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How to solve third degree equations without moving to complex numbers*

Abstract. During the Renaissance, the theory of algebraic equations developed in Europe. It is about finding a solution to the equation of the form

$$a_n x^n + \ldots + a_1 x + a_0 = 0,$$

represented by coefficients subject to algebraic operations and roots of any degree. In the 16th century, algorithms for the third and fourth-degree equations appeared. Only in the nineteenth century, a similar algorithm for the higher degree was proved impossible. In (Cardano, 1545) described an algorithm for solving third-degree equations. In the current version of this algorithm, one has to take roots of complex numbers that even Cardano did not know.
This work proposes an algorithm for solving third-degree algebraic equations using only algebraic operations on real numbers and elementary functions taught at High School.

1. Introduction

Solving algebraic equations was already taken up in the Renaissance period. Such equations have appeared in various practical issues, particularly in optimization problems. The problems leading to higher degree algebraic equations appear, for example, in metallurgy. In the case of a first-degree equation, the situation is simple. The equation

$$ax = b \tag{1}$$

is just solved with the help of the following formula:

$$x = \frac{b}{a}.$$

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Through algebraic operations on coefficients, one determines its solution. In other words, it is enough to find a solution in the smallest field containing a and b. It is important because the formula gives a general procedure. But in the situation when the second power occurs, it is not so simple. In secondary schools, second degree algebraic equations are considered, i.e. equations like

$$ax^2 + bx + c = 0, \tag{2}$$

where a, b, c are constants. The solution is limited to the reduction of the left side of equation (2) to a canonical form, i.e. to the following one:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}, \tag{3}$$

where $\Delta = b^2 - 4ac$. However, solving the equation requires finding a number that, when squared, will give this discriminant. Seeking this number, one needs to go beyond algebraic operations. We are just looking for an element in the field $K(\sqrt{\Delta})$. The general scheme, therefore, requires introducing a new function – the square root.

Similarly, the expression of a higher degree

$$a_n x^n + \ldots + a_1 x + a_0,$$

can be reduced to a canonical form by arranging the polynomial with respect to the variable $x + \frac{a_{n-1}}{na_n}$. We then get the following expression:

$$a_n\left(x + \frac{a_{n-1}}{na_n}\right)^n + Q_{n-2}\left(x + \frac{a_{n-1}}{na_n}\right),$$

where Q_{n-2} is a polynomial of max. $n-2$ degree. In the case of $n = 2$, it is enough to solve the equation. In the case of $n \geqslant 3$, it is not enough to solve the equation, but it slightly simplifies the problem. When starting to look for a solution of a higher-degree equation after reducing it to a canonical form and dividing it on both sides by a_n the problem is reduced to solving the following equation:

$$x^n + p_{n-2}x^{n-2} + \ldots + p_1 x_1 + p_0.$$

It is needless to say we are interested in the so-called elemental solution, i.e., the solution obtained from the coefficients of the equation using only algebraic operations and root extraction. There are no such algorithms for fifth or higher-degree equations. Such formulas exist for third and fourth-degree equations, provided that it is possible to go beyond the real numbers.

In the Renaissance, there were attempts to find solutions for higher degree equations. Tartaglia and Cardano did it for third-degree equations, and Ferrari - for fourth-degree equations. To find a general method, one would need to find formulas that do not change with the permutation of elements. Lagrange (Lagrange, 1770–1771) was the person who dealt with this problem introducing the concept of the resolvent equation. The problem is that the theory went no further than the fifth-degree equation because its resolvent was of the sixth degree. Today we have the Abel- Ruffini theorem, and we know that there is no formula based on four algebraic actions and root extraction, and there cannot be one. Using the so-called special functions can be a new course of action. In 1858 Hermite (Hermite, 1842) showed the possibility of solving a fifth-degree equation through elliptical integrals.

The article aims to present an algorithm for solving third-degree equations based on the method taught in secondary schools. Complex numbers are not in the curriculum, but trigonometric equations are. The presented algorithm applies trigonometry. It can be used, for example, during the classes of a maths club.

2. Cardano formula

The first formulas for solving third-degree equations were derived by the Italian mathematician Tartaglia (this is a nickname, meaning "Stammerer" in Italian), but they were published by another Italian mathematician Geronimo Cardano (Cardano, 1545), thus being called the Cardano's formulas today. Let us consider the equation

$$x^3 + px + q = 0. \tag{4}$$

Based on the considerations of the previous chapter, it can be deduced that any equation of degree after changing the variables is transformed to such an equation. We are looking for the root of equation (4) in the form of

$$x = u - \frac{p}{3u} \tag{5}$$

for some real number u. Based on the formula for the cube of the sum, it turns out that

$$x^3 = \left(u - \frac{p}{3u}\right)^3 = u^3 - 3u^2\frac{p}{3u} + 3u\frac{p^2}{(3u)^2} - \frac{p^3}{(3u)^3} = u^3 - pu + \frac{p^2}{3u} - \frac{p^3}{27u^3}.$$

Putting the formulas for x and x^3 to equation (4) and reducing similar expressions, we obtain the equation

$$u^3 + q - \frac{p^3}{27u^3} = 0.$$

Multiplying equation (4) both sides by u^3 and introducing a new variable $z = u^3$ we obtain the equation

$$z^2 + qz - \frac{p^3}{27} = 0 \tag{6}$$

called the resolvent (Latin: resolvo - *solve*) or the solving equation of equation (4). This is an ordinary quadratic equation, in case of which the number of solutions depends on the value of the discriminant (4). If the discriminant is positive, the roots are expressed by the formula

$$z_{1,2} = \frac{-q \pm \sqrt{\Delta}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Recall

$$z_1 z_2 = -\frac{p^3}{27}. \tag{7}$$

Formula (7), i.e. Viete's formula, will also be important for further considerations. The roots of equation (4) will thus have the form

$$x_i = \sqrt[3]{z_i} - \frac{p}{3\sqrt[3]{z_i}} = \sqrt[3]{z_i} + \sqrt[3]{-\frac{p^3}{27z_i}}, \quad i = 1, 2.$$

It is not difficult to notice that after inserting formula (7) into the last formula, we obtain the equality

$$x_1 = x_2 = \sqrt[3]{z_1} + \sqrt[3]{z_2},$$

meaning, we obtain **one** root of equation (4) by doing so. When all formulas are combined, it is expressed by the formula

$$x_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{8}$$

This is the Cardano's formula in question. In this way, in the case when the discriminant of equation (6) is non-negative, we can determine the root of the equation (4). However, if the discriminant is negative, this procedure goes away. Such a case is referred to as "casus irreducibilis".

3. More about Cardano formulas

The reasoning presented above, however, does not take care of everything. Cardano's formula allows us to determine one solution to the equation (4). But is it the only one? Based on the reasoning presented in the previous chapter, it appears that it is the only one among the numbers that can be represented in the form (5). Nevertheless, is it possible to present every real number in this manner? This is the case when $p \geq 0$ (Figure 1)

Whereas for p < 0, $p < 0$, the only values of x that can be represented in such a form are numbers from the interval $(-\infty, \frac{2}{\sqrt{3}}\sqrt{-p}] \cup [\frac{2}{\sqrt{3}}\sqrt{-p}, \infty)$, i.e. those meeting the inequality

$$x^2 \geq \frac{-4}{3}p. \tag{9}$$

(Figure 2) In this situation, using the fact that x_0 satisfies equations (4) and (9), let us perform the following calculations

$$\begin{aligned} x^3 + px + x &= x^3 + px + q - x_1^3 - px_1 - q = x^3 - x_1^3 + p(x - x_1) \\ &= (x - x_1)(x^2 + xx_1 + x_1^2) + p(x - x_1) = (x - x_1)(x^2 + x_1 x + x_1^2 + p). \end{aligned}$$

After reducing the quadratic polynomial to canonical form, we obtain equation

$$x^3 + px + q = (x - x_1)\left(\left(x + \frac{1}{2}x_1\right)^2 + \frac{3}{4}x_1^2 + p\right)^2$$

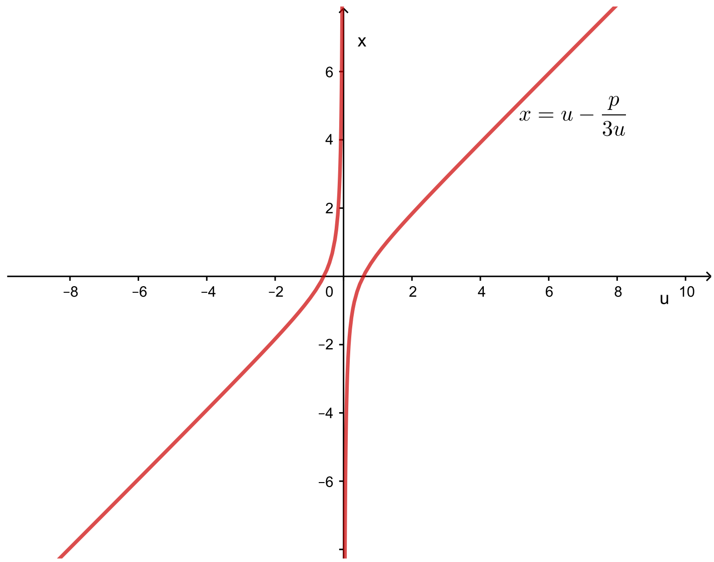

Figure 1: Graph of the function $x = u - \frac{p}{3u}$ for $p > 0$

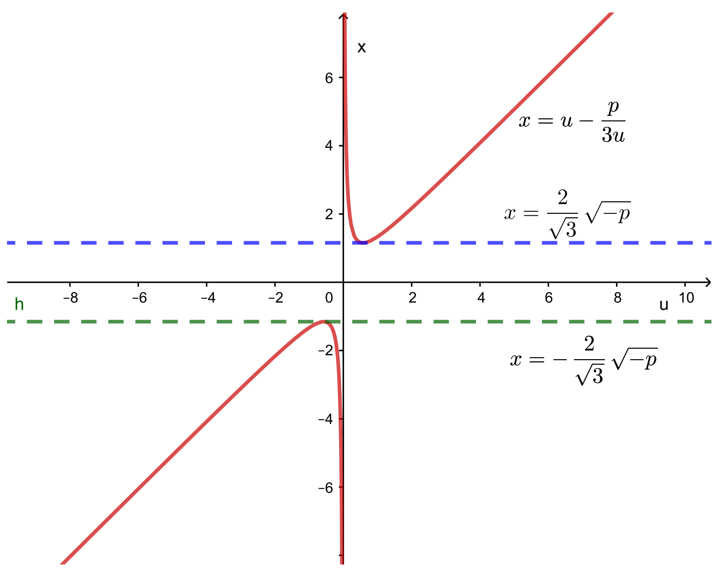

Figure 2: Graph of the function $x = u - \frac{p}{3u}$ for $p < 0$

In case of a sharp inequality in the formula (9), based on that, x_1 is the only root of the equation (4), since the quadratic polynomial which is a divisor of the left-hand side of (4) does not have zeroes. However, when

$$x_1^2 = \frac{-4}{3}p \tag{10}$$

besides x_1, there is also a double root equal $-\frac{x_1}{2}$. However, then there is the equality

$$0 = x_1^3 + px_1 + q = \frac{-4}{3}px_1 + px_1 + q = -\frac{1}{3}px_1 + q,$$

i.e.

$$\frac{1}{3}px_1 = q.$$

After squaring both sides and inserting (10), we obtain the equality

$$\frac{-4}{3} \cdot \frac{1}{9}p^3 = q^2,$$

and this means that the discriminant of the resolvent is equal to zero.

Summing up, if the discriminant of equation (6) is positive, equation (4) has one solution expressed by equation (8), while if this discriminant is zero, the equation has a single root equal to $-2\sqrt[3]{\frac{q}{2}}$, and a double root equal to $\sqrt[3]{\frac{q}{2}}$. Finally, it is necessary to consider a situation where the discriminant is negative.

4. Casus irreducibilis

If the discriminant of equation (6) is negative, it has no real roots and, consequently, equation (4) has no roots which can be presented in the formula (5), which, however, does not exclude it from having roots. We will use trigonometric functions as a tool to create an algorithm. Let us first recall the formula which, as it will turn out, will determine our success. This is the triple-angle cosine formula, i.e

$$\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi.$$

We will look for the root of equation (4) in the form $x = \rho\cos\varphi$, where ρ will be some real number. The equation (4) then takes the form

$$\rho^3\cos^3\varphi + p\rho\cos\varphi = q \tag{11}$$

Now, one can see what needs to be done. Find a ρ such that ρ^3 and $p\rho$ would be proportional to 4 and -3, respectively, i.e., solve the equation

$$-3\rho^3 = 4p\rho. \tag{12}$$

Clearly, $\rho = 0$ satisfies this equation, but this does not lead to a solution, except in the trivial case where $q = 0$. When we assume that $\rho \neq 0$, the equation takes the form

$$\rho^2 = -\frac{4}{3}p$$

It is important to note that this equation has roots, because the discriminant of the equation (6) can only be negative if p is negative. Thus, the non-zero roots of equation (12) can be $\frac{2}{\sqrt{3}}\sqrt{-p}$ and $-\frac{2}{\sqrt{3}}\sqrt{-p}$. Since $\rho\cos\varphi = (-\rho)\cos(\pi-\varphi)$, it can be assumed that

$$\rho = \frac{2}{\sqrt{3}}\sqrt{-p}.$$

By inserting the last equality into equation (11), we obtain the equation

$$\frac{8}{3\sqrt{3}}p\sqrt{-p}\cos^3\varphi - \frac{2}{\sqrt{3}}p\sqrt{-p}\cos\varphi + q = 0$$

Dividing the equation on both sides by $\frac{2}{3\sqrt{3}}p\sqrt{-p}$ we obtain the equation

$$4\cos^3\varphi - 3\cos\varphi = \frac{3\sqrt{3}q}{2p\sqrt{-p}},$$

i. e.

$$\cos 3\varphi = \frac{3\sqrt{3}q}{2p\sqrt{-p}}.$$

This equation has a solution if and only if its right-hand side belongs to the interval $[-1,1]$. This means that the square of this expression is less than or equal to one, i.e.

$$\frac{27q^2}{-4p^3} \leq 1.$$

The last formula is nothing but an inequality

$$\Delta_{(6)} \leq 0,$$

which is something we assumed at the beginning of this chapter. Of course, we are interested, in the situation when the discriminant is negative. In such a case, one can find such an angle ψ that

$$\cos\psi = \frac{3\sqrt{3}q}{2p\sqrt{-p}}. \tag{13}$$

Based on the properties of trigonometric functions, we know that the set of solutions of equation (13) has the following form

$$\{\psi + 2k\pi : k \in \mathbb{Z}\} \cup \{-\psi + 2k\pi : k \in \mathbb{Z}\},$$

therefore the set of roots for equation (4) has the form

$$\left\{\frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi + k\frac{2\pi}{3}\right) : k \in \mathbb{Z}\right\} \cup \left\{\frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(-\frac{1}{3}\psi + k\frac{2\pi}{3}\right) : k \in \mathbb{Z}\right\}.$$

We know that the cos function is even and periodic with the period 2π. Having considered that, the set presented above is actually equal to

$$\left\{\frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi\right), \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi + \frac{2\pi}{3}\right), \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi + \frac{4\pi}{3}\right)\right\}.$$

We still have to determine its number of elements. In order to do this, we will check when two of the three formulas define the same number. In order to do this, it is sufficient to solve the equation

$$\cos\left(\alpha + \frac{2\pi}{3}\right) = \cos\alpha. \tag{14}$$

Putting as α in order $\frac{1}{3}\psi$, $\frac{1}{3}\psi + \frac{2\pi}{3}$, $\frac{1}{3}\psi + \frac{4\pi}{3}$ we keep all the values α for which the set of roots has less than three elements. So let's do the calculation

$$\cos\left(\alpha + \frac{2\pi}{3}\right) - \cos\alpha = -2\sin\left(\alpha + \frac{\pi}{3}\right)\sin\frac{\pi}{3}.$$

The equation (14) is thus satisfied when $\sin\left(\alpha + \frac{\pi}{3}\right) = 0$, that is

$$\alpha = -\frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z}.$$

After successively inserting $\frac{1}{3}\psi$, $\frac{1}{3}\psi + \frac{2\pi}{3}$ for α, it turns out that must be an integer multiple of , that is, $\cos\psi \in [-1, 1]$.

But this means that $\frac{27q^2}{4p^3} \leq 1$, meaning $\Delta_{(6)} = 0$. Based on that, when $\Delta_{(6)} < 0$, the equation (4) has three different real roots.

5. Conclusion

By using this method, we can find the solution of the equation (4), depending on the value of the discriminant of the equation (6). Specifically:

1. When $\Delta_{(6)} > 0$, the equation (4) has one element expressed by the formula (8) i. e.

$$x_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

2. When $\Delta_{(6)} < 0$, the equation (4) has three different roots equal to

   - $$x_1 = \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi\right)$$
   - $$x_2 = \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi + \frac{2\pi}{3}\right)$$
   - $$x_3 = \frac{2}{\sqrt{3}}\sqrt{-p}\cos\left(\frac{1}{3}\psi + \frac{4\pi}{3}\right),$$

   respectively, where ψ is such that $\cos\psi = \frac{3\sqrt{3}q}{2p\sqrt{-p}}$

3. When $\Delta_{(6)} = 0$ the equation (4) has a single root equal to

$$x_1 = -2\sqrt[3]{\frac{q}{2}}$$

and the double root

$$x_2 = \sqrt[3]{\frac{q}{2}}.$$

References

Cardano, G.: 1545, *Artis Magnae sive de Regulis Algebraicis Liber Unus*, Nörnberg.

Cardano, G.: 1968, *The Great Art; or The Rules of Algebra. Translated and edited by T. Richard Witmer*, MIT Press, Cambridge, Massachusetts.

Hermite, C.: 1842, Considérations sur la résolution algébrique de l'équation du 5e degré, *Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale*, Série 1, Tome 1, Carilian-Goeury et Vor Dalmont, 329–336.

Lagrange, J.: 1770–1771, *Réflexions sur la résolution algébrique des équations, OEuvres complètes*, Tome 3, 205–421 (volume), Nouveaux mèmoires de l'Académie royale des sciences et belles-lettres de Berlin.

Lang, S.: 2002, *Algebra*, Springer, New York.

Opial, Z.: 1974, *Algebra wyższa*, PWN, Warszawa.

Więsław, W.: 1988, Rozwój teorii równań algebraicznych, w: S. Fudali (red.), *Matematyka XIX wieku, Materiały z II Ogólnopolskiej Szkoły Historii Matematyki*, Sczecin.

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