugate diameters of a central conic section; the project itself can be used to establish this theory.

However, there is already a standard procedure, described for example in (Nelson, Folley, Borgman, 1949, pp. 162–6) and (Vygodsky, 1975, pp. 85–95, 105–9), for finding center and axes of a central conic. It requires more theory, especially in relating different rectangular coordinatizations of the same plane. Matrix algebra is useful here, and even calculus. We review this now, but shall not need it later, so the reader can also skip ahead to § 2.

Let us denote the left-land side of (4) by $\Phi(x, y)$.

- To find the center of the conic defined by (4), we can translate the axes of coordinates so that the linear part $Dx + Ey$ of the equation disappears; the new origin of coordinates is then the center of the conic. To find this center precisely, we compute

$$\begin{aligned}\Phi(x+e, y+f) = Ax^2 + Bxy + Cy^2 + F' \\ + (2Ae + Bf + D)x + (Be + 2Cf + E)y\end{aligned}$$

for some constant F'. The center of the conic is that point (e, f), in the original coordinate system, for which the coefficients $2Ae + Bf + D$ and $Be + 2Cf + E$ on x and y vanish. Thus, as noted in (Ayoub, 1993), (e, f) is the solution of the system

$$\partial_x \Phi = 0, \qquad \partial_y \Phi = 0.$$

These two lines define diameters of the conic that cut it (or its conjugate, if the conic is an hyperbola) where the tangent to the conic is horizontal and vertical respectively. Thus the diameters are not conjugate unless they are already axes, since the tangent where one diameter cuts a central conic is parallel to the conjugate diameter. In any case, having found the center (e, f), if we make a coordinate translation by defining

$$\begin{aligned}(x, y) &= (x' + e, y' + f), \\ (x', y') &= (x - e, y - e),\end{aligned} \tag{9}$$

and then we cover our tracks by erasing the primes, we obtain in place of (4) the equation

$$Ax^2 + Bxy + Cy^2 + F' = 0 \tag{10}$$

in the translated coordinate system.

• To find the axes of the conic, we can rotate the axes of coordinates so that the mixed term Bxy in (4) disappears; the new axes of coordinates are parallel to the axes of the conic. To find the slopes of the axes of the conic in the original coordinate system, we compute

$$\begin{aligned}\Phi(x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi) &= A'x^2 + C'y^2 + D'x + E'y + F \\ &\quad + \big(B\cos(2\varphi) - (A-C)\sin(2\varphi)\big)xy\end{aligned}$$

for some constants A', C', D' and E'. In the original coordinate system, the slopes of the axes of the conic are the two values of $\tan\varphi$ for which the coefficient $B\cos(2\varphi) - (A-C)\sin(2\varphi)$ on xy vanishes, that is,

$$\tan(2\varphi) = \frac{B}{A-C}.$$

By the double-angle formula for tangents, this equation yields

$$B\tan^2\varphi + 2(A-C)\tan\varphi - B = 0. \tag{11}$$

It is noted in (Ayoub, 1993) that we get this equation if we convert (10) to polar coordinates and differentiate with respect to angle, in order to find the angles that maximize or minimize the radius. In any case, if we make a coordinate rotation by defining

$$\begin{aligned}(x, y) &= (x'\cos\varphi - y'\sin\varphi, x'\sin\varphi + y'\cos\varphi),\\ (x', y') &= (x\cos\varphi + y\sin\varphi, x\sin\varphi - y\cos\varphi),\end{aligned} \tag{12}$$

and again we cover our tracks by erasing the primes, we obtain in place of (4) the equation

$$A'x^2 + C'y^2 + D'x + E'x + F = 0$$

in the rotated coordinate system.

The foregoing procedures (which we can apply in either order) for finding the center and the axial slopes of a central conic are analytic in the original sense of Pappus of Alexandria (Thomas, 1951, p. 597):

> in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle . . .

In our case, in order to “suppose that which is sought to be already done,” we use the place-holding letters e, f, and φ in the procedures sketched above.

The analytic method may be essential to mathematics. It would seem to be what Descartes calls “true mathematics” in the fourth of the *Rules for the Direction of the Mind* (Descartes, 1985, pp. 18–9):

> one can even see some traces of this true mathematics, I think, in Pappus and Diophantus . . . But I have come to think that these writers themselves, with a kind of pernicious cunning, later suppressed

> this mathematics . . . They may have feared that their method, just because it was so easy and simple, would be depreciated if it were divulged . . . In the present age some very gifted men have tried to revive this method, for the method seems to be none other than the art which goes by the outlandish name of 'algebra' . . .

Descartes wrote that in 1628; in 1637, he would publish the *Geometry* (Descartes, 1954), which seems to have initiated our practice of defining curves by equations that use letters x and y as variables, and letters from the beginning of the alphabet as constants.

The present work is all about such equations, especially (3). Our purpose is to connect the equation directly with the conic that it defines. We want to understand the conic *in its place,* with respect to its given coordinate system. Specifically, we want to rewrite (3) as (5), which reveals the axes. In order to find the vectors (α, β) and (γ, δ) that feature in (5), after the rewriting that turns (3) into (4), and after the translation that turns the latter into (10), we still must somehow solve the last equation simultaneously with either of $y = x \tan\varphi$ and $x + y\tan\varphi = 0$. We can do this directly, as we are going to do later; or we can rotate the coordinate axes so that (10) becomes

$$A'x^2 + C'y^2 + F' = 0. \tag{13}$$

When we rotate back, we get

$$A'(x\cos\varphi + y\sin\varphi)^2 + C'(x\sin\varphi - y\cos\varphi)^2 + F' = 0.$$

Then

$$\begin{aligned}(\alpha, \beta) &= (\sqrt{A'}\cdot\cos\varphi, \sqrt{A'}\cdot\sin\varphi),\\ (\gamma, \delta) &= (\sqrt{|C'|}\cdot\sin\varphi, -\sqrt{|C'|}\cdot\cos\varphi).\end{aligned}$$

In § 4, we are going to find A', C', $\sin\varphi$, and $\cos\varphi$ in terms of the coefficients of (4).

1.5. Invariants

In fact we are going to find A', C', $\sin\varphi$, and $\cos\varphi$ in terms of A, B, and C in (4), under the assumption that (4) is the same as (3); for in this case, F' in (10) will be $-|\Delta|/4$, where Δ is as in (6). One can just check this by computation; we shall do the computation now, in the process of clarifying the changes of coordinates that we have just discussed. Again though, none of this is needed for carrying out the project that is our main subject.

We can write the left-hand side of (4) as a matrix product, as follows; see for example (Pettofrezzo, 1978, p. 104):

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = \begin{pmatrix} x & y & 1\end{pmatrix}\cdot M_3 \cdot \begin{pmatrix} x & y & 1\end{pmatrix}^{\mathrm{T}},$$

where

$$M_3 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M_2}} & D/2 \\ & & E/2 \\ \hline D/2 & E/2 & F \end{array}\right),$$

where

$$M_2 = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}.$$

The translation (9) and rotation (12) are given by

$$\begin{pmatrix} x & y & 1 \end{pmatrix} = \begin{pmatrix} x' & y' & 1 \end{pmatrix} \cdot \Lambda,$$

where Λ is respectively

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline e & f & 1 \end{array}\right), \qquad \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right),$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

In every case, $\det \Lambda = 1$ and $\det R = 1$. This shows the invariance of $\det M_3$ and $\det M_2$, noted for example in (Pamfilos, n.d., §§ 2–4) and (Vygodsky, 1975, pp. 99–102), under translation and rotation of the coordinate system. In the notation of (6), which we shall continue to use, we have

$$4 \det M_2 = \Delta. \tag{14}$$

As a special case of the invariance just mentioned, $\det M_3$ and $\det M_2$ stay the same, whether we compute them for (3) or for

$$(dx - cy)^2 \pm (bx - ay)^2 - (ad - bc)^2 = 0, \tag{15}$$

and whether we compute them for (4), (10), or (13). The determinants do *not* stay the same, if an equation is multiplied by a positive constant other than unity. However, if $\det M_2 \neq 0$, then $(\det M_3)^2/(\det M_2)^3$ does stay the same; it is thus an invariant, not only of an equation, but of the conic that it defines.

To accomplish the translation that gives us (10) in the first place, we require of the corresponding Λ that

$$\Lambda \cdot M_3 \cdot \Lambda^{\mathrm{T}} = \left(\begin{array}{c|c} M_2 & 0 \\ \hline 0 & F' \end{array}\right). \tag{16}$$

The conditions on e, f, and F' are

$$\begin{gathered} \begin{pmatrix} e & f \end{pmatrix} \cdot M_2 + \begin{pmatrix} D/2 & E/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \\ \det M_2 \cdot F' = \det M_3, \end{gathered} \tag{17}$$

and, as shown for example in (Pamfilos, n.d., §§ 10, 11), these conditions are satisfiable if $\det M_2 \neq 0$.

Such is the case when (10) is (15), multiplied out, under the nondegeneracy condition (1). In this case, computing

$$(dx - cy)^2 \pm (bx - ay)^2 = (d^2 \pm b^2)x^2 - 2(cd \pm ab)xy + (c^2 \pm a^2)y^2, \tag{18}$$

we can write (15) as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \cdot \begin{pmatrix} d^2 \pm b^2 & -(cd \pm ab) & 0 \\ -(cd \pm ab) & c^2 \pm a^2 & 0 \\ 0 & 0 & -(ad-bc)^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{19}$$

In the notation of (16) for the present case,

$$\det M_2 = \det \begin{pmatrix} d^2 \pm b^2 & -(cd \pm ab) \\ -(cd \pm ab) & c^2 \pm a^2 \end{pmatrix} = \pm(ad-bc)^2 = \mp F'. \tag{20}$$

Using (14), we can now write (15), and thus (19), in the form

$$Ax^2 + Bxy + Cy^2 - |\Delta|/4 = 0. \tag{21}$$

We shall use this in the general computations of § 4. Meanwhile, from (17) and (20),

$$\mp F' = \frac{F'^2}{\mp F'} = \frac{(\det M_3)^2}{(\det M_2)^3}, \tag{22}$$

which is the conic invariant that we mentioned; also, the ambiguous sign here is opposite the sign of $\det M_2$, and this sign (which is that of Δ) is a conic invariant.

When we rotate the coordinate system so as to eliminate the mixed term from (15), we get (13), where $\operatorname{diag}(A', C', F')$ is the diagonalization (by means of the appropriate matrix Λ) of the square matrix in (19). Thus, in particular, A' and C' are the eigenvalues of M_2. Another way to see this is shown in (Ayoub, 1993): when we multiply (11) by $B/4$ and replace $\tan\varphi$ by $2(\lambda - A)/B$, we obtain

$$(\lambda - A)(\lambda - C) - B^2/4 = 0,$$

which is the characteristic equation of M_2, and the roots are A' and C'.

In (13) then, by (22), we have

$$F' = \mp A'C'.$$

Assuming $A' > 0$, we can write (13) as

$$(sx)^2 \pm (ry)^2 = (rs)^2$$

for some positive r and s. Dividing by $(rs)^2$ gives us

$$\frac{x^2}{r^2} \pm \frac{y^2}{s^2} = 1, \tag{23}$$

the "standard" form of the equation of an ellipse or hyperbola, as in (Nelson, Folley, Borgman, 1949, pp. 107, 115) or (Vygodsky, 1975, pp. 59, 65).

2. Example: ellipse

2.1. Creation

In equation (3), if we let ± 1 be 1, and we plug in the values

$$(a, b) = (5, 1), \qquad (c, d) = (2, 4), \qquad (e, f) = (4, 1),$$

we obtain the equation

$$\big(4(x-4) - 2(y-1)\big)^2 + \big((x-4) - 5(y-1)\big)^2 = (5 \cdot 4 - 1 \cdot 2)^2. \tag{24}$$

This equation is satisfied by $(4, 1) \pm (5, 1)$ and $(4, 1) \pm (2, 4)$, which are A, A', B, and B' in Figure 3, where $(4, 1)$ itself is O. We can rewrite (24) as

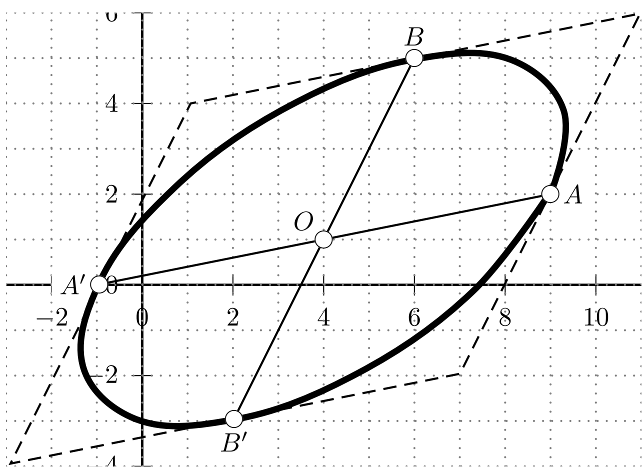

Figure 3: Ellipse, conjugate diameters, bounding parallelogram

$$2^2 \cdot (2x - y - 7)^2 + (x - 5y + 1)^2 = 18^2. \tag{25}$$

In passing from (24) to (25), we have not lost any information. Indeed, by equating the two squared linear terms to zero, we obtain the equations

$$x - 5y + 1 = 0, \qquad 2x - y - 7 = 0,$$

which define the lines AA' and BB' in Figure 3; these lines intersect at O. We are thus able to recover (24) from (25). We multiply out either equation to obtain

$$17x^2 - 26xy + 29y^2 - 110x + 46y - 127 = 0. \tag{26}$$

We have thus created for ourselves the problem of figuring out the curve defined by (26).

2.2. On diameters

In the terminology of Apollonius already mentioned in the Introduction, the lines AA' and BB' are **conjugate diameters** of the curve defined by (24), (25), and (26). The point O is the **center** of the curve (Apollonius of Perga, 1998, p. 36). This means in particular that all chords of the curve that are parallel to one of the two diameters are bisected by the other one. In more detail, we have the following.

1. For all nonzero t, the line AA' is equidistant from the parallel lines given by
$$x - 5y + 1 = t, \qquad\qquad x - 5y + 1 = -t.$$
2. For all nonzero s, the line BB' is equidistant from the parallel lines given by
$$2x - y - 7 = s, \qquad\qquad 2x - y - 7 = -s.$$
3. In case the intersection point of one of the first pair of parallels with one of the second pair of parallels lies on the curve defined by (24), we have
$$2^2s^2 + t^2 = 18^2,$$
and this condition depends only on $|s|$ and $|t|$.

If the conjugate diameters AA' and BB' were at right angles to one another, then they would be **axes** of the curve, and the curve itself would be an **ellipse,** if such a curve is by definition a curve that has a standard equation as in (23). The conjugate diameters are not in fact at right angles; still, we can sketch the curve as in Figure 3.

By one account (Vygodsky, 1975, pp. 58–9), an ellipse is a circle that has undergone *uniform compression* at right angles to a particular *axis of compression.* We are going to find such an axis for our curve.

By the account of Apollonius, suitably interpreted, our curve is already an ellipse, because it has conjugate diameters in the sense defined; Apollonius *proves* that every line through the center is a diameter that has a conjugate, and that in one case that conjugate is orthogonal. We are going to show this independently.

2.3. Solution

2.3.1. Completion of squares

After multiplying (26) by 17 for convenience and rearranging the terms, placing first the terms in x, we obtain

$$(17x)^2 - 2 \cdot 17 \cdot 13xy - 2 \cdot 17 \cdot 55x + 17 \cdot (29y^2 + 46y) - 17 \cdot 127 = 0. \tag{27}$$

Now we complete the squares. First, for the terms in x, we have

$$(17x)^2 - 2 \cdot 17 \cdot 13xy - 2 \cdot 17 \cdot 55x = (17x - 13y - 55)^2 - (13y + 55)^2.$$

We plug this into (27) and complete the square in the terms that are in y but *not* x:

$$\begin{aligned}
&-(13y+55)^2+17\cdot(29y^2+46y)\\
&\qquad\qquad\equiv(-169+493)y^2+(-1430+782)y-3025)\\
&\qquad\qquad\equiv 324y^2-648y-3025\\
&\qquad\qquad\equiv 18^2(y^2-2y)-3025\\
&\qquad\qquad\equiv 18^2(y-1)^2-(324+3025).
\end{aligned}$$

Since $324+3025=3349=17\cdot 197$ and

$$197+127=324=18^2,$$

remembering that we multiplied (26) by 17 to get (27), we can now write the former as

$$\frac{(17x-13y-55)^2}{17}+\frac{18(y-1)^2}{17}=18^2. \tag{28}$$

As from (25), so from (28) we can read off the equations of a pair of conjugate diameters. These equations are

$$y-1=0, \qquad\qquad 17x-13y-55=0.$$

The new conjugate diameters are still not orthogonal, but they still intersect at $(4,1)$, and thus we have recovered the center O. When we make the substitution

$$(x,y)=(x'+4,y'+1)$$

and then erase the primes, (28) becomes

$$\left(x\sqrt{17}-\frac{13y\sqrt{17}}{17}\right)^2+\left(\frac{18y\sqrt{17}}{17}\right)^2=18^2. \tag{29}$$

From this we extract the equations

$$y=0, \qquad\qquad 17x-13y=0$$

of a pair of conjugate diameters, along with their endpoints,

$$\left(\frac{\pm 18\sqrt{17}}{17},0\right), \qquad\qquad \left(\frac{\pm 13\sqrt{17}}{17},\pm\sqrt{17}\right),$$

which are about $\pm(4.37,0)$ and $\pm(3.15,4.12)$ and are the points C, C', D, and D' in Figure 4; they are where the conjugate diameters meet the curve, and we can also just read them from (29), as we read $\pm(a,b)$ and $\pm(c,d)$ from (15), because

$$\det\begin{pmatrix}18\sqrt{17}/17 & 0\\ 13\sqrt{17}/17 & \sqrt{17}\end{pmatrix}=18.$$

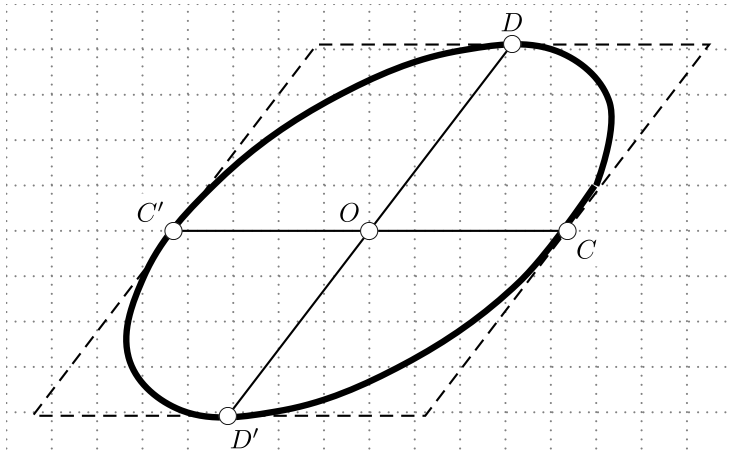

Figure 4: Ellipse with horizontal diameter

2.3.2. Intersection with circle

The circle that shares with our curve the horizontal diameter $C'C$ as in Figure 5 is given by

$$17x^2 + 17y^2 = 18^2. \tag{30}$$

We solve this simultaneously with

$$17x^2 - 26xy + 29y^2 = 18^2, \tag{31}$$

which is what (26) must be after our coordinate change. From our solution of (30) with (31), we obtain the second-degree homogeneous equation

$$17x^2 - 26xy + 29y^2 = 17x^2 + 17y^2,$$

which reduces successively to

$$\begin{gathered} 29y^2 - 26xy = 17y^2, \\ 12y^2 - 26xy = 0, \\ y(6y - 13x) = 0. \end{gathered}$$

This defines two lines through the center of the circle, meeting it and the original curve at four points, which must be the vertices of a rectangle, namely $CEC'E'$ in Figure 5. If we want to plot them to check our work, since

$$6^2 + 13^2 = 205, \tag{32}$$

the points E and E' are

$$\left(\frac{\pm 6 \cdot 18\sqrt{17}}{17\sqrt{205}}, \frac{\pm 13 \cdot 18\sqrt{17}}{17\sqrt{205}}\right),$$

which are about $\pm(1.83, 3.96)$; but we can plot these points also just by drawing the circle given by (30) and the line through O with slope $13/6$.

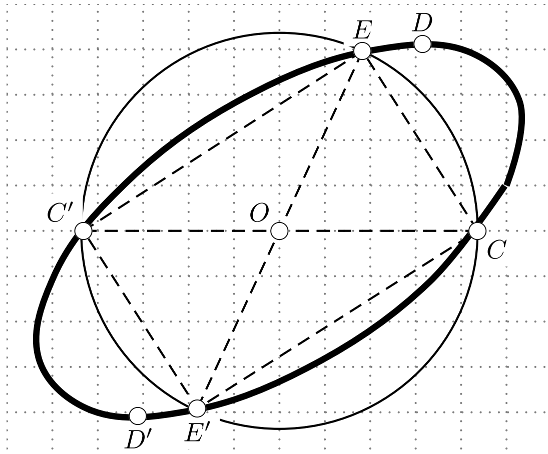

Figure 5: Ellipse and concentric circle

2.3.3. Identification of axes

For the sake of finding the axes of our ellipse, we can work with a circle of more convenient radius, noting the following in turn.

1. The lines CC' and EE' meet the circle that has center O and radius $\sqrt{205}$ at $\pm(\sqrt{205}, 0)$ and $\pm(6, 13)$.
2. The slopes of CE and $C'E$ are $13/(6 \mp \sqrt{205})$.
3. The product of these slopes is -1.

Being parallel to CC' and EE' respectively, the bisectors of angles $C'OE$ and COE are given accordingly by

$$13x = (6 \mp \sqrt{205})y. \tag{33}$$

Solving this simultaneously with (29) will yield the coordinates of F, F', G, and G' in Figure 6, and then we shall have nearly solved our problem, because, by Proposition 47 of Book II of the *Conics* of Apollonius, already mentioned (Apollonius of Perga, 1998, pp. 156–7), FF' and GG' are the axes of our ellipse.

Multiplying (29) for convenience by $17 \cdot 13^2$, we obtain

$$(17 \cdot 13x - 13^2 y)^2 + 13^2 \cdot 18^2 y^2 = 17 \cdot 13^2 \cdot 18^2.$$

Replacing $13x$ with $(6 \mp \sqrt{205})y$ as given by (33) yields

$$\left.\begin{aligned} 17 \cdot 13^2 \cdot 18^2 &= \Big(\big(17 \cdot (6 \mp \sqrt{205}) - 13^2\big)^2 + 13^2 \cdot 18^2\Big)y^2 \\ &= \big((67 \pm 17\sqrt{205})^2 + 13^2 \cdot 18^2\big)y^2 \\ &= (67^2 + 13^2 \cdot 18^2 + 17^2 \cdot 205 \pm 2 \cdot 17 \cdot 67\sqrt{205})y^2. \end{aligned}\right\} \tag{34}$$

Since, as it turns out,

$$67^2 + 13^2 \cdot 18^2 = 17^2 \cdot 205, \tag{35}$$

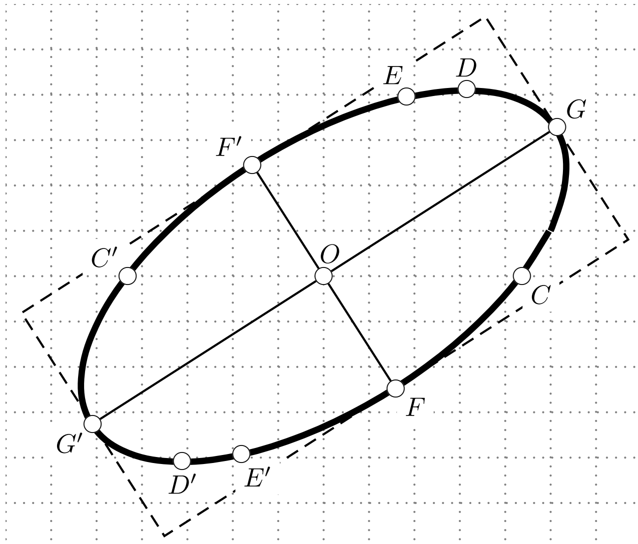

Figure 6: Ellipse and axes

we can rewrite (34) as

$$17 \cdot 13^2 \cdot 18^2 = (2 \cdot 17^2 \cdot 205 \pm 2 \cdot 17 \cdot 67\sqrt{205})y^2.$$

Eliminating the common factor 17, we obtain

$$13^2 \cdot 18^2 = 2\sqrt{205} \cdot (17\sqrt{205} \pm 67)y^2$$

and then, by (35) again,

$$y^2 = \frac{17\sqrt{205} \mp 67}{2\sqrt{205}}. \tag{36}$$

It may not have been obvious that the simplification of (34) that is facilitated by (35) would be possible. However, if one now started over with different points A, B, and O, following the example of the present computations, one would have a similar simplification. One can see this by sticking with the original literal constants in (4), as we shall do in § 4. Meanwhile, using numerical constants allows us to check our work by graphing.

We continue with that work. From (36), using (33), we obtain

$$x^2 = \frac{(\sqrt{205} \mp 6)^2(17\sqrt{205} \mp 67)}{13^2 \cdot 2\sqrt{205}}.$$

We reduce this, using (32), to

$$x^2 = \frac{(\sqrt{205} \mp 6)(17\sqrt{205} \mp 67)}{(\sqrt{205} \pm 6) \cdot 2\sqrt{205}}. \tag{37}$$

To simplify this, we can use

$$\begin{aligned} 17\sqrt{205} \mp 67 &= (\sqrt{205} \pm 6)(23 \mp \sqrt{205}), \\ 29\sqrt{205} \mp 243 &= (\sqrt{205} \mp 6)(23 \mp \sqrt{205}). \end{aligned} \tag{38}$$

Now (37) becomes

$$x^2 = \frac{29\sqrt{205} \mp 243}{2\sqrt{205}}. \tag{39}$$

Keeping in mind that $6 < \sqrt{205}$ in (33), we obtain the coordinates of F, F', G, and G' from (39) and (36) as

$$\begin{aligned} &\left(\pm\sqrt{\frac{29\sqrt{205} - 343}{2\sqrt{205}}}, \mp\sqrt{\frac{17\sqrt{205} - 67}{2\sqrt{205}}}\right), \\ &\left(\pm\sqrt{\frac{29\sqrt{205} + 343}{2\sqrt{205}}}, \pm\sqrt{\frac{17\sqrt{205} + 67}{2\sqrt{205}}}\right). \end{aligned} \tag{40}$$

These are close to $\pm(1.59, -2.48)$ and $\pm(5.15, 3.29)$, as in Figure 6; in particular, they seem to lie on the curve sketched originally in Figure 3. In any case, abbreviating the actual points F, F', G, and G' as $\pm(\alpha, \beta)$ and $\pm(\gamma, \delta)$, we assert that our curve is given by

$$(\delta x - \gamma y)^2 + (\beta x - \alpha y)^2 = (\alpha\delta - \beta\gamma)^2 \tag{41}$$

and therefore by (5) in the original coordinate system.

2.4. Confirmation

Our assertion is theoretically correct by Apollonius: if our curve has axes, then in principle we have found them as the segments of the lines given by (33) that have the endpoints in (40). If we made any mistakes in computing the exact coordinates of any of the points that we have plotted, the errors are likely to be big enough to be obvious on our paper.

Without relying on Apollonius, we can confirm that (41) is the correct equation by multiplying out to see that what we arrive at is (31). The work is not that hard, if we use the factorizations in (38). In particular, first of all,

$$\begin{aligned} \alpha\delta &= \frac{\sqrt{(29\sqrt{205} - 343)(17\sqrt{205} + 67)}}{2\sqrt{205}} \\ &= \frac{\sqrt{(\sqrt{205} - 6)(23 - \sqrt{205})(\sqrt{205} - 6)(23 + \sqrt{205})}}{2\sqrt{205}} \\ &= \frac{(\sqrt{205} - 6)\sqrt{23^2 - 205}}{2\sqrt{205}} = \frac{(\sqrt{205} - 6) \cdot 18}{2\sqrt{205}}. \end{aligned}$$

We can get $-\beta\gamma$ from $\alpha\delta$ by changing $\sqrt{205}$ to $-\sqrt{205}$; thus

$$-\beta\gamma = \frac{(\sqrt{205} + 6) \cdot 18}{2\sqrt{205}}.$$

Now the right-hand side of (41) is that of (31), namely 18^2. The left-hand sides of those equations agree at (α, β) and (γ, δ), if our computations have been correct. Since those left-hand sides obviously agree at a third point, namely $(0, 0)$, they must agree everywhere, since also the left-hand side of (41) does multiply out to $Ax^2 + Bxy + Cy^2$ for some three coefficients A, B, and C.

Theoretically speaking, we can confirm Apollonius as follows. After multiplying (31) by 29, we can complete first the square in the terms in y to obtain

$$(29y - 13x)^2 + (18x)^2 = 29 \cdot 18^2.$$

This shows that the line $x = 0$ is a diameter, and its conjugate is $29y = 13x$. In the same way, for any constant s, the line $y = sx$ is a diameter with a conjugate; for letting $t = y - sx$, replacing y in (31) with $t + sx$, we get

$$(17 - 26s + 29s^2)x^2 - (26 - 2 \cdot 29)xt + 19t^2 = 18^2;$$

completing the square in the terms in x yields the diameter that is conjugate to $t = 0$, that is, to $y = sx$. This requires $17 - 26s + 29s^2$ not to be 0, and it never is, since $4 \cdot 17 \cdot 29 - 26^2$, which is Δ for (31), is positive. The same argument works for any ellipse.

3. Example: hyperbola

Change the sign on one of the squares in (24), we get the two possibilities

$$\pm\bigl(4(x-4) - 2(y-1)\bigr)^2 \mp \bigl((x-4) - 5(y-1)\bigr)^2 = (5 \cdot 4 - 1 \cdot 2)^2,$$

which multiply out, respectively, to

$$\begin{gathered} 15x^2 - 6xy - 21y^2 - 114x + 66y - 129 = 0, \\ -15x^2 + 6xy + 21y^2 + 114x - 66y - 519 = 0 \end{gathered}$$

and define the solid and dashed curves shown in Figure 7. The curves are conjugate hyperbolas that share, with the ellipse of § 2, the conjugate diameters $A'A$ and $B'B$. Performing on the first hyperbola the same procedure that we used for the ellipse, we obtain the points shown in Figures 8 and 9. The endpoints of the diameters DD' and GG', as well as of BB', lie on the other hyperbola; however, we do not actually need BB' and DD', in order to find the axes. We find F and F' of our chosen hyperbola, as before we found them on the ellipse; they are

$$\left(4 \pm \sqrt{\frac{3(43 + 7\sqrt{37})}{2\sqrt{37}}}, 1 \mp \sqrt{\frac{3(31 - 5\sqrt{37})}{2\sqrt{37}}}\right).$$

Then G, and G' are

$$\left(4 \pm \sqrt{\frac{3(43 - 7\sqrt{37})}{2\sqrt{37}}}, 1 \pm \sqrt{\frac{3(31 + 5\sqrt{37})}{2\sqrt{37}}}\right).$$

If we worked instead with the second equation, we would interchange the roles of x and y, finding first the conjugate diameters, one of which is vertical, and then intersecting with a circle that shares this diameter.

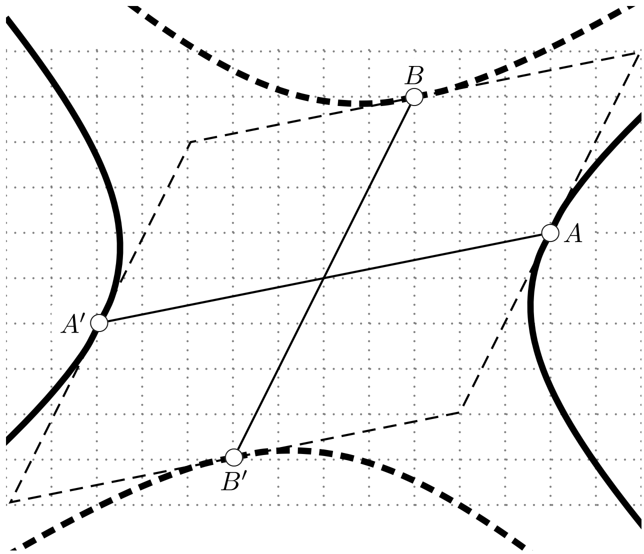

Figure 7: Conjugate hyperbolas and shared axes

4. General solution

We now work out the curve defined by (21), namely $Ax^2+Bxy+Cy^2 = |\Delta|/4$. As we noted, this is the equation that we obtain from (4) after translating the origin of coordinates to the center of the conic, provided also that (4) is a rewriting of (3).

THEOREM 1
The curve defined by (21)*, under the assumption* $A > 0$ *and* $\Delta \neq 0$*, is an ellipse or hyperbola, defined also by*

$$(\delta x - \gamma y)^2 \pm (\beta x - \alpha y)^2 = (\alpha\delta - \beta\gamma)^2, \tag{42}$$

where the ambiguous sign is that of Δ*, and, under the abbreviation*

$$\vartheta = \sqrt{B^2 + (A-C)^2}, \tag{43}$$

the parameters α*,* β*,* γ*, and* δ *are well defined by*

$$\begin{aligned}\alpha &= \frac{1}{2}\sqrt{\frac{(\vartheta + A - C)\cdot|\vartheta - A - C|}{\vartheta}},\\ \beta &= \frac{B}{2\,|B|}\sqrt{\frac{(\vartheta - A + C)\cdot|\vartheta - A - C|}{\vartheta}},\end{aligned} \tag{44}$$

and

$$\begin{aligned}\gamma &= \frac{1}{2}\sqrt{\frac{(\vartheta - A + C)(\vartheta + A + C)}{\vartheta}},\\ \delta &= \frac{-B}{2\,|B|}\sqrt{\frac{(\vartheta + A - C)(\vartheta + A + C)}{\vartheta}}.\end{aligned} \tag{45}$$

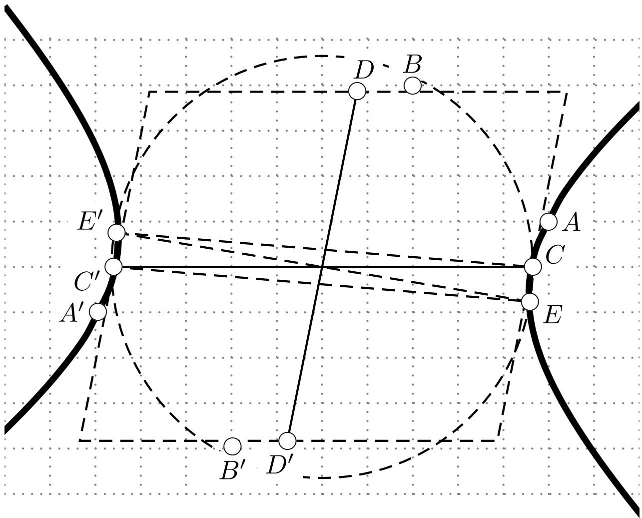

Figure 8: Conjugate hyperbolas and shared axes

Then also (α, β) *and* (γ, δ) *are orthogonal to one another, so that* $\pm(\alpha, \beta)$ *and* (γ, δ) *are endpoints of the axes of the conic.*

Proof. One can prove the claim by multiplying out (42) with the given values of α, β, γ and δ to get (21). Instead we shall *derive* those values. We want them to be such that

$$\alpha\gamma + \beta\delta = 0$$

and the equation

$$(\delta x - \gamma y)^2 \pm (\beta x - \alpha y)^2 = Ax^2 + Bxy + Cy^2$$

is an identity. One can use these conditions directly to solve for (α, β) and (γ, δ). However, this algebraic approach does not seem to save any of the labor of the following more geometric approach, generalizing what we did in the previous sections.

If we multiply (21) by $4A$, we can then complete squares to obtain

$$(2Ax + By)^2 + \Delta y^2 = A\,|\Delta|\,.$$

This defines the same curve, since we assume $A \neq 0$. Thus, if $\Delta < 0$, then (21) must define an hyperbola. In this case, there is another hyperbola, given by

$$Ax^2 + Bxy + Cy^2 = -\,|\Delta|\,/4. \tag{46}$$

The new hyperbola is conjugate to the first, in the sense of sharing the conjugate diameters given by $2Ax + By = 0$ and $y = 0$ respectively, as well as being tangent, at points lying on one of those diameters, to the ellipse given by

$$(2Ax + By)^2 - \Delta y^2 = A \cdot |\Delta|\,.$$

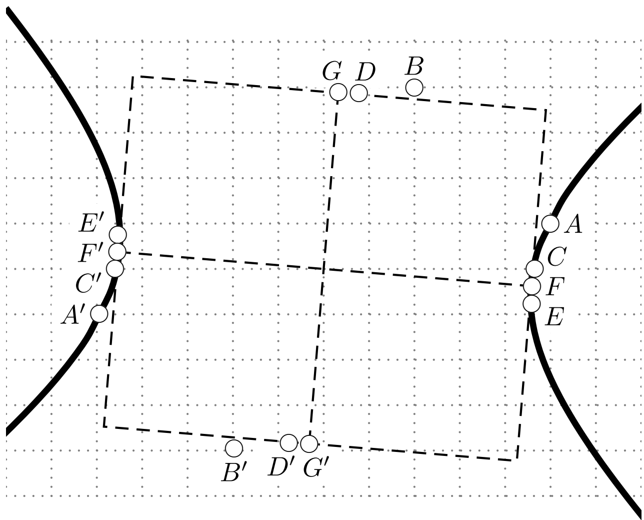

Figure 9: Conjugate hyperbolas and shared axes

Regardless of whether Δ is positive or negative, we cut our ellipse, or hyperbola, or its conjugate, with the circle given by

$$Ax^2 + Ay^2 = |\Delta|/4.$$

The points of intersection lie on the two lines whose union is given by

$$Bxy + Cy^2 = Ay^2.$$

Thus the lines themselves are given respectively by

$$y = 0, \qquad Bx = (A - C)y. \tag{47}$$

These two lines meet the circle that has center O and radius ϑ at the points

$$\pm(\vartheta, 0), \qquad \pm(A - C, B).$$

Then the bisectors of the angles formed by the two lines defined by (47) have slopes m and m', defined by

$$\begin{aligned} m &= \frac{B}{\vartheta + A - C} = \frac{\vartheta - A + C}{B}, \\ m' &= \frac{-B}{\vartheta - A + C} = \frac{\vartheta + A - C}{-B}. \end{aligned} \tag{48}$$

Then $mm' = -1$ and

$$m^2 = \frac{m}{-m'} = \frac{\vartheta - A + C}{\vartheta + A - C}, \tag{49}$$

$$m'^2 = \frac{m'}{-m} = \frac{\vartheta + A - C}{\vartheta - A + C}. \tag{50}$$

We now solve (21) simultaneously with

$$y = mx. \tag{51}$$

The solution is that of (51) with

$$(A + mB + m^2C)x^2 = |\Delta|/4. \tag{52}$$

From (49),

$$\begin{aligned}
A + mB + m^2C &= A + (\vartheta - A + C) + \frac{(\vartheta - A + C)C}{\vartheta + A - C}\\
&= \frac{(\vartheta + A - C)(\vartheta + C) + (\vartheta - A + C)C}{\vartheta + A - C}\\
&= \frac{\vartheta^2 + (A - C)\vartheta + 2\vartheta C}{\vartheta + A - C}\\
&= \frac{\vartheta^2 + (A + C)\vartheta}{\vartheta + A - C} = \frac{(\vartheta + A + C)\vartheta}{\vartheta + A - C}.
\end{aligned} \tag{53}$$

By (43),

$$0 < \vartheta - |A - C| \leqslant \vartheta \leqslant \vartheta + |A - C|.$$

Moreover,

$$\Delta = (A + C)^2 - \vartheta^2 = (A + C + \vartheta)(A + C - \vartheta),$$

and this gives us the following cases.

- If $\Delta > 0$, then, both A and C are positive, and so both factors $A + C \pm \vartheta$ are positive.
- if $\Delta < 0$, then just one of those factors is positive, namely $A + C + \vartheta$.

Thus

$$|\Delta| = (A + C + \vartheta)\,|A + C - \vartheta|,$$

and so (52) is equivalent to

$$x^2 = \frac{(\vartheta + A - C) \cdot |\vartheta - A - C|}{4\vartheta},$$

and the right-hand side of this is positive. The positive solution of this last equation is α as in (44). Also, m has the sign of B by (48), so that

$$m = \frac{B}{|B|}\sqrt{m^2},$$

and therefore

$$m\alpha = \frac{B}{2\,|B|}\sqrt{m^2 \cdot \frac{(\vartheta + A - C) \cdot |\vartheta - A - C|}{\vartheta}} = \beta$$

by (49). Thus (α, β) satisfies (21) and (51).

In the same way, m' has the opposite sign of B, and so, with γ and δ as in (45),

$$m'\gamma = \delta.$$

As for γ itself, the pattern of (53) gives us

$$A + m'B + m'^2C = \frac{(A + C - \vartheta)\vartheta}{\vartheta - A + C}.$$

Because of this, the equation

$$(A + m'B + m'^2C)x^2 = \Delta/4$$

is equivalent to

$$x^2 = \frac{(A + C + \vartheta)(\vartheta - A + C)}{4\vartheta};$$

the right-hand side is positive, and γ solves the equation. Thus (γ, δ) is on

- the ellipse that (21) defines, if $\Delta > 0$;
- the conjugate hyperbola that (46) defines, if $\Delta < 0$.

In any case, we have derived the theorem.

Thanks to the referees for reading and helping to show what could be clarified.

References

Apollonius of Perga: 1998, *Conics. Books I–III*, revised edn, Green Lion Press, Santa Fe, NM. Translated and with a note and an appendix by R. Catesby Taliaferro, with a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue, edited by Densmore.

Ayoub, A.B.: 1993, The central conic sections revisited, *Mathematics Magazine* **66**(5), 322–5. `www.jstor.org/stable/2690513`, accessed February 31, 2015.

Descartes, R.: 1954, *The Geometry of René Descartes*, Dover Publications, New York. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

Descartes, R.: 1985, *The Philosophical Writings of Descartes*, Vol. I, Cambridge University Press. Translated by John Cottingham, Robert Stoothoff, and Dugald Murdoch.

Nelson, A.L., Folley, K.W., Borgman, W.M.: 1949, *Analytic Geometry*, The Ronald Press Company, New York.

Pamfilos, P.: n.d., The quadratic equation in the plane, `http://users.math.uoc.gr/~pamfilos/eGallery/problems/Quadratic_Equation.pdf`, accessed December 30, 2021. “This is a short review of the corresponding chapter in an analytic geometry lesson.” 17 pages, 23 sections, 11 figures.

Pettofrezzo, A.J.: 1978, *Matrices and Transformations*, Dover, New York. Original published 1966.

Pirsig, R.M.: 1961, Unorthodox teaching of writing to college freshman, `venturearete.org/ResearchProjects/ProfessorGurr/Documents/FormativePrecursorOfZMM`, accessed August 21, 2020. Letter to Professor Edith Buchanan, Department of English Language and Literature, University of New Mexico, Albuquerque.

Pirsig, R.M.: 1974, *Zen and the Art of Motorcycle Maintenance*, Bantam, Toronto. New Age Edition 1981.

Thomas, I. (ed.): 1951, *Selections Illustrating the History of Greek Mathematics. Vol. II. From Aristarchus to Pappus*, nr 362 in *Loeb Classical Library*, Harvard University Press, Cambridge, Mass. With an English translation by the editor.

Vygodsky, M.: 1975, *Mathematical Handbook: Higher Mathematics*, Mir Publishers, Moscow. Translated from the Russian by George Yankovsky. Fifth printing 1987.

Weeks, A.W., Adkins, J.B.: 1971, *Second Course in Algebra With Trigonometry*, Ginn and Company, Lexington MA.

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