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Fixed point theorems of contraction type mappings and some of their applications in fractal geometry*

Abstract. The paper contains several theorems about the Browder type contraction fixed points and some of their applications.

1. Introduction

The paper demonstrates several theorems about fixed points of so-called ψ-contractive mappings, that is mappings of type „$d(f(x), f(y)) \leqslant \psi(d(x,y))$", where a function ψ meets certain conditions. It also contains the application of these theorems to approximate the golden number, thus complementing works (Barcz, 2019) and (Barcz, 2020). Some generalization of Hutchinson's fixed point theorem is given further. The constructions of two new fractals is also presented, which are:

(i) a Sierpinski type carpet,

(ii) a graph of a continuous function nowhere-differentiable (other than in (Katsuura, 1991)).

These constructions use the system of iterated functions, which in (i) consists of several contractions with the same contraction constant, and in (ii) not all contraction constants are equal. Theorem 10, which is a version of Browder's theorem, was helpful in the construction of fractal (i), and Hutchinson's theorem was helpful in the construction of fractal (ii).

Let us add that fractals such as those described in the work have a number of connections with other mathematical objects. For example, Sierpinski's triangle (which is also presented in this paper) has surprising connections with objects such as Pascal's triangle, chaos game, L-systems, some cellular automata as game

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of life described by John Conway in 1970. Topics related to fractals were initiated by Polish mathematicians such Sierpinski, Banach and Ulam. Let us also add that in many areas fractal applications can be found, for example as a method of data compression and the principle of construction of cell phone antennas. In view of the above connections, the educational significance of many themes of the (fashionable and promising) fractal theory is important. Therefore, it is worth popularizing fractals and thus getting closer to modern mathematics lessons, which will also be associated with fractal geometry objects.

For students interested in chaos and fractals, the item (Peitgen, Jurgens, Saupe, 1992) and publications on the subject matter may be useful.

2. Fixed points of generalized contractions and examples of their application

Definition 1
A golden section of the segment of length d is called a division into smaller sections of lengths x and $d-x$, in which

$$\frac{d}{x} = \frac{x}{d-x}.$$

Definition 2
For a given rectangle with side lengths in the ratio $1 : x$, we will call the golden proportion of the only ratio $1 : \varphi$ at which the original rectangle can be divided into a square and a new rectangle which has the same ratio of sides $1 : \varphi$.

Definition 3
The golden rectangle is called a rectangle in which the ratio of the lenght of its sides is $1 : \varphi$.

Definition 4
Fibonacci sequence is a sequence defined recursively as follows:

$$f_1 = f_2 = 1, f_{n+1} = f_{n-1} + f_n, n \geq 2$$

(sometimes formally accepted $f_0 = 0$ and then the recursive formula is valid for $n \geq 1$).

Definition 5
Fibonacci numbers are called consecutive terms of the sequence (f_n).

Definition 6
A mapping f of a metric space (X, d) into itself satisfying the condition

$$d\left(f(x), f(x')\right) \leqslant \psi\left(d(x, x')\right) \text{ for all } x, x' \in X$$

with the function $\psi : \langle 0, \infty) \to \langle 0, \infty)$ which is non-decreasing, right continuous and such that $\psi(t) < t$ for each $t > 0$ we call a Browder contraction.

Definition 7
Let (X, d) be a metric space and $f : X \to X$. We say that a mapping f is ψ-contractive if it meets the condition

$$d\left(f(x), f(x')\right) \leqslant \psi\left(d(x, x')\right) \text{ for all } x, x' \in X,$$

where $\psi : \langle 0, \infty) \to \langle 0, \infty)$ is any function such that

(i) ψ is non-decreasing and right-continuous,

(ii) $\psi^n(t) \to 0$ for each $t > 0$.

Lemma 1
(see (Barcz, 1983)) *Let $\psi : \langle 0, \infty) \to \langle 0, \infty)$ be a non-decreasing function such that $\psi^n(t) \to 0$ for each $t > 0$. Then $\psi(t) < t$ for each $t > 0$.*

Lemma 2
Let $\psi : \langle 0, \infty) \to \langle 0, \infty)$ be a non-decreasing and right continuous function such that $\psi(t) < t$ for each $t > 0$. Then $\psi^n(t) \to 0$ for each $t > 0$.

Proof. From $\psi(t) < t$ for fixed $t > 0$ we have $\psi^n(t) \leqslant \psi^{n-1}(t) \leqslant \cdots \leqslant \psi(t) < t$. So we have a non-increasing sequence of non-negative numbers $\psi^n(t)$, therefore convergent to $l \geqslant 0$. Assumption $l = \lim_{n\to\infty} \psi^n(t) > 0$ leads to a contradiction: $l > \psi(l) = \psi\left(\lim_{n\to\infty} \psi^n(t)\right) = \lim_{n\to\infty} \psi^{n+1}(t) = l$.

On the basis of Lemmas 1 and 2 we get the following

Fact 1
If f is ψ-contractive on the metric space (X, d), then f is a Browder contraction on this space, and conversely.

Theorem 1
(Hillam, 2018) *Let $x \in \langle 0, 1\rangle$ and a continuous function $T : \langle 0, 1\rangle \to \langle 0, 1\rangle$ be given. The iterative sequence $\{x_n = T^n x\}$ converges to a fixed point of the mapping T if and only if*

$$\lim_{n\to\infty} |T^{n+1}x - T^n x| = 0.$$

It turns out that this result does not transfer to the higher-dimensional cases (compare J. Górnicki, Okruchy matematyki, PWN, Warszawa, 2009, p. 187). Theorem 1 will be useful in proof of Theorem 3.

Theorem 2
(Edelstein, 1962) *Let X be a compact metric space and let $f : X \to X$ be a contractive mapping, that is $d\left(f(x), f(y)\right) < d(x, y)$ for all $x \neq y$ in X. Then f has a unique fixed point. Further, for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point.*

Theorem 3
Let $f : \langle a, b\rangle \to \langle a, b\rangle$ be ψ-contractive. Then f has a unique fixed point u, and $f^n(x_0) \to u$ for each $x_0 \in \langle a, b\rangle$.

Proof. Since f is a continuous mapping, so the function $F : \langle a, b\rangle \to \mathbb{R}$ given by the formula $F(x) = f(x) - x$ is a continuous function, moreover, it satisfies inequalities $F(a) \geqslant 0,\ F(b) \leqslant 0$. Therefore on the basis of the Darboux property F has a point $u \in \langle a, b\rangle$ such that $F(u) = 0$. It means that $f(u) = u$. That u is the unique fixed point results from the following reasoning: for if $u = f(u) \neq f(v) = v$ we get a contradiction

$$|u - v| = |f(u) - f(v)| \leqslant \psi\left(|u - v|\right) < |u - v|.$$

Now let us take any $x_0 \in \langle a, b\rangle$ and let us create a sequence $(x_n),\ x_n = f(x_{n-1})$, $n = 1, 2, \ldots$ Then

$$|f^{n+1}(x_0) - f^n(x_0)| \leqslant \psi\left(|f^n(x_0) - f^{n-1}(x_0)|\right) \leqslant \cdots \leqslant \psi^n\left(|f(x_0) - x_0|\right) \to 0.$$

Therefore on the basis of Theorem 1 we get a convergence $f^n(x_0) \to u$.

Remark 1
From Theorem 3 it follows Banach Contraction Principle for the space $X = \langle a, b\rangle$ in the case when $\psi(t) = qt,\ t \geqslant 0$, where $q < 1$ is a contraction constant.

Corollary 1
Let a mapping $f : D(x_0, r) \to \mathbb{R}$ be a ψ-contractive mapping, where $D(x_0, r) = \langle x_0 - r, x_0 + r\rangle$. If $|f(x_0) - x_0| \leqslant r - \psi(r)$, then f has a unique fixed point u, and $f^n(x) \to u$ for each $x \in D(x_0, r)$.

Proof. Let x be any point in $D(x_0, r)$, i.e. $|x - x_0| \leqslant r$. Then we get the inequality $|f(x) - f(x_0)| \leqslant \psi\left(|x - x_0|\right)$, so

$$|f(x) - x_0| \leqslant |f(x) - f(x_0)| + |f(x_0) - x_0| \leqslant \psi\left(|x - x_0|\right) + |f(x_0) - x_0| \leqslant$$

$$\leqslant \psi\left(|x - x_0|\right) + r - \psi(r) \leqslant \psi(r) + r - \psi(r) = r,$$

which means that $f : D(x_0, r) \to D(x_0, r)$. The conclusion follows from Theorem 3.

Remark 2
Because $\psi(t) < t$, $t > 0$ for a ψ-contractive mapping $f : \langle a, b\rangle \to \langle a, b\rangle$, so $|f(x) - f(x')| < |x - x'|$ for $x \neq x'$ in $\langle a, b\rangle$, and we also get Theorem 3 from Edelstein's fixed point theorem (Theorem 2).

Example 1
Because $f(x) = 1 + \frac{1}{x}$ defined on the set $\langle 1, 2\rangle$ is a contractive mapping, so by Theorem 2 it has a unique fixed point $u = 1 + \frac{1}{u}$ in $\langle 1, 2\rangle$, therefore $u = \varphi$. Moreover, for $1 = \frac{f_2}{f_1}$ and $2 = \frac{f_3}{f_2}$ which are ends of $\langle 1, 2\rangle$ we get $\lim_{n\to\infty} f^n(1) = \lim_{n\to\infty} f^n(2) = \varphi$. Note that the last equality can be obtained from Banach Contraction Principle (which follows here from Theorem 3) considering that $f(x) = 1 + \frac{1}{x}$ is a contraction on the interval $\left\langle \frac{3}{2}, 2\right\rangle$.

Theorem 4

Let $f : \langle a, b\rangle \to \langle a, b\rangle$ be a map such that $f^N : \langle a, b\rangle \to \langle a, b\rangle$ is ψ-contractive for some $N > 1$. Then f has a unique fixed point u, and the sequence of iterates $f^n(x) \to u$ for each $x \in \langle a, b\rangle$.

Proof. The first part of the proof concerning the existence of exactly one fixed point $u = f(u)$ has already been presented in the paper (Barcz, 2020). Below, in order to obtain a complete proof, we present this part with an added part showing the convergence $f^n(x) \to u$ $(x \in \langle a, b\rangle)$.

Based on Theorem 3 f^N has a unique fixed point $u = f^N(u)$. However $f^N(f(u)) = f\left(f^N(u)\right) = f(u)$, therefore $f(u)$ is also a fixed point of f^N. Because the fixed point of f^N is only one, so $f(u) = u$. If for another point $v = f(v)$, then from $f^n(v) = v$, $n = 1, 2, \dots$ we have $f^N(v) = v$, so $v = u$.

The convergence $f^n(x) \to u$ for each $x \in \langle a, b\rangle$ is not too difficult to prove. We will present a sketch of the proof of this fact.
Consider the sequence $\left(f^k(x)\right), k = 1, 2, \dots$ We can choose subsequence $f^{kN+i}(x)$ for $i = 0, 1, \dots, N-1$ from it. For each such i we have $f^{kN+i}(x) = f^{kN}\left(f^i(x)\right)$. Since $u = f^N(u)$, so subsequences $\left(f^{kN}(x)\right), \left(f^{kN+1}(x)\right), \dots, \left(f^{kN+N-1}(x)\right)$ converge to u. It can be shown that $f^n(x) \to u$.

Example 2

The mapping $f : \langle 1, 2\rangle \to \langle 1, 2\rangle$, $f(x) = 1 + \frac{1}{x}$ is not a contraction on $\langle 1, 2\rangle$, but its second iteration f^2 is a contraction (with the constant $q = \frac{1}{4}$). Therefore f^2 has only one fixed point $u = f^2(u) = 1 + \frac{u}{u+1}$ in $\langle 1, 2\rangle$, i.e. $u = \varphi$. At the same time $u = \varphi$ is the unique fixed point for f, and $f^n(x_0) \to u = \varphi$ for each $x_0 \in \langle 1, 2\rangle$. So for any $x_0 = \frac{f_{k+1}}{f_k}$, where k is a fixed natural number we have $x_0 \in \langle 1, 2\rangle$ and $f^n(x_0) \to \varphi$.

Theorem 5

Let (X, d) be a complete metric space and let $f : X \to X$ be a map such that $d(f(x), f(x')) \leqslant \psi(d(x, x'))$ for all $x, x' \in X$, where $\psi : \langle 0, \infty) \to \langle 0, \infty)$ is any function satisfying the condition (i), (ii) from Definition 7 (i.e. f is ψ-contractive) and

$$(iii)\quad \psi(t) = \psi(1) \cdot t,\ \psi(t + t') \leqslant \psi(t) + \psi(t') \text{ for all } t, t' \in \langle 0, \infty).$$

Then f has a unique fixed point $u \in X$.

Proof. I (non-constructive) Let $a = \inf\{d(x, f(x)); x \in X\}$. We prove that $a = 0$. Let $a > 0$. For $\varepsilon > 0$ we choose $x \in X$ such that $d(x, f(x)) \leqslant a + \varepsilon$. Then

$$a \leqslant d\left(f(x), f^2(x)\right) \leqslant \psi\left(d(x, f(x)\right) \leqslant \psi(a + \varepsilon) < a + \varepsilon,$$

which leads to a contradiction as $\varepsilon \to 0$. Therefore $a = 0$. By the Cantor's theorem we will show that this a is achieved. For this, we define the sets

$$D_n = \{x \in X; d(x, f(x)) \leqslant \frac{1}{n}\},\ n = 1, 2, \dots$$

They are non-empty and closed $D_1 \supset D_2 \supset D_3 \supset \dots$ Moreover, for $x, y \in D_n$ we have

$$d(x,y) \leqslant d\left(x, f(x)\right) + d\left(f(x), f(y)\right) + d\left(f(y), y\right) \leqslant \frac{2}{n} + \psi\left(d(x,y)\right),$$

and hence $d(x,y) \leqslant \frac{2}{n} + \psi(1) \cdot d(x,y)$ (from (*iii*)). So we have $(1-\psi(1))\, d(x,y) \leqslant \frac{2}{n}$, therefore $d(x,y) \leqslant \frac{\frac{2}{u}}{1-\psi(1)}$. It follows that $\operatorname{diam}(D_n) \to 0$. Based on Cantor's theorem we get $\bigcap_{n \geqslant 1} D_n = \{u\}$, and hence $f(u) = u$.

Proof. II (constructive) We choose any point $x_0 \in X$ and we create a sequence $(x_n),\ x_n = f(x_{n-1}),\ n = 1, 2, \dots$ Then

$$d\left(x_{n+1}, x_n\right) \leqslant d\left(f^{n+1}(x_0), f^n(x_0)\right) \leqslant \psi\Big(d\left(f^n(x_0), f^{n-1}(x_0)\right)\Big) \leqslant \dots \leqslant$$

$$\leqslant \psi^n\Big(d\left(f(x_0), x_0\right)\Big) \to 0 \text{ as } n \to \infty.$$

Because

$$d(x_{n+p}, x_n) \leqslant d(x_{n+p}, x_{n+p+1}) + d(x_{n+p+1}, x_{n+1}) + d(x_{n+1}, x_n) \leqslant$$

$$\leqslant d(x_{n+p}, x_{n+p+1}) + \psi\left(d(x_{n+p}, x_n)\right) + d(x_{n+1}, x_n) \ (p \in \mathbb{N}),$$

so we have

$$d(x_{n+p}, x_n) \leqslant \psi(1) \cdot d(x_{n+p}, x_n) + d(x_{n+p}, x_{n+p+1}) + d(x_{n+1}, x_n),$$

and hence

$$d(x_{n+p}, x_n) \leqslant \frac{1}{1-\psi(1)} \ \left(d(x_{n+p+1}, x_{n+p}) + d(x_{n+1}, x_n)\right) \to 0 \text{ as } n \to \infty.$$

So (x_n) is a Cauchy sequence. Since the space X is complete, there is a point $u \in X$ such that $u = \lim_{n\to\infty} x_n$. Given the continuity of f we have

$$f(u) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

There is exactly one such fixed point. If $u = f(u) \neq f(v) = v$, we get a contradiction:

$$0 < d(u,v) = d\left(f(u), f(v)\right) \leqslant \psi\left(d(u,v)\right) < d(u,v).$$

Remark 3

Additionally, from $d\left(f^n(x_0), u\right) \leqslant \psi^n\left(d(x_0), u)\right) \to 0$ *we obtain more than the thesis from Theorem 5. Namely, for every* $x_0 \in X$ *the iteration sequence* $(x_n),\ x_n = f^n(x_0)$ *converges to a unique fixed point* u *of* f*. Therefore, this fixed point may be given as a limit of an iterative sequence with any (starting) point* x_0*, i.e.* $u = \lim_{n\to\infty} f^n(x_0)$*.*

Let us add that it is easy to see that the conditions (i) - (iii) of Theorem 5 are satisfied when $\psi(t) = qt,\ q < 1, t \geqslant 0$*.*

Moreover, Theorem 5 is also true without assumption (iii); in this case we have Browder's fixed point theorem which follows from Matkowski's fixed point theorem (see (Dugundji, Granas, 1982)).

3. Fixed point theorem of the Hutchinson type operator determinated by the system of iterated ψ-contractive functions with its application to the construction of fractals

Let $K(X)$ be a family of non-empty and compact subsets of a metric space (X,d). In the set $K(X)$ we define the metric using the set

$$A_\varepsilon = \{x \in X; d(a,x) \leqslant \varepsilon \text{ for some } a \in A\},$$

A_ε is called an ε-environment of the set $A \subset X$.

It can be shown that the function $d_H : K(X) \times K(X) \to \langle o, \infty)$ given by the formula

$$d_H(A,B) = \inf\{\varepsilon \geqslant 0; A \subset B_\varepsilon \wedge B \subset A_\varepsilon\}$$

is metric. We call it the Hausdorff metric on the set $K(X)$. If (X,d) is a complete metric space, then $(K(X), d_H)$ is a complete metric space.

Let $F_i : X \to X,\ i = 1, \ldots, k$ are functions and let a mapping $F : K(X) \to K(X)$ be given by the formula

$$F(A) = F_1(A) \cup \cdots \cup F_k(A) \text{ for } A \in K(X). \tag{$*$}$$

Theorem 6

If all functions $F_i : X \to X,\ i = 1, \ldots, k$ are ψ-contractive for the same function $\psi : \langle o, \infty) \to \langle o, \infty)$, then the mapping $F : K(X) \to K(X)$ given by the formula $(*)$ is ψ-contractive with respect to the Hausdorff metric.

Proof. Since all functions F_i $(i = 1, \ldots, k)$ are ψ-contractive with the same function ψ, so for any $p, q \in X$ and $i = 1, \ldots, k$ we have $d\left(F_i(p), F_i(q)\right) \leqslant \psi\left(d(p,q)\right)$. Let $A, B \in K(X)$ and let $\delta = d_H(A,B)$. Then for every $p \in A$ there exists such $q \in B$ that $d(p,q) \leqslant \delta$. Therefore for each i we have $d\left(F_i(p), F_i(q)\right) \leqslant \psi\left(\delta\right)$. It follows that $F_i(A)$ is a set contained in the ε-environment of $F_i(B)$ for $\varepsilon = \psi(\delta)$. So we have $F(A) = \bigcup_{i=1}^k F_i(A) \subset \bigcup_{i=1}^k \left(F_i(B)\right)_\varepsilon = \left(F(B)\right)_\varepsilon$. Similarly we prove that $F(B) \subset \left(F(A)\right)_\varepsilon$. Therefore

$$d_H\left(F(A), F(B)\right) \leqslant \varepsilon = \psi(\delta) = \psi\left(d_H(A,B)\right).$$

We will further use the following

Theorem 7

(Jachymski, Gajek, Pokarowski, 2000) *Let X be a topological space (not necessarily Hausdorff), $F_1, \ldots, F_k$ be continuous selfmaps of X and F be defined by*

$$F(A) = \bigcup_{i=1}^k F_i(A) \text{ for } A \subset X.$$

The following conditions are equivalent:

(i) there exists a non-empty compact set A_0 such that $F(A_0) = A_0$

(ii) there exists a non-empty compact set A such that $F(A) \subset A$.

Theorem 8
Let $F_1, \ldots, F_k : \mathbb{R}^n \to \mathbb{R}^n$ be ψ-contractive (due to the usual metric in $\mathbb{R}^n$) with the same function ψ, and F be defined by $F(A) = \bigcup_{i=1}^{k} F_i(A)$ for each $A \subset \mathbb{R}^n$. Then there exists a non-empty compact set $C \subset \mathbb{R}^n$ such that

$$F(C) = \bigcup_{i=1}^{k} F_i(C) = C.$$

Proof. I (without using the Hausdorff metric)

Because $F_i,\ i = 1, \ldots, k$ are ψ-contractive, there is a closed ball $D = D(\theta, r)$ $\big(\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n,\ r$ is large enough$\big)$ such that $F_i(D) \subset D$ for every $i = 1, \ldots, k$. Therefore $F(D) \subset D$.

The set D is a closed bounded subset of $\mathbb{R}^n$, so it is a compact set. Functions F_i $(1 = 1, \ldots, k)$ are continuous, so $F_i(D)$ are compact and hence $F(D)$ is also a compact set. Therefore, since $F_1, \ldots, F_k$ are continuous selfmaps of $\mathbb{R}^n$ and $F(D) \subset D$, so based on Theorem 7 we get the thesis.

Proof. II (using the Hausdorff metric)

Proceeding in the same way as in the above proof, we get a non-empty compact set $D,\ D = D(\theta, r)$ such that $F(D) \subset D$. Thus $F^n(D),\ n = 1, 2, \ldots$ are compact sets such that

$$F(D) \supset F^2(D) \supset F^3(D) \supset \ldots$$

Hence by Kuratowski's generalization of Cantor's theorem $C = \bigcap_{n \geqslant 1} F^n(D)$ is a non-empty compact set, and $F^n(D) \to C$ as $n \to \infty$ in the sense of the Hausdorff metric. Using the continuity of ψ-contractive F (with respect to the Hausdorff metric) we have

$$F(C) = F\left(\lim_{n\to\infty} F^n(D)\right) = \lim_{n\to\infty} F^{n+1}(D) = C.$$

Moreover, since $(F^n(D))$ is a descending sequence, therefore due to the convergence $F^n(D) \to C$ in the Hausdorff metric ($\lim_{n\to\infty} F^n(D) = \bigcap_{n \geqslant 1} F^n(D)$ from Kuratowski's generalization of Cantor's theorem) we have

$$F(C) = F\left(\bigcap_{n \geqslant 1} F^n(D)\right) = \bigcap_{n \geqslant 1} F^{n+1}(D) = \bigcap_{n \geqslant 2} F^n(D) = \bigcap_{n \geqslant 1} F^n(D) = C.$$

It is known that $(K(\mathbb{R}^n), d_H)$ is a complete metric space, and F defined in the following theorem is a contraction (compare (Sękowski, 2007)), so as a conclusion from Banach Contraction Principle we have the following

Theorem 9
(Hutchinson's theorem). *The mapping* $F : K(\mathbb{R}^n) \to K(\mathbb{R}^n)$ *determined by contractions* $F_1, \dots, F_k$ *of the Euclidean space* $\mathbb{R}^n$ *(with contraction constants* $q_1, \dots, q_k$*), i.e.* F *given by the formula* $F(A) = \bigcup_{i=1}^k F_i(A)$ *for* $A \in K(\mathbb{R}^n)$ *has exactly one fixed point, i.e. such a compact subset* C *of* $\mathbb{R}^n$ *for which* $F(C) = C$*.*

We present below theorem useful in the construction of fractals.

Theorem 10
Let E *be a closed subset of* $\mathbb{R}^n$*. For* ψ*-contractive* $F_1, F_2, \dots, F_k : E \to E$ *(with the same function* ψ*) that determine* F, $F(A) = \bigcup_{i=1}^k F_i(A)$ *for* $A \subset E$ *there exists exactly one non-empty compact set* $A_0 = F(A_0) = \bigcup_{i=1}^k F_i(A_0)$*. Moreover, for each* $A \in K(E)$ *the iteration sequence* $(F^n(A))$ *converges to* A_0 *with respect to the Hausdorff metric.*

Proof. Let A be a set in $K(E)$ with $F_i(A) \subset A$ for $i = 1, \dots, k$. (Such a set may be the set E, if it is bounded.) Then $A_n = F^n(A) \subset F^{n-1}(A) = A_{n-1}$, $n > 1$. Therefore $(F^n(A))$ is decreasing sequence of compact set. On the basis of Kuratowski's generalization of Cantor's theorem $A_* = \bigcap_{n \geqslant 1} F^n(A)$ is a non-empty and compact set. Moreover

$$F(A_*) = F\left(\bigcap_{n \geqslant 1} A_n\right) \subset \bigcap_{n \geqslant 1} F(A_n) \subset \bigcap_{n \geqslant 1} A_n = A_*.$$

On the basis of Theorem 7, there exists a non-empty compact set A_0 such that $F(A_0) = A_0$.

Now we will start to use Hasdorff metric. It can be shown that $A_0 = A_*$ taking into account the convergence $F^n(A) \to \bigcap_{n \geqslant 1} F^n(A)$ in the Hausdorff metric. You can see that the set A_0 is only one, because if there were another set $B_0 = F(B_0)$ $(B_0 \neq A_0)$ we would have a contradiction:

$$d_H(A_0, B_0) = d_H\left(F(A_0), F(B_0)\right) \leqslant \psi\left(d_H(A_0, B_0)\right) < d_H(A_0, B_0).$$

Moreover, for any $A \in K(E)$ we have

$$d_H\left(F(A), A_0\right) = d_H\left(F(A), F(A_0)\right) \leqslant \psi\left(d_H(A, A_0)\right),$$

and hence

$$d_H\left(F^n(A), A_0\right) \leqslant \psi^n\left(d_H(A, A_0)\right) \to 0 \text{ as } n \to \infty.$$

A compact and non-empty set $A_0 = F(A_0)$ is called a fractal or a self-similar set (in relation to $F_1, \dots, F_k$).

Remark 4
If in Theorem 10 we assume that $F_1, \dots, F_k$ *are contractions with constants* q_i*, then for* $q = \max\{q_1, \dots, q_k\}$ *we get Hutchinson's theorem on the fixed point of the map* F *being a contraction (see (Hutchinson, 1981), (Sękowski, 2007)).*

Here are some examples of sets that are self-similar. Examples (I)-(III) present in turn: (I) Cantor's set, (II) Sierpinski's triangle, (III) the Sierpinski type carpet, (IV) the design of the graph of a continuous function nowhere-differentiable. The first two examples can also be found in appropriate books (for example in (Peitgen, Jurgens, Saupe, 1992)); we present these examples to remind them and illustrate the theory presented.

Let us add that each of the examples (I)-(III) uses a system of iterated functions consisting of contractions $f_1, \dots, f_k$ with the same constant $q = q_1 = \cdots = q_k$, which changes in these examples; then we use Theorem 10 taking $\psi(t) = qt$ (e.g. in example (I), $\psi(t) = \frac{1}{3}t$). In these examples we can also use Banach Contraction Principle, because it is easy to see (based on Theorem 6) that F determined by the respective contractions is also a contraction. However, in example (IV), not all constants $q_1, \dots, q_k$ are equal; here we use Hutchinson's fixed point theorem (see Remark 4).

Examples:

(I) *Consider a set* $E = \langle 0, 1\rangle \subset \mathbb{R}$ *with a natural metric and two contractions* $f_1, f_2 : E \to E$ *given by formulas:* $f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$. *Then* $F(E) = f_1(E) \cup f_2(E) = \left\langle 0, \frac{1}{3}\right\rangle \cup \left\langle \frac{2}{3}, 1\right\rangle$,

$F^2(E) = f_1(F(E)) \cup f_2(F(E)) = \left\langle 0, \frac{1}{9}\right\rangle \cup \left\langle \frac{2}{9}, \frac{1}{3}\right\rangle \cup \left\langle \frac{2}{3}, \frac{7}{9}\right\rangle \cup \left\langle \frac{8}{9}, 1\right\rangle$, *etc. The limit set* $C = \lim_{n\to\infty} F^n(E)$ *in the Hausdorff metric is a unique fixed point of the mapping* F, *known as the Cantor set. The length of the Cantor set is*

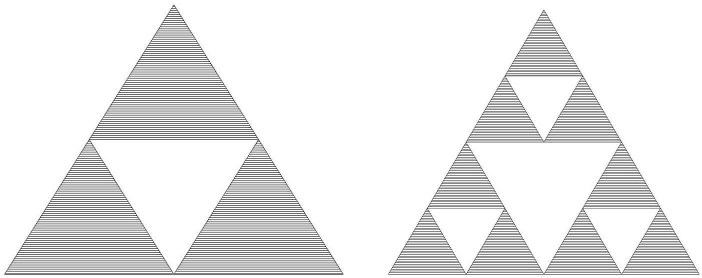
$$1 - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 1 - \frac{\frac{1}{3}}{1-\frac{2}{3}} = 0.$$

(II) *In this example we will deal with the construction of one of the first classical fractals, which is Sierpinski's triangle. Let's first consider an equilateral triangle* S_0 *with vertices* $(0,0), (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. *Consider affine transformations*

$f_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$, $f_3(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)$ *of points of this triangle and transformation* $F(A) = f_1(A) \cup f_2(A) \cup f_3(A)$, $A \subset S_0$.

Applying the transformation F *to* S_0 *we get the triangle* $S_1 = F(S_0) = f_1(S_0) \cup f_2(S_0) \cup f_3(S_0)$ *(see Figure 1(a)). Now applying the transformation* F *to the result obtained, we get* $S_2 = F^2(S_0) = f_1(S_1) \cup f_2(S_1) \cup f_3(S_1)$ *(see Figure 1(b)). The result of continuing this procedure will be the triangles* $S_1, S_2, S_3, \dots, S_k, \dots$ *obtained by removing the triangle in the middle of each larger triangle, whereby* $S_1 \supset S_2 \supset S_3 \supset \cdots \supset S_k \supset \dots$, *where* $S_k = F^k(S_0)$, $k = 1, 2, \dots$ *There is a unique set* $S = F(S)$ *which is a self-similar set called the Sierpinski triangle, and* $F^k(S_0) \to S$ $(k \to \infty)$ *with respect to the Hausdorff metric.*

It is worth calculating the area and perimeter of the Sierpinski triangle. Since in each subsequent step we remove $\frac{1}{4}$ *of the whole we will have* $\frac{3}{4}$ *of the previous area; hence in the second step* $\left(\frac{3}{4}\right)^2$ *of the area of* S_0 *will remain.*



(a) (b)

Figure 1

It is easy to check that in the n-th step there will be $\left(\frac{3}{4}\right)^n$ *of the area of* S_0*. So the area of the Sierpinski triangle is equal to* 0*. Moreover, it can be shown that this triangle has an infinite perimeter.*

It is worth adding that the Sierpinski triangle can also be made from Pascal's triangle by coloring its odd numbers block. Note that there is also a relationship between the Pascal triangle and Fibonacci numbers.

(III) *Sierpinski added one more object to the set of classic fractals, namely the Sierpinski carpet. We start its construction by dividing the unit square into 9 congruent squares from which we remove the middle one. After removing it with the remaining squares, we proceed in the same way and continue this procedure. The figure that we get as a result of this infinite process is the Sierpinski carpet.*

The construction of the Sierpinski type carpet that we present here differs from the construction of the classical Sierpinski carpet. Now we first remove the interior of the square in the lower right corner, then after dividing the remaining 8 squares into 9 smaller squares we remove the small squares in the lower right corners (see Figure 3). Continuing this procedure, we approach the Sierpinski type carpet. Whereas the construction of this type of carpet with iterated function system is shown below.

Let $E = \langle 0,1\rangle \times \langle 0,1\rangle$ *and let* E *be equipped with the Euclidean metric. Consider the operator* F *defined by the system of eight functions* $f_{ij} : E \to E$ *by the formulas* $f_{ij}(x,y) = \left(\frac{1}{3}x + \frac{i}{3}, \frac{1}{3}y + \frac{j}{3}\right)$ *for each of pair* $(i,j) \in P = \{0,1,2\}^2 \setminus \{(2,0)\},\ (x,y) \in \langle 0,1\rangle \times \langle 0,1\rangle$ *. (compare Figure 2). Which means that* $F(A) = \bigcup_{(i,j)\in P} f_{ij}(A),\ A \subset E$*. The functions* f_{ij} *transform the unit square* E *into 8 congruent smaller squares (because to get the image* $E_1 = F(E)$ *we exclude everything that comes from function* f_{20}*). The sequence* (E_n) *defined recursively* $E_n = F(E_{n-1})$ $(n \geqslant 1),\ E_0 = E$ *will converge to the set* $D = F(D)$*, which is the Sierpinski type carpet.*

(IV) *In 1861 K. Weierstrass gave an example of a real continuous function defined on an interval that has no derivative at any point.*

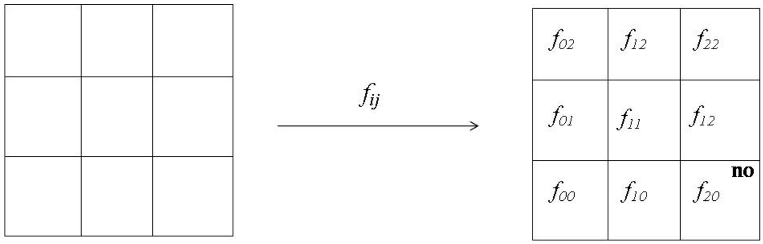


Figure 2

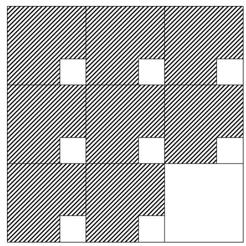


Figure 3: The second stage of the construction of the Sierpinski type carpet

There are many examples of such functions. Below we will present an example based on the use of the iterated function system, and we will get a graph of a continuous function nowhere-differentiable, which is a self-similar set.

Let $E = \langle 0,1\rangle \times \langle 0,1\rangle$. *In* E *we have the Euclidean metric. We define three functions* $f_i : E \to E$ *which are contractions:*

$$f_1(x,y) = \left(\frac{x}{3}, \frac{3}{4}y\right), f_2(x,y) = \left(\frac{2-x}{3}, \frac{1+2y}{4}\right), f_3(x,y) = \left(\frac{x+2}{3}, \frac{3y+1}{4}\right).$$

Denote by F *the Hutchinson operator determined on the family* $K(E)$ *sets compacted in* E. *Define*

$$F(A) = f_1(A) \cup f_2(A) \cup f_3(A),\ A \in K(E).$$

$F : K(E) \to K(E)$ *is a contraction with respect to the Hausdorff metric. Because* $K(E)$ *with a Hausdorff metric is a complete space, so based on the Hutchinson fixed point theorem (comp. Remark 4)* F *has exactly one fixed point* D *being a compact non-empty set. Moreover, a sequence of sets* $D_n = F(D_{n-1})$ $(n \geqslant 1)$, $D_0 = \{(x,x) \in E\}$ *that are graphs of continuous functions* $g_n : \langle 0,1\rangle \to \langle 0,1\rangle$ *converges to* D, *while the limit function* $g = \lim_{n\to\infty} g_n$ *is defined by its graph, which is a self-similar set* D. *The function* g *is continuous and has no derivative at any point (its graph has no tangent at any point).* As already mentioned in the introduction, this example of a nowhere-differentiable function graph differs from the example from (Katsuura, 1991), because the contraction system $\{f_1, f_2, f_3\}$ used here is different.

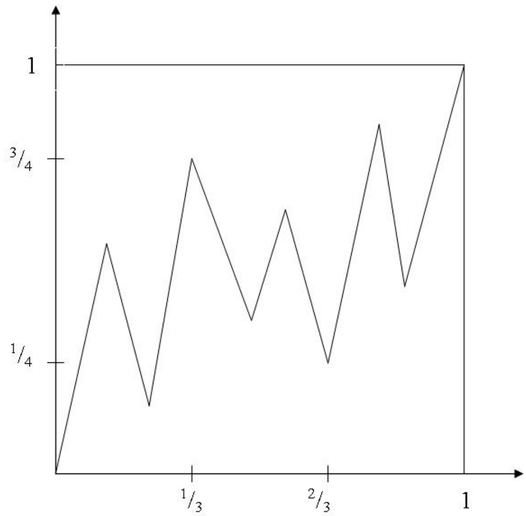


Figure 4: The set D_3 which is the graph of the function g_3 and one of the stages of constructing the function nowhere differentiable

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