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Students' approaches while solving a non-typical geometrical problem*

Abstract. In the paper we present the results of two teaching episodes, which took place in two middle school classes with 13- and 14-year-old students. The students in both classes were asked to solve the same geometrical problem; then a discussion followed, in which they had to justify their solutions. In both cases the students had no prior experience in solving non-typical mathematical problems. Additionally, the students were asked to justify their answers, which is not a common characteristic of a 'typical' mathematics classroom at that level. The problem was chosen from a wider study, in which twenty classes from twenty different schools were analysed. One of the aims of the present study was to analyse the skills that require a deeper understanding of mathematical concepts and properties. Particularly, we aimed to investigate students' different solution methods and justifications during problem solving. The results show considerable differences among the two classes, not only concerning the depth of investigating (which was expected due to the different age groups), but also concerning the relationship between achievement (as assessed by the mathematics teacher) and success in solving the problem. These results demonstrate the need for re-directing mathematics education from a pure algorithmic to a deeper thinking approach.

Introduction

Mathematics is seen as an activity and as ways of thinking and acting in the world. By accepting the uncertainty and complexity of the modern world, the importance of engaging students in problem solving situations is acknowledged by all scholars in mathematics education and is manifested in numerous contemporary

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curricula and research reports. Additionally, the process of identifying, designing or implementing challenging problems has led to a large number of relevant publications (e.g., Henningsen & Stein, 1997; Sullivan, Clarke, & Clarke, 2013).

Our research was initiated by the following three ideas: First, that problem solving is one of the most important mathematical skills. Second, that solving non-routine problems enhances creativity and independent thinking. Third, that geometry, because of its representational nature, is a fruitful field for creative mathematical explorations. In that line, the focus of this paper is students' different approaches, including solution methods and justifications, during solving a non-typical geometric problem.

Theoretical background

Tasks play an essential role in mathematics education. Krygowska (1977, p. 3) states that "A student creates a concept of mathematics that appears to him through the prism of the tasks he solves. The student's attitude and learning motivations depend on that". One of the basic types of tasks, which are extremely important in mathematics education and the development of mathematical thinking, are the tasks-problems, i.e., problems cannot be solved by known algorithms, but they create a research stance and lead to the enrichment of mathematical and non-mathematical knowledge and skills (Krygowska, 1977). It is important to stress the distinction between typical and non-typical problems, especially since in mathematics education literature sometimes the term 'problem' denotes simple exercises (Shoenfeld, 1992). Therefore, problem solving is about trying to reach the solution of a mathematical situation by actions which are not immediately known (Cooper, 1986). More specifically,

> . . . problem solving performance is influenced by five factors: knowledge acquisition and utilization, control, beliefs, affects and socio-cultural contexts. Problem solvers must be able to connect their own knowledge representation and the problem situation at hand and the extent to which they are able to do this, in turn, impacts on their success with solving the problem (Lester & Kroll, 1993, as cited in Muir, Beswick, & Williamson, 2008, p. 229).

Solving non-typical tasks

There are different types of tasks that can be met in mathematics classrooms; we find non-typical (or non-routine) tasks to be among the most effective for enhancing originality and students' creative thought (Pólya, 1966). The term non-routine tasks or problems, in juxtaposition with routine tasks, is used in order to stress the fact that such problems require a novel approach by the potential solver, i.e., "one that emphasizes the making of new meanings through construction of new representations" (Lester, 2013, p. 255). The uncertainty of non-routine problems and the doubt they bring to students' existing ways of thinking and solving facilitates creativity and out-of-the-box thinking. Therefore, the teachers

should be able to distinguish among typical and non-typical problems and be able to transform typical to non-typical problems, in order to enhance their students' engagement (Beghetto, 2017). Experiencing non-typical problems leads students to thinking and acting in new ways; these ways were the focus of our study.

The special nature of geometry and its teaching

Geometry was born out of action and out of humans' need for developing and structuring the space around them. Geometry is one the basic components of school mathematics and yet it is the source of difficulties for many students. One of its main characteristics is the importance of visualization, whether it includes geometrical signs and registers or mental processes (Arcavi, 2003; Presmeg, 2006). Therefore, in order to take advantage of the full potential of geometry in education, one needs to change the approach to its teaching. In this line, an important element in the teaching of geometry should be the students' actions and reasoning in relation to geometrical objects. In the next section we describe a non-typical geometrical problem that was designed to elicit students' solutions approaches, as well as their reasoning.

Context of the study and methodology

In the paper we present the results of two teaching episodes, which took place in two groups of pupils from two different schools: 16 pupils from the second grade of lower secondary school (aged 14) and 17 pupils from the seventh grade of primary school (aged 13); a total of 33 students. The classes in the study were taught by different mathematics teachers. The same mathematical problem was provided to both classes and the students were asked to solve it. In both cases the students had no experience in solving non-typical mathematical problems, i.e., problems which cannot be solved by following particular steps. Before starting the assignment, students were informed that they would receive a working sheet with the task to be solved and their work will not be evaluated and will not affect their marks in mathematics. After the work was finished, a discussion was initiated, in which the students were talking about their solutions. This part of the lesson was recorded. Students were able to compare their solutions and verify their results. The research material includes written solutions of the students and transcripts of recorded conversations with students. Both the works and the students' statements have been coded as S1 – S17 (students of lower secondary school) and S18 – S33 (students of primary school).

The problem was chosen (and modified) from a wider study done by The Educational Research Institute in Poland (Karpiński, Grudniewska & Zambrowska, 2013), in which twenty classes from twenty different schools were analysed. The problem used was a geometric one, in which the students had to compare perimeters, areas and dimensions of two given figures, one of which was formed by slit along the diagonals of the other. The problem did not require knowledge beyond grade 6 of primary school; however, it can be classified as a non-typical, since students were required to show independent mathematical thinking and engagement

in problem solving. For the purpose of the study, the task was modified by the authors of the paper: the students were asked to justify their answers, which is not a common characteristic of a 'typical' mathematics classroom at that level. Below, the working sheet is presented (every time after the "Why? Justify your choice" the students were provided with space in order to write their answers).
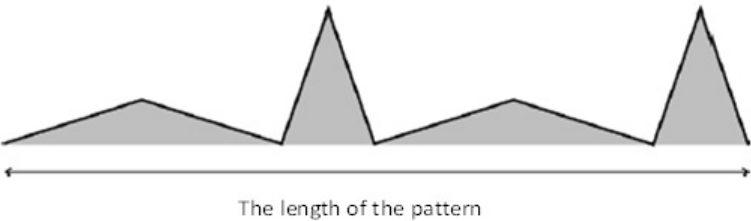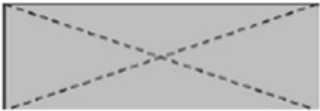
Working sheet

Think about and answer the following questions. Cross-mark True or False, and then justify your choice.

PATTERN

The rectangle shown on the right was cut along its diagonals in four triangles. Then a pattern was created by these triangles as shown below.

Assess if the sentences in the table are true or false.

The length of the pattern

I.	The length of the pattern is equal to the perimeter of the rectangle.	☐ TRUE / ☐ FALSE

Why? Justify your choice.

II.	The length of the bold line along the top edge of the pattern is equal to the sum of the lengths of the diagonals of the rectangle.	☐ TRUE / ☐ FALSE

Why? Justify your choice.

III.	The area of the pattern is twice bigger than the area of the rectangle.	☐ TRUE / ☐ FALSE

Why? Justify your choice.

Figure 1: The working sheet used in the study

The task consisted of three questions to which the student was expected to answer true/false and justify his or her answer. The first question concerned the perimeter of the two given figures. The second question concerned the diagonals' length of the first figure (rectangle) and the upper edge of the second figure

(after cutting the rectangle along the diagonals and unfolding it, a figure called 'pattern' was obtained; the lower edge of the 'pattern' was equal to the perimeter of the rectangle, and the top was formed by the lines previously contained in its diagonals). In the third question, the areas of the two figures had to be compared.

The problem can be considered as non-typical also because of its dynamic aspect. Cutting the rectangle and 'spread out' created motion. Some segments 'doubled', others only changed position, combining into one longer one. The ability to see this change was crucial for the correct solution of the given task.

The main purpose of the study was to diagnose the ability to solve tasks that require a deeper understanding of geometric concepts (length, perimeter and area of a plane figure) and the properties of these concepts. We also aimed to investigate students' different approaches during solving the given problem. Particularly, we were interested in:

1. What are the students methods of the solutions?

2. How do students justify their answers?

3. What are the common misconceptions of the concepts of length, perimeter and area?

Results

Among a total of 33 students only 23 answered the first question correctly (14 from the second grade of lower secondary school – 87% and 9 from seventh grade primary school – 52%). The second question was correctly answered by 25 students (15 from the second grade of the lower secondary school – 93% and 10 from seventh grade – 58%). The third question was best handled by students, with as many as 28 people answering correctly (15 from the lower secondary school – 93% and 13 from the seventh grade of primary school – 76%) (see the table 1). The pupils of the lower secondary school performed much better than the pupils in the seventh grade of the primary school, which was not surprising because of the age and the greater mathematical experience of the lower secondary school students. It should be clarified here that the correct answer to the question does not imply a correct justification. An analysis of students' arguments will be described in the following sections of this article.

Analysis of results for question 1

The correct answer to question 1 was 'True'. As already mentioned, 23 out of 33 students indicated the correct answer. The following are the justifications of the students who considered the length of the pattern to be equal to the perimeter of the rectangle:

[S4:] Because after putting these triangles together you will get the same rectangle.

[S5:] True because every base of the triangle is the side of the rectangle.

	I. question: The length of the pattern is equal to the perimeter of the rectangle (correct answer: TRUE)	II. question The length of the bold line along the top edge of the pattern is equal to the sum of the lengths of the diagonals of the rectangle (correct answer: FALSE)	III. question The area of the pattern is twice bigger than the area of the rectangle (correct answer: FALSE)
Grade 2 of lower secondary school (16 in total)	14	15	15
Grade 7 of primary school (17 in total)	9	10	13
SUM (33 in total)	23	25	28

Table 1: Quantitative results of the written solutions (correct answers)

[S6:] The length of the pattern is equal to the perimeter of the rectangle because the edges which make the length of the pattern have the same length and at the same time they make the perimeter of the rectangle.

[S13:] It is made from the sides of this rectangle.

[S17:] Yes, because the elements are the same just differently arranged.

A characteristic feature of the above presented justifications is the students imagining a certain movement: folding and unfolding the rectangle cut along the diagonals. On the basis of this movement, the students were able to compare the length of the pattern and the perimeter of the rectangle.

Apart from the written justification we could also observe some graphical ones. Below some examples of such justifications are presented.
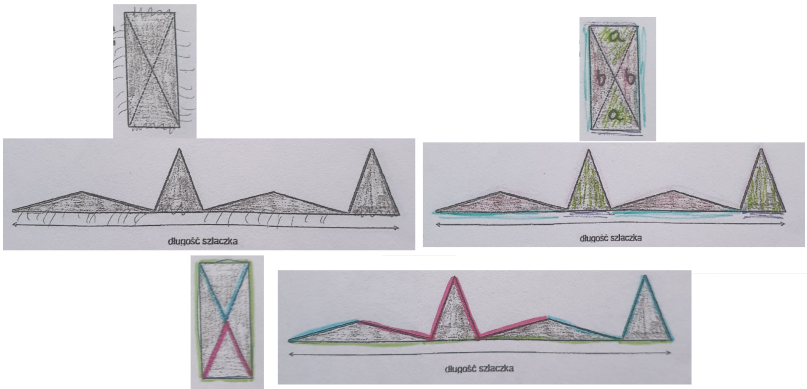

Figure 2: The works of students S9, S10 and S12

In the works of S9 and S10 (Figure 2), instead of a verbal justification the students marked the relevant segments, which are equal in length, and one of their written responses was "It is equal to the perimeter of rectangle" (S10). S12, in addition to the coloured graphic solution she wrote the following answer: "The length of the pattern is equal to the perimeter of the rectangle, since the bases of these triangles are arranged in a single line, which can be obtained by unfolding the rectangle along its diagonals", thus referring to a dynamic approach to the presented task situation.

As many as 10 out of 33 students (30%) could not answer this question correctly. They probably could not imagine that if we put the elements of the pattern together, we will create exactly the same rectangle as in the picture next to it. The corresponding sides of the rectangle are in turn fragments of the length of the pattern. The pupils also had a description for the drawing, which clearly marked "length of the pattern". This may suggest students' difficulty in understanding mathematical text.

Figure 3: The work of student S16

Student S16 (Figure 3) measured the length of the sides of the rectangle probably by using a ruler and calculated the perimeter of the rectangle with a result of 9.6 cm. Then he measured the perimeter of the figure forming a pattern with a base that is the length of the pattern along with eight sides 2 cm long. By adding these lengths he obtained 26.8 cm. He then stated:

> S16: The length of the pattern is not equal to the perimeter of the rectangle because the downer line of the pattern looks like it has a bigger value than the perimeter of the rectangle.

The solution of this student is an example of an incorrect understanding of the task: the student identified the length of the pattern with its perimeter. Perhaps he looked at the pattern as a plane figure. The length of the figure is the length of the line limiting it. S16 measured the lengths of the sides of the rectangle and the pattern with a ruler and compared the results. The perimeter of the figure is not perceived as length, it is perceived as a certain concept, so it can be compared only with the same concept, that is, the perimeter of another figure – in this case, the perimeter of the pattern. In the solution of this student two different types of

argumentations can be noticed. The first is associated with the calculated perimeters of both figures. There is a certain logical argumentation here: you can compare the sizes of the same types (here: the perimeter of the figures), and the larger is the one which is assigned a greater numerical value. The process of measuring perimeters can be a sign of empirical argumentation; by acting appropriately, the student wanted to see that his hypothesis is correct. However, for S16 the calculation of perimeters and the comparison of the results obtained did not appear to be definitive evidence of the correctness of the hypothesis. In his justification, he referred to the visual aspect of the whole task, claiming that "the downer line of the pattern looks like it has a bigger value". This is, in a sense, further evidence that the student did not fully understand the concepts of length and perimeter, and misinterpreted the content of the task, and treated the pattern as a plane figure.
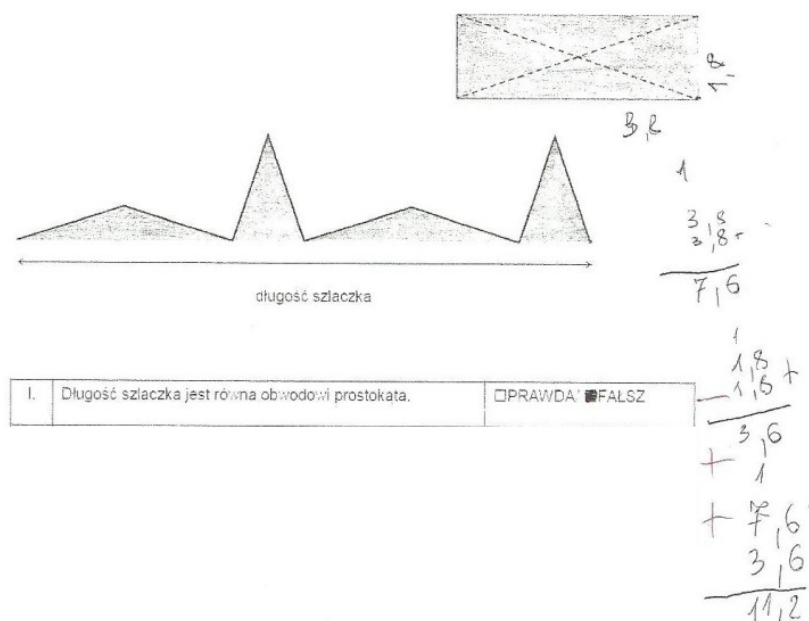

Figure 4: The work of student S24

The Figure 4 presents the work of student S24, which shows some calculations. The student clearly signed the lengths of the sides of the rectangle: 3.8 and 1.8, and then calculated the perimeter of the rectangle (11.2). Most likely, the student measured the length of the pattern. The lengths did not correspond, so he ticked the answer "False" without giving a justification.

The other students who incorrectly answered for the question 1, like S16, referred to the perimeter of the pattern:

> S14: Because the given pattern can have only the same area, not the perimeter because the pattern counts as a double perimeter.

> S20: Because the pattern is built not only from the sides of the rectangle but also from its diagonals.

> S22: No, because if we add these two triangles – the first and the second, and 3rd and 4th there will not be the same perimeter.

The student S14 referred in her utterance to her previous experience with geometry, that when you cut a figure the area is the same but the perimeter changes. The students S20 and S22 treated the pattern as a geometric figure of a particular shape and tried to assess its perimeter.

Analysis of results for question 2

To correctly answer the second question, the students had to notice that the length of the bold line along the top edge of the pattern is equal to the double sum of the diagonals' lengths of the rectangle. In the rectangle, the diagonals are divided into halves, so there are four segments of the same length equal to the half of one diagonal of the rectangle, and the upper edge of the pattern consists of eight such segments. This means that it is double length of the rectangle diagonals. In this case the students had to imagine and notice that fragments of the upper line of the pattern will overlap if we fold the pattern into a rectangle. In this question, 25 students marked the answer "False" which was the correct one. As in the previous question, there were verbal and graphical justifications, in addition, some students used algebraic expressions.

> S2: Not because only the upper parts of the triangles (1, 3) are merged with diagonal lengths.

> S8: The length of the bold line along the top edge is twice as long as the diagonals of the rectangle, because the two symmetrical triangles connecting the vertices represent the diagonal length arms in this rectangle, and we have two pairs.

> S14: False – because some lines overlap together creating diagonals, so when we sum up all the lines then we get a bigger length than the sum of the diagonals.

In the utterances of S2 and S8 a reference to folding the pattern into a rectangle can be found together with the description on how the individual segments will overlap on each other. However, the description does not concern the folding process, but rather the two situations – the initial and the final one. The language used by students is full of colloquial formulations combined with mathematical concepts. S14's answer is an example of a dynamic attitude: he uses verbs connected to movement, like: *overlap*, *create*, *sum up* (in the sense of summing up

the segments). However, more than the description of the movement by itself, this student gives the result, the effect of performing a certain action, e.g., the joint overlapping of the lines forming diagonals is the result of folding the pattern into a rectangle.

S10 formulated her justification by using algebraic language, indicating the unknown x as the length of the mid-diagonal.

S10: $\frac{1}{2}$ of diagonal $= x$

diagonals (together) $= 4x$

the length of the bold line $= 8x$

$4x \neq 8x$

This student used algebraic expressions to justify the answer; she referred to algebra, which was lately taught in the class and used it in a new context (operational knowledge).

Among the answers which were considered as correct, there were some erroneous justifications:

S13: There are two diagonals of the rectangle included.

S16: In my opinion, the length of the bold line is not equal because the diagonals of the rectangle are 7 cm, and the sum of all segments is 16 cm.

The utterance of the S13 is not precise, it is possible that for him "two diagonals" means the same as the double sum of the lengths of both diagonals. The phrase "There are (...) included" may apply to the upper line of the pattern and the relation of inclusion. The student S16, as in question 1, measured all the segments and added their lengths. Hence he answered "False".

Students who marked "True" either did not give justifications or did not notice that the length of the bold line along the top edge of the pattern is equal to double the sum of the diagonal lengths of the rectangle, even though they looked at the situation dynamically and noticed that the particular segments corresponded to the rectangle's diagonals:

S15: It is true, because after folding the upper edges become the diagonals.

The student S15 noticed that the pattern becomes a rectangle when folded, and vice versa. However, she referred to the initial and final situations. It seems that for this student, the problem of measurement is not relevant here. Perhaps for her, the important element was the congruence of the segments. During the process of folding the pattern into the rectangle, the corresponding segments overlapped, resulting in diagonals of the rectangle. The fact of doubling the segments was not taken into account at all.

Analysis of results for question 3

The last of the three questions in the task consisted in comparing the areas of both figures. It turned out that 5 out of 33 students think that after cutting the rectangle into pieces and folding these pieces into an another figure, a figure with a different area will be created. Perhaps, these students did not read this question carefully or they did not have a sufficiently formed understanding of the concept of the area of a figure. It is also possible that they analysed the task visually, trying to estimate the areas of both figures, without referring to their formal knowledge.

Among the justifications for answering question 3 there were references to algebraic expressions as well as verbal and graphic justifications.

Figure 5: The work of the student S10

The work of S10 is an example of algebraic and graphic justification (Figure 5). This student firstly marked by colours the congruent triangles in the rectangle and the corresponding triangles in the pattern. She then designated the areas of each triangle with the letters a and b and wrote the corresponding algebraic expressions describing area of the rectangle and the pattern. The colouring of the elements would be a sufficient justification. Apparently, however, the student wanted to present a more formal justification.
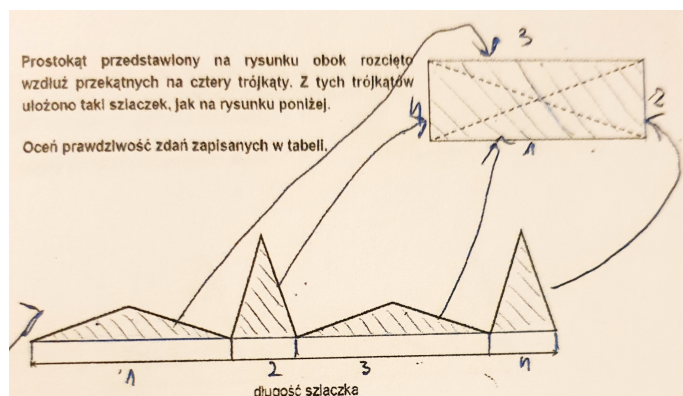

Figure 6: The work of the student S22

The student S22 (Figure 6) applied the decomposition of the rectangle into four parts and referred to the property of invariance of the area. The drawn arrows referred to a dynamic approach, from fragments of the figure the entire composition is created. For the student, this was so clear and obvious that he did not feel the need to give another, more formal justification. The proposed solution is a geometric proof of the "see" type.

After completing the task, students shared their reflections on the solutions. The following fragment of the discussion shows that students had a good intuition of the figure area and its properties, while also referring to everyday life experience:

> S1: It is false because if we have a rectangle and we build something from it then it is not possible that the area of this which we built was bigger than the area of that which we used.
>
> S4: The same!
>
> S5: The area will not change at all!
>
> S1: Because we didn't add anything, and we didn't subtract anything.
>
> S4: It is like puzzles – when they are thrown around and we build the picture.

The students in this dialog did not allow themselves any other option than that the area of the pattern and the rectangle from which this pattern was made may have a different value. The students S1 and S4 referred to the intuition of change, in addition, S4 supported this with a concrete example from everyday life – the laying of puzzles.

Discussion

An important element of geometry teaching should be linking a dynamic approach to geometric objects and formal knowledge (Swoboda, 2011) and the ability of students to conduct mathematical reasoning in relation to these objects. Such an opportunity is created by non-typical tasks. The students who took part in the study did not have much experience with these types of tasks. Therefore, the purpose of the study was to diagnose the students' ability to solve tasks that require a deeper understanding of geometric concepts, such as the perimeter and area of a plane figure, and the properties of these concepts. Significant differences were observed in the solutions of the two research groups.

The analysis of the works of the students of the second grade of lower secondary school, has shown that the intuition of the figure area is well formed; students were aware of the property of invariance of the area. However, some students had some difficulties with differentiating the concepts: the length of the pattern and the perimeter of the figure. The visual nature of the task, the pattern as a plane figure with two dimensions caused some problems in answering question 1. In most cases, when the students were referring to "an action" (e.g., folding up pattern into a rectangle), the answers to all questions were correct – the dynamic attitude was more helpful than the static one.

The results of the works of the students of the seventh grade of primary school indicate that some of them did not have enough skills related to a deeper understanding of the concepts of length, perimeter and area of the figure. The argumentations used for question 1 showed a misconception related to the perimeter of the figure. It was not seen as a length, but as a polyline which delimits the shape of the figure. Many of the good and very good students (according to their teacher) failed in this task; the weak students performed better. The good students tried to find some known schemas for the solution – which was hard in case of non-typical task. The weaker students referred more to their intuition. This tendency was also observed in the general results of the Educational Research Institute in Poland (Karpiński, Grudniewska, & Zambrowska, 2013).

In both groups, many students struggled to properly justify their reasoning, despite the correct answer. Among the provided justifications several types can be distinguished: verbal, graphic and algebraic. The most common type of argument was a verbal description. In these cases, the students often referred to colloquial language, combining it with mathematical formulations. A lack of precision of expressions was also evident. This tendency was observed in the solutions of both groups studied. Some students chose a graphical way of justification; this type of argumentation often referred to a dynamic approach to the task. Only in a few cases there was an algebraic justification and it appeared only among lower secondary school students. This may have been because students in this class were recently taught algebra.

Our study has shown that there is a need for more detailed methodological tools which can analyse and compare different solution strategies, especially in Geometry. There is big research in the field, but we believe that what is still missing is some tools that can be used even by a school teacher, in order to better understand his or her students' way of thinking.

Secondly, and concerning thinking, we believe that there is a big need for redirecting mathematics education from a pure algorithmic to a deeper thinking approach. Solving non-typical tasks is a good way to achieve this, but we also need a clear and understandable method to analyse the students' works. Especially in Geometry we need more 'action-oriented' tasks, that is tasks which require the students to perform some actions in order to solve them.

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