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# Decoding Book II of the *Elements**

**Abstract.** The paper is a commentary to the Polish translation of the *Elements* Book II, included in this volume. We focus on relations between figures represented and not represented on diagrams and identify rules which enable Euclid to bridge these two kinds of objects. Also, we argue that the main mathematical problem addressed in Book II is constructing a leg of a right-angled triangle, given its hypotenuse and the other leg. In proposition II.14, Euclid solves it through the construction called the geometric mean. We trace the problem in Book III and beyond the *Elements*: in Heron's *Metrica*, Descartes' *La Géométrie*, and modern foundations of mathematics. We show that Descartes, by novel interpretation of the Pythagorean theorem, provides a modern solution to this problem.

## 1. Overview of Book II

### 1.1. Visible and invisible figures

Book II consists of two definitions and fourteen propositions. The first definition introduces the term *parallelogram contained by*, the second – *gnomon*.[1]

Same as *triangle* and *circle*, *gnomon* refers to an individual figure featured on a diagram. *Parallelogram contained by* is the context-sensitive term. It may refer

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[1] All English translations of the *Elements* after (Fitzpatrick, 2007). Occasionally we modify Fitzpatrick's version by skipping interpolations, most importantly, the words related to addition or sum. Still, these amendments are easy to verify, as this edition is available on the Internet and also provides the Greek text of the classic Heiberg's edition (Heiberg, 1883). As regards diagrams, we also rely on Heiberg's edition. (Saito, 2011) replicates diagrams from Greek manuscripts of Book II. They are not as different to those from Heiberg's edition to undermine our interpretation.

to a figure represented (visible) or not represented on a diagram (invisible). That is a new phenomenon with no analog in Book I, which discusses only visible figures.[2]

To elaborate, all parallelograms considered as *contained by* are rectangles. And indeed, *rectangle contained by* (τὸ περιεχὸμενον ὀρθογώνιον ὑπό) and its twin term *square on* (τετράγωνον ἀπὸ) are basic concepts of Book II. Let start with the latter. Proposition I.46, introducing it, constructs a square on a given line segment. *Square on* is an abbreviation of the phrase 'square described on the line', say $AB$. At times the square on $AB$ is represented on the diagram accompanying a proposition, other times, the diagram includes only the line $AB$. A square featured on a diagram is set apart from other lines by capital letters located next to its vertices. In a proposition, it is denoted by these four letters, or letters designating a diagonal, or as the *square on AB*. A square not represented on a diagram is named solely as a *square on*. It is easy to complete it, so the absence of respective sides is seemingly motivated by a diagrammatic economy.

Rectangles on diagrams are bordered by letters placed next to their vertices. In the text, they are named by these letters, or letters standing on diagonals, or as *rectangles contained by*, then names of sides containing a right angle follow. Usually the term *rectangle contained by* is shortened to *contained by* or simply one-word *by*. However, this term also refers to contiguous line segments. Such an object can not be featured on a diagram at all. Euclid seeks to demonstrate that it is equal to some figure featured on the diagram. That is the substance of problems addressed in propositions II.1–8.

To illustrate this naming technique and its relation to visible and invisible figures, let us turn to proposition II.2 (see Fig. 1). Therein, the square $ADEB$ is also called the square $AE$, as well as the *square on AB*. The rectangle $AF$ is also called the rectangle (contained) by $AD, AC$. Euclid claims that it is equal to the invisible rectangle contained by $AB, AC$.

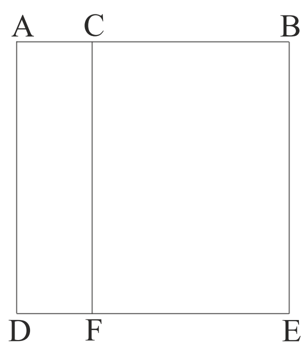


Figure 1: *Elements*, II.2.

[2]Ken Saito's (Saito, 2004) introduced the term invisible figures. It was explored further by Leo Corry's (Corry, 2013). In Appendix II, we present rules relating visible and invisible figures. One may view them as a next step in the study of invisible figures.

While naming squares by vertices or diagonal letters is a convention, the term *square on* links-up with an implicit rule concerning area, namely

$$X = Y \Rightarrow X^2 = Y^2,$$

where $X^2$ stands for the square on the line $X$.

With rectangles, similarly, naming by letters on a diagonal or as *contained by* lines containing a right angle is a convention. Yet, the term *rectangle contained by* is related to the following (implicit) rule

$$X.Y \ \& \ Y = Z \Rightarrow X.Y = X.Z,$$

where symbol $X.Y$ stands for rectangle contained by line segments $X, Y$ containing a right angle, while lines $X, Z$ are contiguous. A diagram features the rectangle $X.Y$, yet, the rectangle contained by $X$, $Z$ is invisible. The following argument exemplifies this rule "And AF is the rectangle contained by BA, AC. For it is contained by DA, AC, and AD is equal to AB". In symbols

$$AF = DA.AC \ \& \ AD = AB \Rightarrow AF = BA.AC.$$

In Appendix II, we present implicit rules relating visible and invisible figures. To our knowledge, it is the first try to unravel that strange relationship. The problem of visible-invisible figures finds its final solution, we believe, through the modern axioms for commutative rings. Yet, we are not developing this thread. It requires a separate study. Let us just note that contemporary accounts of the Euclid system underestimate that problem. Here is how, for example, Robin Hartshorne views Book II: "In Book II, all of the results make statements about certain figures having equal content to certain other, and all of these are valid in our framework" (Hartshorne, 2000, p. 203). He defines the figure as follows: "A rectilinear figure (or figure in short) is a subset of the plane that can be expressed as a finite nonoverlapping union of triangles" (Hartshorne, 2000, p. 196). However, Euclid's invisible figures are not subsets of Euclid's plane. They also do not meet Euclid's definitions of the figure, Book I, def. 14, 19, which applies but to visible figures.

### 1.2. Three groups of propositions

We identify three groups of propositions in Book II: II.1–8 are lemmas, II.9–10 exercise use of the Pythagorean theorem, II.11–14 provide substantial results by combining lemmas and the Pythagorean theorem.

Although *Elements* is considered an epitome of the axiomatic derivation of propositions, II.1–8 do not meet this stereotype. Surprisingly, they all start with relations based on visual evidence rather than axioms or previous results. Proofs of II.9–10 apply I.47, II.11–14 start with one of II.4–7. Thus, II.9–14 fulfill, more or less, current rigors of deduction.

### 1.3. Starting from visual evidence. II.1–8

In this section, we go through propositions II.1–8. They share the same general pattern, namely: starting from an obvious relation spotted in a figure represented on a diagram, Euclid finds a similar one but rephrased in terms of invisible figures.

II.1 starts with equality between rectangles (named by letters on diagonals) featured on the accompanying diagram: "BH is equal to BK, DL, and EH". Its conclusion consists of equality of invisible figures: rectangles contained "by A and BC is equal to by A and BD, by A and DE, and, finally, by A and EC" (see Fig. 2).

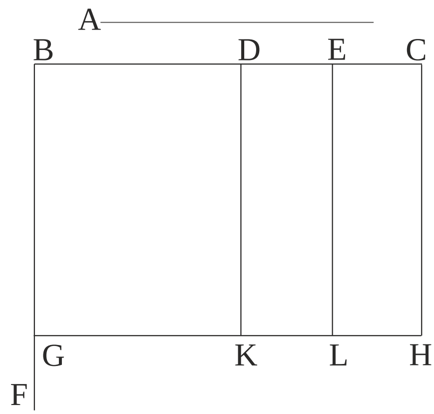


Figure 2: *Elements* II.1.

Below we present it in a more schematic form, following the convention that symbol $A.BC$ stands for the phrase "rectangle contained by A,BC".

$$BH = BK, DL, EH$$
$$\vdots$$
$$\rightarrow \quad A.BC = A.BD,\ A.DE,\ A.EC.$$

The above scheme, to emphasize the visible-invisible relationship, represents only the starting point and conclusion while skips in-between argument.

The proof, thus, begins with self-evident equality between rectangles, which rests on the dissection of $BH$ into $BK, DL, EH$. The conclusion concerns invisible objects. They inhabit a domain where dissection plays no role.

Schemes of proposition II.2,3, similarly, start with equations between visible rectangles: "AE is equal to the AF, CE" (II.2), "AE is equal to the (rectangle) AD and the (square) CE" (II.3). Their conclusions involve invisible rectangles characterized as *contained by* and squares (see Fig. 3).

$$AE = AF,\ CE \qquad\qquad AE = AD,\ CE$$
$$\vdots \qquad\qquad \vdots$$
$$\rightarrow \quad AB.BC,\ BA.AC = AB^2. \qquad\qquad \rightarrow \quad AB.BC = AC.CB,\ BC^2.$$

In II.4, the visually obvious assertion occurs at the end of the proof. From the logical perspective, it is the starting point of that argument. Like in previous

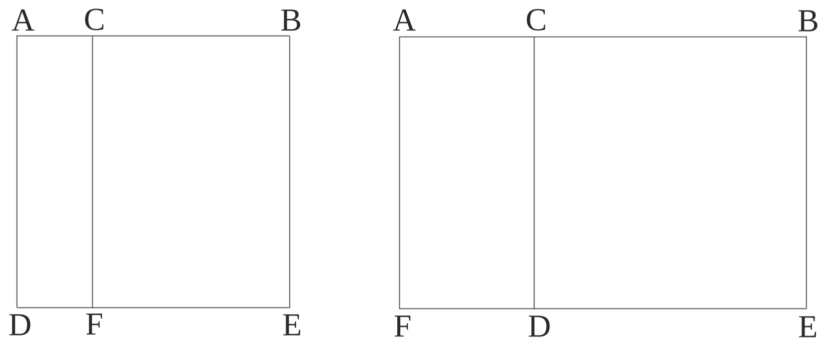


Figure 3: *Elements* II.2 (left) and II.3 (right).

schemes, we focus on the relation between visible and invisible figures. The phrase "HF, CK, AG, and GE are the whole of ADEB, which is the square on AB" refers to figures represented on the diagram. The following one "the square on AB is equal to squares on AC and CB, and twice the rectangle contained by AC and CB" – to not represented (see Fig. 4).

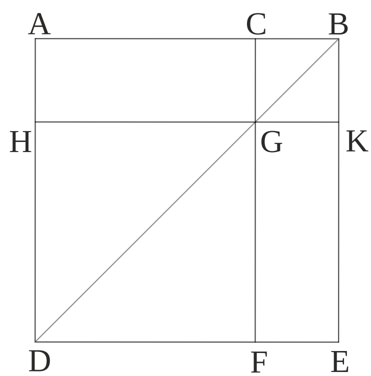


Figure 4: *Elements* II.4.

$$HF, CK, AG, GE = ADEB = AB^2$$
$$\vdots$$
$$\rightarrow \quad AB^2 = AC^2, CB^2, 2AC.CB.$$

Diagrams II.1–3 are pretty simple. They represent dissections of rectangles and squares. II.4, additionally, includes a diagonal. As we proceed further, enriched with dashed circle-like lines identifying gnomons, they get more complicated.

Euclid defines gnomons as follows: "And for any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon". To give an example, in Fig. 5, the gnomon $NOP$

on the left consists of the square $DHMB$ plus rectangles $CLHD$ and $HGFM$. A modern reader could view it also as a difference between the squares $CEFB$ and $LEGH$. Strangely enough, for Euclid, these two options are not equivalent – we detail it in section §2.1 below discussing axiom Common Notions 3.

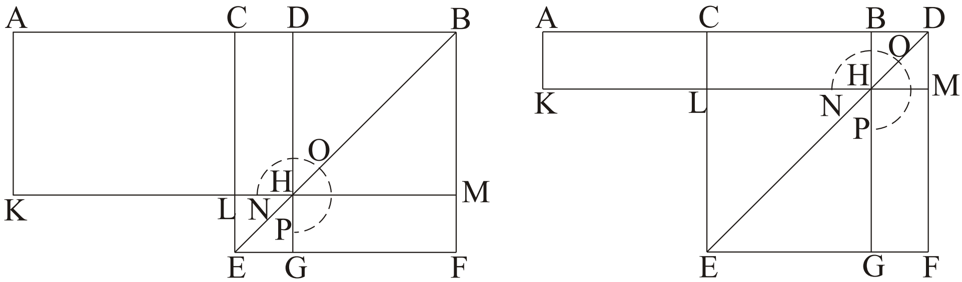


Figure 5: *Elements* II.5 (left) and II.6 (right).

Proofs II.5, 6 are substantially the same in terms of the role of gnomons (see Fig. 5). At first, Euclid shows that a rectangle is equal to a gnomon: "the whole (rectangle) AH is equal to the gnomon NOP". Then, gnomon $NOP$ taken together with a square $LEGH$ forms a bigger square. In II.5, "the gnomon NOP and the (square) LG is the whole square CEFB",

$$NOP,\ LG = CEFB.$$

Identical phrase and equality occurs in II.6, namely

$$NOP,\ LG = CEFD.$$

II.7,8 include yet more complex compositions (see Fig. 6). Here are symbolic formulations of their theses:

$$AB^2, BC^2 = 2AB.BC,\ CA^2 \qquad 4AB.BC,\ AC^2 = (AB + BC)^2.$$

In II.7, squares $AB^2$ and $CB^2$ overlaps. Diagram II.8 does not represent neither the square $(AB + BC)^2$, nor the line $(AB + BC)$.

Proof of II.7 includes the following visual argument: "the gnomon KLM, and the square CF, is double the (rectangle) AF",

$$KLM,\ CF = 2AF.$$

In II.8, the crucial step reads: "Thus, (rectangle) AG is also equal to (rectangle) RF. Thus, the four (rectangles) AG, MQ, QL, and RF are equal to one another. Thus, the four (taken together) are quadruple AG. And it was also shown that the four (squares) DK, CK, GR and RN (taken together) are quadruple (square) CK. Thus, the eight (figures taken together), which comprise the gnomon STU, are quadruple (rectangle) AK". In symbols

$$DK, CK, GR, RN, AG, MQ, QL, RF = STU = 4AK.$$

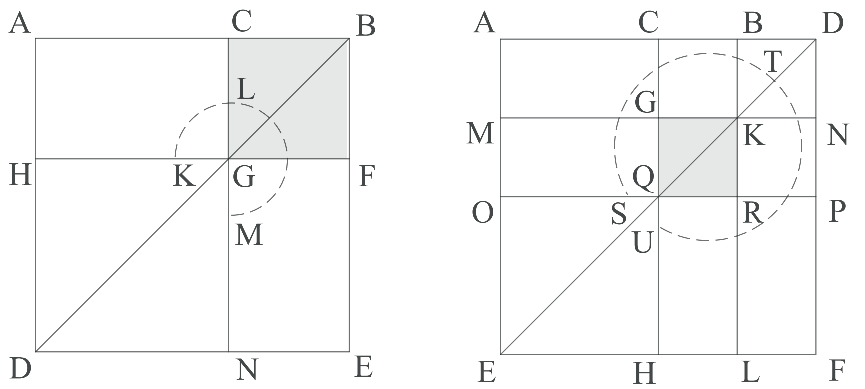


Figure 6: *Elements* II.7, 8.

The first equality $DK, ..., RF = STU$ is based on visual evidence. The equality $STU = 4AK$ is controversial. We can pair rectangles and squares as follows

$$(AG, CK)\ (MQ, GR)\ (RF, KP),\ (QL, GR).$$

However, getting four copies of $AK$ in this way, $GR$ is counted twice. Another option is to combine the square $BN$ and the rectangle $QL$. That proof, yet, requires a dissection combined with a translation.

A geometrical characteristic of arguments we call visual evidence is sometimes related to dissections. In the 19th and 20th century geometry, the term dissection referred to the decomposition of a figure as a union of non-overlapping figures that can be reassembled into another figure.[3] That technique is related to the famous Wallace-Bolayi-Gerwien theorem.[4] Euclidean dissections are quite different. They provide separate views on the same figure: it is a simple whole, say, a rectangle, or consists of two rectangles. Thus, one compares two aspects of the same figure rather than two figures.

### 1.4. Starting from the Pythagorean theorem. II.9–10

Propositions II.9,10 do not apply visual evidence (see Fig. 7). Despite they concern squares, their diagrams represent only sides of right-angled triangles. Symbolic representations of their theses are identical,

$$AD^2, DB^2 = 2(AC^2,\ CD^2), \qquad AD^2, DB^2 = 2(AC^2,\ CD^2),$$

yet, they realize different division modes, and relate other squares.

Their proofs (tallied below in columns) exercise variations on the Pythagorean theorem. Below we focus on the reference schemes. In II.9, the following phrases

[3]See (Hartshorne, 2000, chapter 22).
[4]See (Giovannini, 2021).

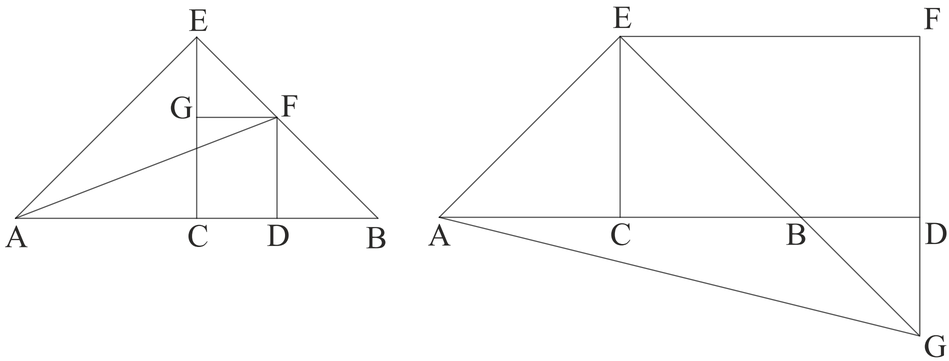


Figure 7: *Elements* II.9, 10.

evoke I.47: "And the square on EA is equal to the squares on AC, CE. For angle ACE (is) a right-angle". In II.10, they are in an abridged form: "the (square) on EA is equal to the (squares) on EC, CA".

$$
\begin{array}{ll}
\underset{I47}{\longrightarrow} & AE^2 = 2AC^2 \\
\underset{I47}{\longrightarrow} & EF^2 = 2CD^2 \\
\underset{I.47}{\longrightarrow} & AE^2, EF^2 = AF^2 \\
\rightarrow & AF^2 = 2(AC^2, CD^2) \\
\underset{I47}{\longrightarrow} & AF^2 = AD^2, DB^2 \\
\rightarrow & AD^2, DB^2 = 2(AC^2, CD^2).
\end{array}
\qquad
\begin{array}{ll}
\underset{I47}{\longrightarrow} & AE^2 = 2AC^2 \\
\underset{I47}{\longrightarrow} & EG^2 = 2CD^2 \\
\underset{I.47}{\longrightarrow} & AE^2, EG^2 = 2(AC^2, CD^2) \\
\rightarrow & AG^2 = 2(AC^2, CD^2) \\
\underset{I47}{\longrightarrow} & AG^2 = AD^2, DB^2 \\
\rightarrow & AD^2, DB^2 = 2(AC^2, CD^2).
\end{array}
$$

These proofs do not rely on visual evidence. Yet, when we step back to proposition I.47, one of its arguments is of that kind (see Fig. 8). The relevant part reads: "Thus, the parallelogram BL is also equal to the square GB. So, similarly, [...] the parallelogram CL can be shown, (to be) equal to the square HC. Thus, the whole square BDEC is equal to two squares GB, HC".

Euclid's theory of equal figures justifies equalities of non-congruent figures like rectangle $BL$ and square $GB$, on the one hand, and rectangle $CL$ and the square $HC$, on the other. However, the equality $BL, CL = BDEC$, actually skipped by Euclid, exemplifies visual evidence.

## 1.5. Starting from II.4–7

In the next section, discussing the main results of Book II, we focus on references to propositions II.4,7. They form pairs such as II.11,6, II.12,4, II.13,7, II.14,5, which share the way of cutting, let us name it the basic line.

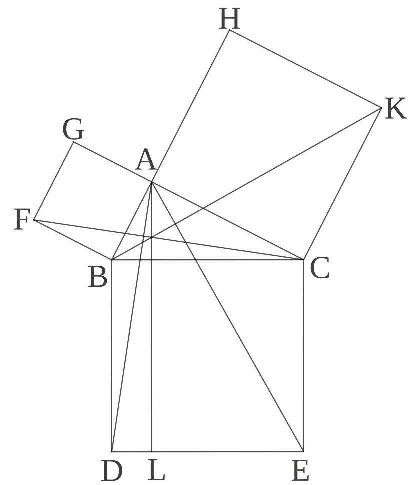


Figure 8: *Elements* I.47

Euclid's technique of citing propositions II.4–7 is this: a specification on placing new points on the basic line follows the phrase *for since the straight-line ... has been cut ...*. It can be cut in three ways: *at random* (*unequally*), *in half* (*equally*), or a new line is added to the basic one. Finally, the thesis of the cited proposition adopts the names of points employed in the accompanying diagram.

For example, the respective part of II.14 reads: "since the straight-line BF has been cut equally at G, and unequally at E, the rectangle contained by BE, EF, together with the square on EG, is thus equal to the square on GF". While II.5 states: "For let any straight-line AB have been cut equally at C and unequally at D. I say that the rectangle contained by AD, DB, together with the square on CD, is equal to the square on CB". These statements differ only in names of points; the way $G$, $E$ or $C$, $D$ cut the line $BF$ or $AB$ is the same.

Similarly, the following phrases link II.11 and II.6: "the straight-line AC has been cut in half at E, and FA has been added to it" (II.11). The respective phrase in II.6 is this: "let any straight-line AB have been cut in half at point C, and let any straight-line BD have been added to it straight-on".

Both II.12 and II.13 share the same word pattern, namely "the straight-line CD has been cut, at random, at point A" (II.12), "let the straight-line AB have been cut, at random, at C" (II.4,7).

## 1.6. Main results. II.11–14

Book II is commonly identified with its main results – propositions II.11–14. Indeed, II.11, the so-called golden ratio construction, is the crucial step in the cosmological plan of the *Elements*. It enables the construction of the regular pentagon, and finally, the dodecahedron. II.12,13 provide the cosine rule. They do not play any role in the general plan of the *Elements*, but have gained importance with the development of trigonometry in early modern mathematics.[5] II.14, the squaring of a polygon, crowns Euclid's theory of equal figures.

Proof II.11 builds on II.6. Fig. 9 presents a diagrammatic scheme of a relation between these propositions. Grey square on the left represents "the square on half" (II.6), on the right – "the square on AE" (II.11), given $CE = EA$. In this and the next figures, we add dashed lines to the original diagrams.

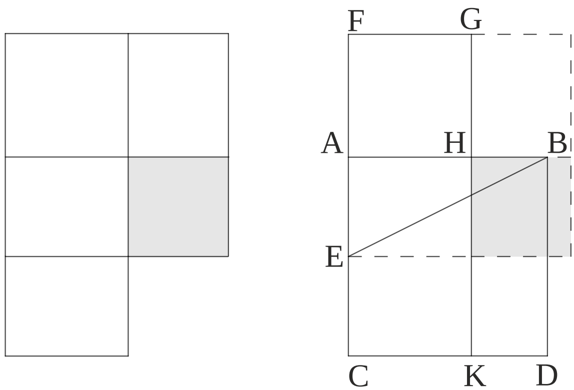


Figure 9: The scheme of application of II.6 in II.11.

The following scheme interprets the beginning of proof II.11.[6] Here, the basic line, $CA$, is cut in half at $E$ and $AF$ is added. As a result, by II.6, rectangle contained by $CF$ and $FA$ and the square on $AE$ are equal to the square on $EF$.

$$\begin{array}{rcl} & \xrightarrow[II.6]{} & CF.FA,\ AE^2 = EF^2 \\ EF = EB & \rightarrow & CF.FA,\ AE^2 = EB^2 \\ & \xrightarrow[I.47]{} & EB^2 = AE^2, AB^2 \\ \angle A = \pi/2 & \rightarrow & CF.FA,\ AE^2 = \ AE^2, AB^2 \\ & \xrightarrow[CN3]{} & CF.FA = AB^2. \end{array}$$

In this and other schemes, we adopt the following conventions (some of these symbols we already applied):

| | |
|---|---|
| $CF.FA$ | interprets the phrase "rectangle contained by CF, FA" |
| $EF^2$ | interprets the phrase "square on [the line] EF" |
| $\rightarrow$ | stands for a conjunction, usually it is γάρ |
| $\xrightarrow[II.6]{}$ | signals the explicit reference to proposition II.6 |
| $CN$ | stands for Common Notions |
| $\angle A = \pi/2$ | stands for "the angle at A (is) a right-angle". |

[5] In §5.1, we discus Heron's interpretation of these propositions.

[6] In our schemes, we keep the original order of the letters. It happens that in the same sentence, Euclid renames the line segment from AC to CA, etc.

The rest of this proof could proceed like that: since $CF.FA$ is the rectangle $FCKG$, and square $AB^2$ is the square $ACDB$, subtracting the rectangle $ACKH$ from both figures, the equality $AH^2 = HKDB$ obtains. Euclid's actual proof does not stop here, for he aims to show $AH^2$ is equal to the rectangle contained by $AH$ and $BH$.[7]

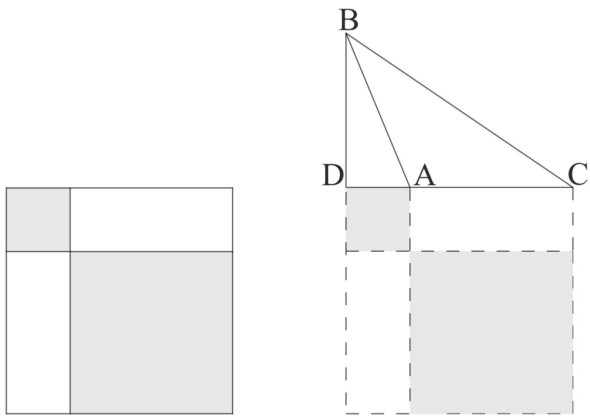


Figure 10: The scheme of application of II.4 in II.12.

Propositions II.12–13 prove the cosine rule (in the ancient Greek disguise) for an obtuse- and acute-angled triangle, respectively. Fig. 10 presents diagrammatic relations between II.4 and II.12. Grey squares on the left represent "squares on the pieces" (II.4), on the right – "squares on CA, AD" (II.12). It means that the basic line, $DC$, is cut randomly, at $A$. Here is the scheme of the proof II.12 starting with the application of II.4.

$$\begin{array}{rcl}
 & \xrightarrow[II.4]{} & DC^2 = AC^2, AD^2,\ 2CA.AD \\
 & \xrightarrow[CN2]{} & DC^2 + DB^2 = AC^2, AD^2 + DB^2,\ 2CA.AD \\
\angle D = \pi/2 & \xrightarrow[I47]{} & CD^2, DB^2 = CB^2 \\
 & & AD^2, DB^2 = AB^2 \\
 & \rightarrow & CD^2, DB^2 = CA^2, AD^2, DB^2,\ 2CA.AD \\
 & \xrightarrow[CN3]{} & CD^2 = CA^2, DB^2,\ 2CA.AD.
\end{array}$$

Fig. 11 presents diagrammatic relations between II.7 and II.13. Grey squares represent square on "one of the pieces" (II.7), and square on BD (II.13). In this case, similarly, the basic line, $BC$, is cut at random, at $D$. Here is the scheme of its proof, starting with the application of II.7.

[7]For schemes of complete proofs, see Appendix III.

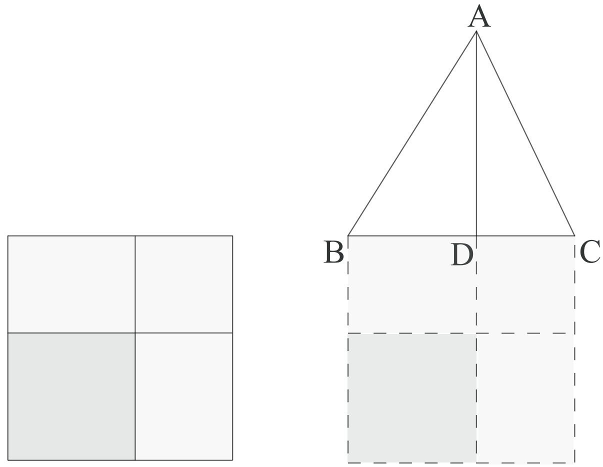


Figure 11: The scheme of application of II.7 in II.13.

$$
\begin{array}{rcl}
 & \xrightarrow[II.7]{} & CB^2, BD^2 = 2CB.BD,\ DC^2 \\
 & \xrightarrow[CN2]{} & CB^2, BD^2 + DA^2 = 2CB.BD,\ DC^2 + DA^2 \\
\angle D = \pi/2 & \xrightarrow[I47]{} & AB^2 = BD^2, DA^2 \\
 & & AC^2 = DC^2, DA^2 \\
 & \rightarrow & CB^2, AB^2 = 2CB.BD,\ AC^2.
\end{array}
$$

II.14 provides the general result of the theory of equal figures as developed in propositions I.35–45.[8] In Book I, Euclid shows how to build a rectangle equal to a given polygon. II.14 transforms a rectangle into an equal square. Thus, it is the final stage in the train of propositions leading to the squaring of a polygon. A reference to II.5 begins the proof. Grey squares in Fig. 12 represent "the square on the (difference) between (equal and unequal) pieces" (II.5), and "the square on EG" (II.14). Here, the basic line, $BF$, is cut in half at $G$, and $E$ is a random point between $G$ and $F$.

Below is the crucial part of the proof II.14 in a schematized form.

$$
\begin{array}{rcl}
 & \xrightarrow[II.5]{} & BE.EF,\ GE^2 = GF^2 \\
GF = GH & \rightarrow & BE.EF,\ GE^2 = GH^2 \\
\angle E = \pi/2 & \xrightarrow[I47]{} & GH^2 = GE^2, EH^2 \\
 & \rightarrow & BE.EF,\ GE^2 = GE^2, EH^2 \\
 & \xrightarrow[CN3]{} & BE.EF = EH^2.
\end{array}
$$

[8]See (Błaszczyk, 2018).

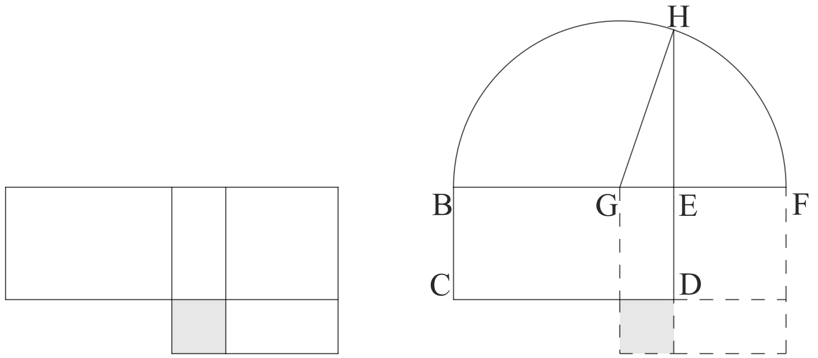


Figure 12: The scheme of application of II.5 in II.14.

## 2. Patterns

### 2.1. Following the pattern of references

We already showed that in propositions II.11–14, Euclid refers to previous theorems through the way of cutting of the basic line. Owing to that pattern, we can reveal the use of II.5 in III.35 and II.6 in III.36. Indeed, III.37, the reverse of III.36, is indispensable in constructing the regular pentagon. Therefore, Book II turns out to be the vital stage in Euclid's construction of Platonic solids.

In section §4.1 below, we discuss proofs III.35,36 with some extra analytic tools, namely an interpretation of Common Notions and a characteristic of II.14 in terms of difference of squares.

### 2.2. Division modes of line segment

This section systemizes observations on how Euclid cuts the basic line, let name it $AB$. He employs three modes of division (cutting). Line $AB$ is cut

(a) at random (II.2, 3, 4, 7, 8, 12, 13),

(b) in half at $C$ and $D$ is placed between $C$ and $B$ (II.5, 9, 14); we refer to this case by the following notation $AC = CB,\ C - D - B$,

(c) in half at $C$ and the line $BD$ is added to $AB$ (II.6, 10, 11); we refer to this case by the following notation $AC = CB,\ C - B - D$.

Names of points in the same mode can differ. In II.11, the line $AC$ is cut in half at $E$, and $AF$ is added. In II.14, $BF$ is cut in half at $G$, and $E$ is placed between $G$ and $F$.

In section §4.3, we present a hypothesis that modes (b) and (c) derive from a relationship discovered in proposition III.36, currently known as a power of a point with respect to a circle. Cutting a line in half relates to the diameter and its center. Then one considers another point lying on the diameter, or its prolongation, outside the circle.

A division mode, being tied to an individual proposition, encodes a specific relationship between squares and rectangles, as exemplified in the previous section. Below, we pair propositions per the division of the basic line.

(a) II.4, 12; II.7, 13,

(b) II.5, 14; II.9, 14,

(c) II.6, 11; II.10, 11.

Interestingly, pairs II.9,14 and II.10,11 also share the same mode of division, and indeed II.9,10 provide the basis for alternative proofs of II.14,11 respectively. In Appendix I, we base the justification of the thesis of II.14, $EH^2 = BCDE$, on II.9, instead of II.5.

Furthermore, modes (b), (c) are applied in Book III and seem more effective than (a). In section §4, we show that (b) is the fundamental mode, as it is related to the basic geometric problem addressed in Book II.

## 2.3. Basic geometric scheme

Proposition II.11 and III.36 play a crucial role in Euclid's construction of the regular pentagon. In section §4, we show that derivation of II.11 and III.36 require only I.47, II.14, and Common Notions. Therefore, from the logical perspective, II.14 is the crucial result of Book II. Moreover, in Appendix I, we show how to derive II.14 from II.9. Adopting that line of derivations, II.11 and II.14 depend on visual evidence as much as I.47 does.

Problem II.14 consists of finding a line $h$ such that the square $h^2$ is equal to a given rectangle $BCDE$. Owning to II.5, Euclid transforms it into finding sides of a right-angled triangle with hypotenuse $c$ and leg $a$ such that

$$BCDE + a^2 = c^2.$$

We may view the main problem addressed in Book II as a constructing a leg of a right-angled triangle with a hypotenuse $c$, given another leg, $a$,

$$h^2 + a^2 = c^2.$$

Euclid pays no attention to uniqueness questions, yet we can show that only one $h$ satisfies the above equation. In other words, right-angled triangles with congruent two sides (freely chosen) are congruent.

Modern solution to this problem is as simple as it can be,

$$h = \sqrt{c^2 - a^2}.$$

For the first time, it was applied in 1637 by Descartes,[9] so how Greeks dealt with it?

The solution given in II.14 seemingly consists of constructing the geometric mean $\sqrt{(c+a)(c-a)}$. Thus, still informally

$$\left[\sqrt{(c+a)(c-a)}\right]^2 = c^2 - a^2.$$

However, proposition III.35 does not support that interpretation. One can view it as solving that problem for two pairs of lines simultaneously. Still, in modern terms, in III.35, Euclid seeks to show that $b^2 - a^2 = d^2 - c^2$ (see Fig. 13 and Fig. 17). When differences of squares are allowed, that equality follows from I.47 applied to right-angled triangles with hypotenuse $d$ and $c$ (see Fig. 13),

$$c^2 - a^2 = h^2 = d^2 - b^2.$$

Then, by finding the geometric mean for the left and right side, namely

$$(b+a)(b-a) = (d-c)(d+c),$$

one gets the desired solution.[10] Yet, Euclid's actual solution does not apply a difference of squares and the geometric mean construction. Instead of II.14, he refers to I.47, II.5, and Common Notions 3.

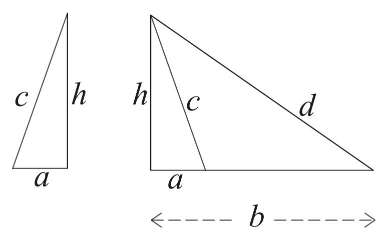


Figure 13: Basic problem of Book II.

The problem phrased in modern terms seems simple, in the *Elements* is complicated. It is because Euclid does not subtract squares. In other words, formula like $c^2 - a^2$ finds no counterpart in the *Elements*. Therefore, from the Greek perspective, the basic problem of Book II consists of finding $h$ such that $h^2 + a^2 = c^2$, although from the modern perspective, it is equivalent to finding $h$ such that $h^2 = c^2 - a^2$.

Fig. 14 illustrates the procedure of finding $h$ presented in II.14. It also represents the grey gnomon $c^2 - a^2$, equal to the grey rectangle. From the modern perspective, construction II.14 is a squaring of the grey rectangle, as well the grey gnomon. From Euclid's perspective, a gnom is only a mean for squaring a rectangle. Therefore it occurs only in lemmas II.5–8.

[9]See §6.2 below.

[10]See §4.1 below for proof of III.35 by this method.

Moreover, a justification of a solution to the main problem has to apply Euclid's means. The following identity reveals the crucial difference between the modern and ancient approach regarding the discussed problem

$$(c+a)(c-a) = c^2 - a^2.$$

In the modern framework, it is evident, in ancient, does not occur. We can view it as equality between rectangle $(c+a)(c-a)$ and gnomon $c^2 - a^2$, yet for Greeks, a gnomon is never the difference of squares.[11]

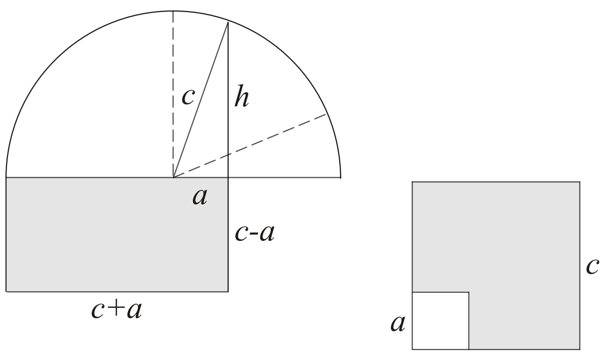


Figure 14: Finding $h$ such than $h^2 + a^2 = c^2$.

## 3. Preliminary interpretations

### 3.1. Common Notions 1 to 3

Propositions II.11,14 include references to *Common Notions*. It is a group of five axioms applicable in geometry and arithmetic. In geometry, they apply both to congruence and equal area. Here are the first three:

(CN1) "Things equal to the same thing are also equal to one another."

(CN2) "And if equal things are added (προστεϑῇ) to equal things then the wholes are equal."

(CN3) "And if equal things are subtracted from equal things then the remainders are equal."

These are, arguably, kind of foundational rules. Algebra seems to provide natural tools to interpret them. However, in modern geometry, algebra applies only to a system with a measure of figures. In other words, it applies to geometric objects through real numbers. Since Euclid's geometry develops without numbers, we

[11] See §3.2 for further discussion.

need to restrain the straightforward use of algebra. Below formalization is based on Euclid's practice captured in schemes of propositions II.1–14 included in Appendix III.

(CN1) $A = C \;\&\; B = C \Rightarrow A = B$.

(CN2) $A = B \;\&\; C = D \Rightarrow A + C = B + D$.

(CN3) $A, C = B, C \Rightarrow A = B$.

CN1 accounts for the transitivity of congruence or equal areas.

CN2 allows adding squares to squares, or rectangles, or gnomons, or rectangles *contained by*, or rectangles to gnomons, etc. Therefore it is used quite freely. The provisos $A = B$ and $C = D$ mean congruence or equal area. The only implicit restriction on the uses of CN2 is dimensional homogeneity, meaning: do not add figures to line segments or solids to figures.

On the received reading of CN3, it enables subtracting less from greater.[12] In our formalization, it gets a more restricted form of the cancellation rule. The difference makes the substance of our reading of Book II.

Propositions II.11,14 apply CN3 clearly as the cancellation rule. It is also a crucial rule in the theory of equal figures in Book I. Then, it takes a slightly more liberal form, namely

(CN3') $A, C = B, C' \;\&\; C \equiv C' \Rightarrow A = B$.

This means it enables the cancellation of congruent figures, not only the same figure. In that form, it is used in I.43.

In a more abstract perspective, we could also consider the following version

(CN3") $A, C = B, C' \;\&\; C = C' \Rightarrow A = B$.

It enables the cancellation of equal figures.

### 3.2. A piece in an alternative history of mathematics

In modern geometry, subtracting less from greater is an obvious move, is there a reason, then, for other restrictions on subtraction in the Euclid system? The analysis of Euclid's propositions, specifically our diagrams of his proofs, motivates our interpretation of CN3. But it also drives to the problem of finding a leg of a right-angled triangle given hypotenuse and another leg.

To elaborate, suppose, $r^2 = s^2 + x^2$, where $x, s$ are legs and $r$ – the hypotenuse of a right-angled triangle (see Fig. 15).[13] Can one get from it to the formula $x^2 = r^2 - s^2$? Explicitly, it is the difference between less and greater. However, for some reason, it finds no counterpart in Euclid's text. Neither expression nor sentence exemplifies this formula. Our schemes of propositions from Book II also attest the observation that Euclid does not consider a difference of squares described on sides of a right-angled triangle.

[12] See, for example, (Mueller, 2006), or (De Risi, 2021).

[13] Fig. 15 is taken from Van der Waerden's (Van der Waerden, 1961) and illustrates his interpretation of Euclid's II.14. We will discuss it at the end of this section.

Let us go beyond mere observation and try to answer why Euclid does not refer to a difference of squares. Let start our guesses with a system that allows such differences, for example, high school geometry.[14]

The arithmetic of real numbers makes the algebraic background of this system. It enables to turn the Pythagorean equality $r^2 = s^2 + x^2$ into $r^2 - s^2 = x^2$. Now, there are two possible geometric interpretations of this difference. The first is the square described on the side $\sqrt{r^2 - s^2}$, the second is a gnomon. The first solution can not be universally applied. It is possible – for example, on a Hilbert plane – that the Pythagorean formula, $r^2 = s^2 + x^2$, makes sense, yet, the other, $r^2 - s^2 = x^2$, does not.[15] So it requires further scrutiny.

The arithmetic sense of the formula $\sqrt{r^2 - s^2}$ relies on the completeness of real numbers since the existence of the square root is proved through that axiom. Its geometric substance rests on Euclid's construction II.14. More to the point, $r^2 - s^2$ is a real number, say $h$, therefore the construction consists of finding $\sqrt{h}$ by straightedge and compass. To this end, textbooks in geometry apply Descartes' interpretation of II.14, or his geometric interpretation of the formula $\sqrt{r^2 - s^2}$.[16]

Regarding the difference $r^2 - s^2$, high school geometry does not provide any geometric insights. It is justified by the rules of an ordered field, as a difference of two numbers.

An interpretation of $r^2 - s^2$ as a gnomon refers to a diagram rather than the completeness of real numbers. Turning to Euclid's II.5 and II.14, one can discern gnomon $NOP$ and its twin figure in II.14 – the square $GF^2$ minus the grey square – as a difference of squares (see Fig. 5, 12, and 14 respectively). In Euclid's approach, gnomon $NOP$ is formed by CN2 rather than subtraction. Here is the related part of II.5.

$$
\begin{aligned}
AC = CB \quad & \xrightarrow[I.36]{} \quad CM = AL \\
& \xrightarrow[CN1]{} \quad AL = DF \\
& \xrightarrow[CN2]{} \quad AL + CH = DF + CH \\
& \rightarrow \quad AH = NOP.
\end{aligned}
$$

Still in II.5, adding the square $LH^2$ to both sides of the equation $AH = NOP$, Euclid completes the gnomon $NOP$ to the square $CB^2$. The conclusion equates the rectangle $AKHD$ plus the square $LH^2$ and the square $CB^2$.

In II.14, the same relation (modulo new names of points) is the starting point, namely

$$\textit{rectangle } BCDE + \textit{grey square} = GF^2.$$

[14] We choose this model because of the grounding assumption of Victor Blåsjö's defense of the so-called geometrical algebra interpretation of Book II. It reads: "The Greeks possessed a mode of reasoning analogous to our algebra, in the sense of a standardized and abstract way of dealing with the kinds of relations we would express using high school algebra" (Blåsjö, 2016, 326).

[15] See §6.2 below.

[16] See §6.1 for Descartes construction of $\sqrt{h}$, or §6.3 for his interpretation $\sqrt{r^2 - s^2}$.

The proof proceeds further by I.47 as follows: the square $GF^2$ equals $GH^2$, and the other represents the sum of squares

$$\textit{rectangle } BCDE + \textit{grey square} = GF^2 = GH^2 = EH^2 + GE^2 = EH^2 + \textit{grey square}.$$

Finally, by CN3,

$$\textit{rectangle } BCDE = EH^2.$$

The crux of II.14, thus, consists of relating the rectangle $BCDE$ with the square on leg $EH$. To put it metaphorically, the square $GF^2$ mediates between the rectangle $BCDE$ and the squares on the sides of the triangle $GEH$. That is Euclid's actual proof.

An alternative proof could rest on the result established in II.5

$$\textit{rectangle } BCDE = \textit{gnomon } (GF^2 - \textit{grey square}).$$

Now, the question is whether the gnomon $GF^2 - \textit{grey square}$ can mediate between rectangle $BCDE$ and squares on the sides of the triangle $GEH$. To this end, one should know that

$$EH^2 = GH^2 - GE^2 = GH^2 - \textit{ grey square}.$$

It does not follow from I.47. In Euclid's proof, the square on the hypotenuse is not dissected into two squares. Yet, a proof of the Pythagorean theorem by a dissection is possible.[17] Ancient mathematical traditions like Babylonian, Chinese, Hindu, or Arabic knew only that way of proving. Euclid's I.47 is unique in terms of the technique involved.

In sum, gnomon interpretation of $r^2 - s^2$ trips us over far beyond the framework of Euclid's system, specifically his proof of the Pythagorean theorem.

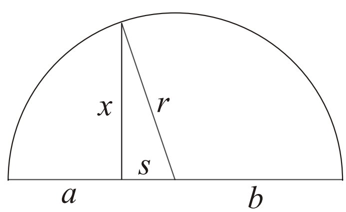


Figure 15: Van der Waerden's Fig. 32.

We can turn now to Van der Waerden's interpretation of II.14. He identifies this proposition as "the construction of the mean proportional $x = \sqrt{ab}$, by means of a semi-circle as illustrated in Fig. 32". And that is how he sums up its proof: "Euclid's proof of the proposition $x^2 = ab$ proceeds as follows:

$$x^2 = r^2 - s^2 = (r - s)(r + s) = ab.$$

We see the 'Theorem of Pythagoras' $r^2 = x^2 + s^2$, is applied here" (Van der Waerden, 1961, 118).

[17] The most suggestive in this context seems Bhaskara's proof; see (Hartshorne, 2000, p. 218).

This supposed proof has nothing to do with Euclid's actual proof. Nevertheless, let us decode it as it stands. The equality $(r-s)(r+s) = ab$ adds no information, as it simply denotes the rectangle with sides $a, b$ by different letters. The equality $r^2 - s^2 = (r-s)(r+s)$, justifiable by high school algebra, makes no sense within the Euclid system. It equates the rectangle $(r-s)(r+s)$ with an unspecified figure: a difference of squares or a gnomon. Finally, Van der Waerden does not explain how to derive the equation $x^2 = r^2 - s^2$ from the Pythagorean formula $r^2 = x^2 + s^2$. It is not self-evident. On some Hilbert planes, such derivation is invalid. Therefore, we have to get certain why the Euclid system allows such an argument.

In sum, from the perspective of credibility with the text of the *Elements*, Van der Waerden's interpretation is ill-informed. Mathematically it is uninformed. Regarding academic standards, incomplete, for it does not explain how to process the formula $r^2 = s^2 + x^2$ into $x^2 = r^2 - s^2$.

## 4. Beyond Book II

### 4.1. Proposition III.35. Euclid's and alternative proofs

In Book II, one can identify the use of II.5 and II.6 through a division mode of the basic line. The same is with propositions in Book III. In III.35, Euclid refers to II.5 as follows: "since the straight-line AC is cut equally at G, and unequally at E, the rectangle contained by AE, EC, and the square on EG, is thus equal to the (square) on GC". In symbols,

$$AG = GC,\ G - E - C \underset{II.5}{\longrightarrow} AE.EC,\ GE^2 = GC^2.$$

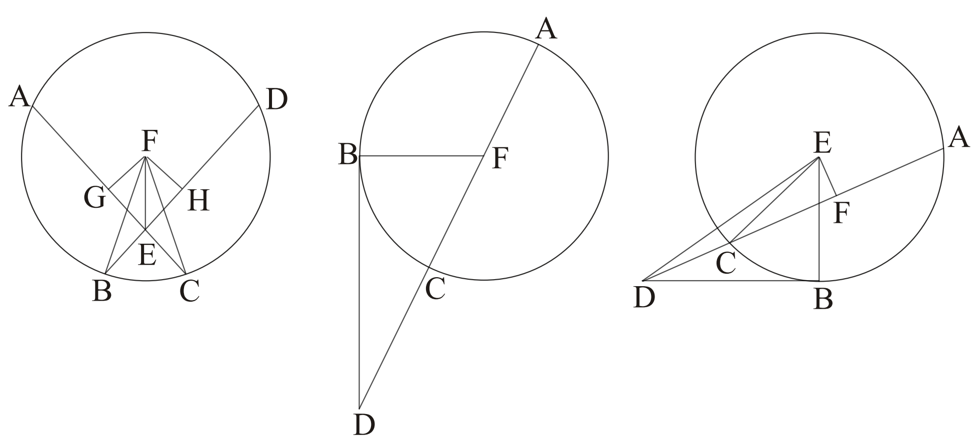


Figure 16: *Elements* III.35 (on the left), III.36 (in the middle and on the right).

The proof drives to conclusion $AE.EC = BE.ED$ (see Fig. 16). It proceeds by applying Common Notions and I.47 like that :

$$\begin{aligned} AE.EC,\ GE^2 = GC^2 &\xrightarrow[CN2]{} AE.EC,\ GE^2 + GF^2 = GC^2 + GF^2 \\ GE^2, GF^2 = EF^2 & \\ GC^2,\ GF^2 = FC^2 & \\ FB^2 = FC^2 &\xrightarrow[CN2]{} AE.EC, EF^2 = FC^2 = FB^2 \\ &\xrightarrow[CN3]{} AE.EC, EF^2 = FB^2. \end{aligned}$$

Next, II.5 applied to the line $BD$ produces similar result

$$BE.ED, EF^2 = FB^2.$$

Finally,

$$\begin{aligned} &\xrightarrow[CN1]{} AE.EC, EF^2 = FB^2 = BE.ED, EF^2 \\ &\xrightarrow[CN3]{} AE.EC = BE.ED. \end{aligned}$$

Euclid's proof is long and tiresome. It applies subtraction in the CN3 version and all throughout reiterates I.47. Note, however, that III.35 justifies a general rule regarding rectangles contained by, namely[18]

$$X = Z \ \&\ Y = W \Rightarrow X.Y = Z.W.$$

Below we present an alternative proof that applies a difference of squares and represents it by a geometric mean. Instead of two chords, we take a chord and the diameter through the point $E$ – a supposed intersection of chords (see Fig. 17).

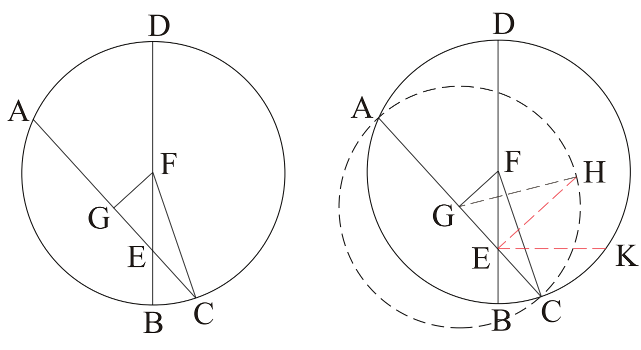


Figure 17: From the geometric mean to III.35

The right-angled triangle $FGC$ realizes the pattern described in section §2.3. Thus,

$$GC^2 - GE^2 = FC^2 - EF^2.$$

[18] In III.35, $AC$ and $BD$ are chords of a circle. To prove this rule, we need a geometric observation to the effect that lines $X + Y$ and $Z + W$ make chords.

The difference of squares $GC^2 - GE^2$ is equal to rectangle $AE.EC$. Similarly, $FC^2 - EF^2$ is equal to $DE.EB$. Therefore, the desired equality $AE.EC = BE.ED$ easily follows

$$AE.EC = EH^2 = GC^2 - GE^2 = FC^2 - EF^2 = EK^2 = BE.ED.$$

By relating a chord with the diameter through $E$, we can set the measure of any rectangle contained by line segments determined by the cutting the chord at the point $E$. Each of them is equal to $EK^2$.

The above proof, combined with Euclid's original argument suggest yet another alternative proof of III.35. This one builds on II.14. In Fig. 17, we represent geometric means of $AE$, $EC$ and $BE$, $ED$ – these are red lines $EH$ and $EK$. By II.14

$$AE.EC = EH^2, \quad DE.ED = EK^2.$$

By I.47 we obtain

$$FC^2 = EK^2 + EF^2,$$

and in addition, noting that $GC = GH$, we get

$$FC^2 = GC^2 + GF^2 = GH^2 + GF^2 = EH^2 + GE^2 + GF^2 = EH^2 + EF^2.$$

Thus,

$$EH^2 + EF^2 = EK^2 + EF^2.$$

By CN3, $EK^2 = GH^2$, which gives the thesis of III.35.

### 4.2. Proposition III.36. Euclid's proof and through III.35

In proposition III.36, Euclid seeks to show the equality $DC.DA = DB^2$ (see Fig. 16). Its proof begins with a characteristic of a division mode: "since the straight-line AC is cut in half at point F, let CD have been added to it". Then, II.6 is applied: "the (rectangle contained) by AD, DC and the (square) on FC is equal to the (square) on FD". In symbols

$$AF = FC,\ D - C - F \ \underset{II.6}{\longrightarrow}\ AD.DC,\ CF^2 = FD^2.$$

When the line $DA$ goes through the center of the circle, the proof is short and simple:

$$\begin{array}{rcl} FC = FB & & \\ FD^2 = DB^2, FB^2 & \rightarrow & AD.DC,\ FB^2 = DB^2, FB^2 \\ & \underset{CN3}{\longrightarrow} & AD.DC = DB^2. \end{array}$$

In the second case, $E$ is the center of the circle. Then, the basic line $AC$ is cut in the same way as in II.6: "the straight-line AC is cut in half at point F, let CD have been added to it. Thus, by AD and DC plus the (square) on FC is equal to the (square) on FD".

The scheme of the starting point is logically the same as in the previous case and employs identical letters

$$AF = FC,\ D - C - F \xrightarrow[II.6]{} AD.DC,\ CF^2 = FD^2.$$

The justification that follows is a long and tiresome reiteration of I.47. Below we present an alternative proof. It is based on II.14 and re-shapes III.36 as a variant of III.35, as follows

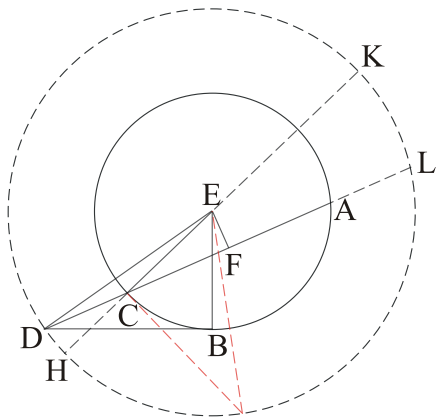


Figure 18: III.36 by III.35.

$$DC.DA = DC.CL$$

By III.35

$$DC.CL = HC.CK.$$

By II.14

$$HC.CK = DB^2.$$

The last equality follows from the fact that we can construct a right-angled triangle with the leg $CE$ congruent to $\triangle DBE$.

### 4.3. Seeking a rationale for division modes

Division modes applied in proposition II.5,6 seem haphazard. Viewed from the perspective of III.36, they find a rationale. Fig. 19 represents three versions of III.36 depending on the relationship between the line $AB$ and the radius $AE$. In each case, the equality holds

$$AB^2 = FCKG.$$

Moreover, in each case, the proof is the same, given the result of III.36, namely[19]

$$AB^2 = NB.LB = FC.AF = FCKG.$$

Therefore, we can view the division mode (c) as modeling the relationship discovered in III.36: line $AC$ is cut in half at $E$ and line $AF$ is added. Since Euclid derives III.36 from II.6, he possibly reached this mode by reverse engineering.

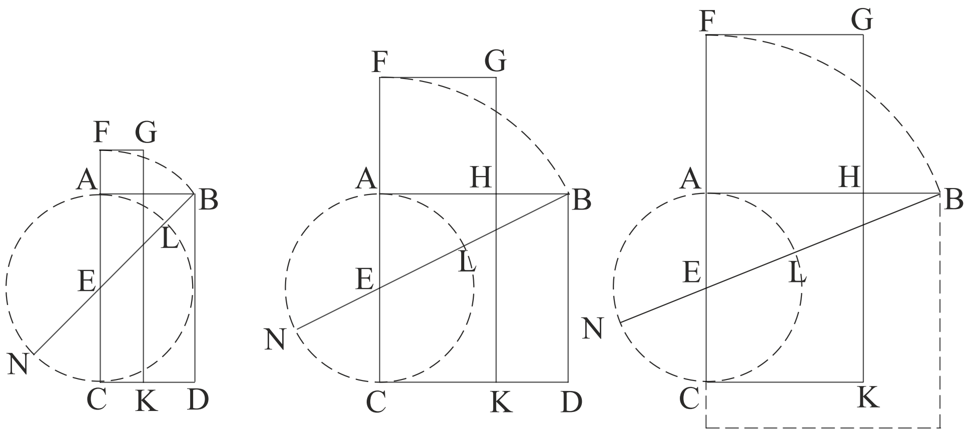


Figure 19: Three versions of III.36: $AB = AE$, $AB = 2AE$, and $AB > 2AE$.

We can also consider variants of III.36 with the line parallel to the diameter $AC$ going through the point $L$ (see Fig. 20). Then, projecting $L$ on $AC$, we gain the relationship discovered in II.14. More specifically, by III.36, the equality $AH^2 = HL.HN$ obtains. Since $AH = GL$, $HL = AG = FG$, and $HN = GC$, we can make over this result as the thesis of II.14, namely[20]

$$HN.HL = AH^2 = GL^2 = AG.GC = FKCG.$$

Again, one can view the division mode (b) as modeling the relationship discovered in this specific variant of III.36: the line $AC$ is cut in half at $E$, and $G$ is placed between $A$ and $E$.

### 4.4. Beyond visual evidence

Alternative proofs and speculations of this section aim to reveal the fundamental role of proposition II.14 and the division mode (b) it exploits when viewed from a logical perspective. Thus, III.35 follows from II.14. Then, III.36 is a particular case of III.35. Next, II.11 follows from III.36.

[19]Points $F, B, C, N$ can be shown to lie on a circle, thus, by III.35, $NL.LB = FE.EC$. Adding to $NB$ line $LB$, and $AF$ to $CF$, by the same argument, we get $NB.LB = FC.AF$.

[20]Within Euclid's framework, equality like $AG.GC = HL.HN$ requires substitution rule or some extra arguments. In geometries with an arithmetic of line segments, such as Descartes' or Hilbert's, the rule if $a = c,\ b = d$, then $a \cdot b = c \cdot d$ is provable.

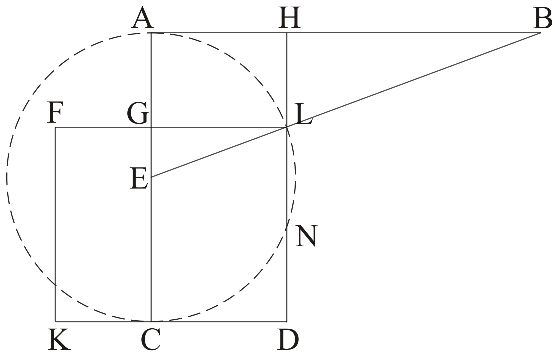


Figure 20: From III.36 to II.14.

We can build yet another net of logical dependencies which omits II.14's reliance on II.5. In Appendix I, we show how to derive II.14 from II.9. Since the proof of II.9 is based solely on I.47, that set of results relies only on visual evidence involved in the proof of I.47 (see §1.4 above).

Euclid's actual train of derivations begins with II.5 and 6. Yet, the inclusion of II.9 in the content of Book II suggests an alternative justification of the regular pentagon construction.

## 5. Beyond the *Elements*

### 5.1. Heron's numerical interpretation of II.12,13

Heron's *Metrica* (Schöne, 1903) is a study on measurement of plane figures that applies a unit square (μονάς) to approximate areas of triangles, polygons, ellipses, circles. In that process, line segments, represented by natural numbers, are multiples of a unit line (μονάς).

In proposition I.8, Heron derives his famous formula for the area of a triangle (ἐμβαδόν), $\triangle$ for short. We can explain that feat as follows

$$\triangle\triangle = s^2r^2 = sabc,$$

or

$$\triangle = \sqrt{sabc}, \tag{1}$$

where $a, b, c$, and $r$ are introduced through Fig. 21. Since the half of the perimeter of the triangle, marked by $s$, is given by the following equations $s = GC = a + b + c$, numbers $a, b, c$ can be determined by $s$ and sides of the triangle as follows $a = s - (b + c) = s - BC$, etc.

Although formula (1) has only numerical sense, to justify it, Heron applies all the means of geometry Euclid develops in Books I to VI.[21]

[21] See (Błaszczyk, 2021a, §8).

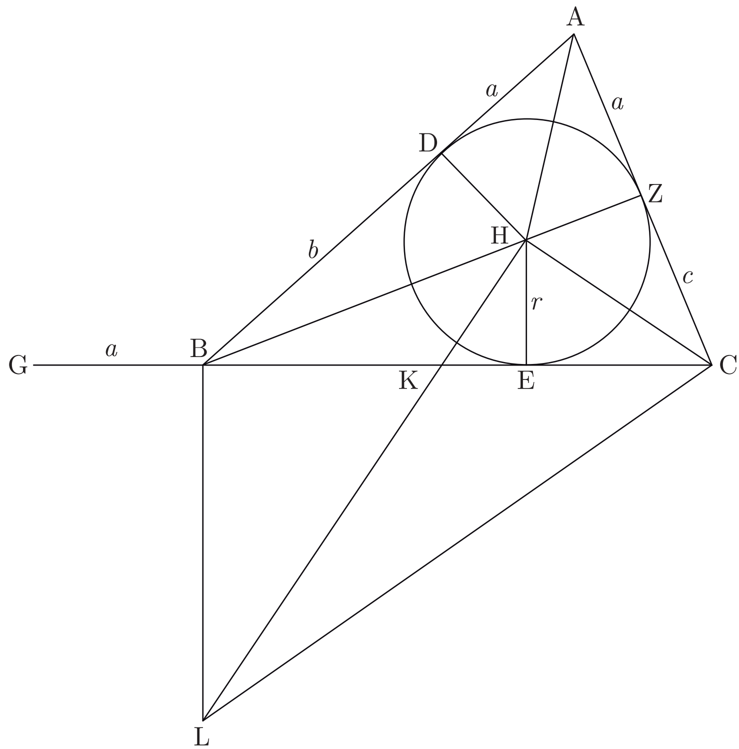


Figure 21: Heron, *Metrica* I.8 (modified: lower-case letters added).

In propositions I.5,6, Heron examines a triangle, given numerical measures of its sides. In this case, yet, he seeks the area of a triangle by determining its height. To this end, he adopts a numerical interpretation of Euclid's propositions II.12,13, and I.47. Regarding an acute-angle triangle, Euclid's II.13, let us recall, brings in the equation (see Fig. 22):

$$AC^2,\ 2BC \cdot BD = AB^2,\ BC^2.$$

Since in Heron's framework $AB, AC$, and $BC$ are numbers, he implicitly turns Euclid's proposition into the following formula

$$AC^2 + 2BC.BD = AB^2 + BC^2. \tag{2}$$

In the next step, owing to the peculiar use of I.47, Heron determines $AD$, namely $AD = \sqrt{AB^2 - BD^2}$. It is a novel motion: throughout the *Elements*, Euclid does not refer to a difference of figures.

In Heron's numerical example, $AB = 13$, $BC = 14, AC = 15$. Based on (2), line $BD$ equals 5. Through the formula $AD = \sqrt{AB^2 - BD^2}$, $AD = 12$. Then $BC \cdot AD = 168$, and finally triangle $ABC$ is 84 of μονάδων.

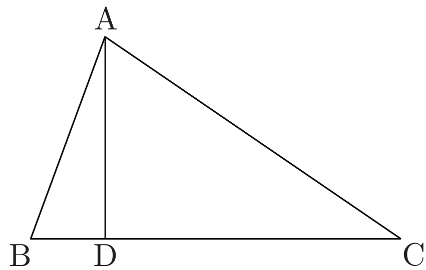


Figure 22: Heron, *Metrica* I.5.

## 6. Descartes' interpretation of II.14

Our historical digressions spotlight the specific question addressed in Euclid's proposition II.14: how to find a leg of a right-angled triangle given its hypotenuse and the other leg. Viewed from that perspective, Heron made a substantial push towards the modern solution. Leo Corry's (Corry, 2013) explores the reception of Book II in the Late Antiquity, Islamic, early Latin, and medieval texts. Based on this study, these mathematical traditions added nothing to this issue. In this section, we portray Descartes' innovative achievements.

In book I of *La Géométrie*, Descartes employs two diagrams to demonstrate how to construct roots of the second-degree polynomials with straightedge and compass.[22] In Fig. 26, two first diagrams from the left represent *La Géométrie*'s original woodcuts. The first is a remake of Euclid's III.36, second employs a novel geometric pattern. Descartes indicates lines which are to be positive roots of specific polynomials. In Book III, he discusses a geometric interpretation of negative roots – negative lines of a sort. Below we do not consider negative cases. Nevertheless, some lines on these diagrams represent negative roots. Then, for completeness reasons, an account should explicate sign rules, such as $(-1)(-1) = 1$, $(-1)(+1) = -1$.[23]

### 6.1. Descartes' arithmetic of line segments

Descartes interprets Euclid's diagrams regularly. On the very first pages of *La Géométrie*, he employs his figures VI.12 and VI.13 to introduce brand new operations on line segments. His concept is this (see Fig 23): assuming $AB = 1$, $BE$ is the product $BD \cdot BC$, and $BC$ – the quotient $BE/BD$, and when $FG = 1$, then $GI$ is the square root of $GH$.

In Fig. 24, we represent Descartes' ideas in modern attire. The procedure is as follows. On the arms of an angle, lay down lines $a, b$, and 1. To get the product, draw through $a$ the line parallel to the line $b, 1$, for the quotient – draw through 1 the parallel to the line $a, b$, for the square root, find $\frac{a+1}{2}$ and follow the construction devised in Euclid's II.14.

[22]See (Descartes, 1637, p. 302–303).
[23]Actually, Descartes applies these rules in book III of *La Géométrie*; see (Błaszczyk, 2021a).

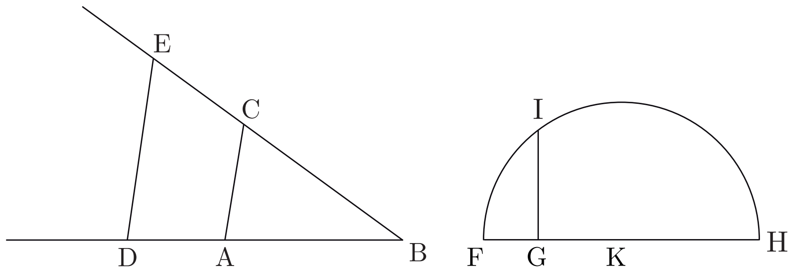


Figure 23: *La Géométrie*, p. 297.

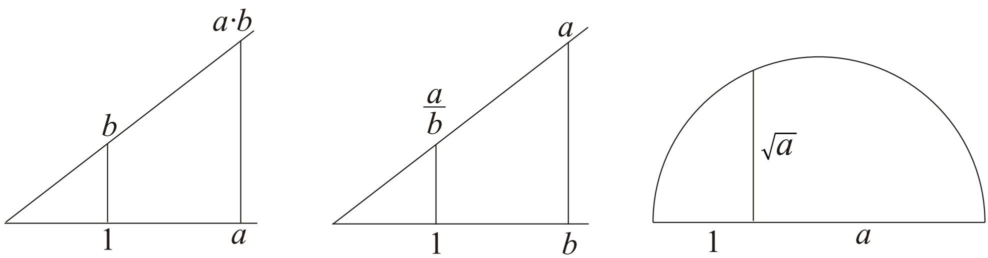


Figure 24: Descartes' arithmetic of line segments based on proportions

Operations on line segments, such as multiplication, division, square root extraction, and addition are defined in such a way, that $a \cdot b$, $\frac{a}{b}$, $\sqrt{a}$, and $\frac{a+1}{2}$ are line segments, given $a, b$ are line segments.[24]

In the 17 century, it was a common practice to interpret terms such as $\sqrt{ab}$ or $\frac{ab}{c}$ geometrically. The first stood for the geometric mean determined via II.14. The second, rooted in I.44, for the side of a rectangle with side $c$ and equal to the rectangle $ab$. Thus, one could also base an arithmetic of line segments on results included in Euclid's Books I – II. Fig. 25 illustrates that alternative.[25] However, Descartes' original arithmetic is related to Euclid's proportions.

Proportions and similar figures (plus some of Apollonius' observations concerning tangents to hyperbola and parabola) make mathematical foundations for Descartes' *Diotrique*. In *La Géométrie*, he transcends Greek mathematics by turning proportions into the arithmetic of line segments according to the following rules

$$a : b :: c : d \Rightarrow a \cdot d = c \cdot b, \quad a : b :: c : d \Rightarrow a = \frac{c}{d} \cdot b.$$

[24]The line $\frac{a+1}{2}$ attests integers and fractions belong to the structure of line segments. Book III testifies that this structure also includes surds. They can be determined by the second diagram, taking, for example, $a = 2$, we get $\sqrt{2}$.

[25]Hilbert's product of line segments builds on the first of these diagrams.

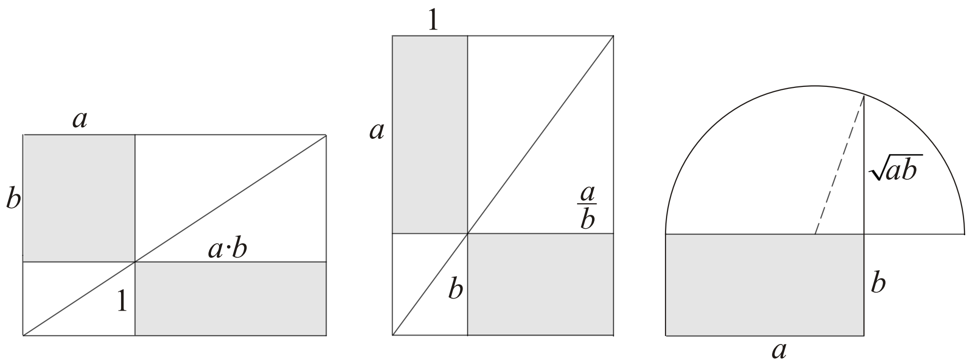


Figure 25: Arithmetic of line segments based theory of equal figures

One can show that founding Descartes definitions on Euclid's proportion, rules such as

$$ab = ba, \quad a(b+c) = ab + ac, \quad \sqrt{a^2} = a, \quad (\sqrt{a})^2 = a, \quad a \cdot 1 = a, \quad a < b \Rightarrow ac < bc,$$

obvious for a modern reader are also valid in Descartes' framework.[26] Descartes applies them implicitly all throughout *La Géométrie*.

In book III, the structure of lines expands by *negative* elements. Since it includes the unit line 1 and the zero 0, it is an ordered field closed under the square root operation.

## 6.2. Solving equations by a difference of squares

Descartes shows how to construct solutions to the following three equations

$$x^2 = ax + b^2, \quad x^2 = -ax + b^2, \quad x^2 = ax - b^2,$$

given $a$ and $b$ are lines, that is, $a, b > 0$. The first two rely on the diagram on the left in Fig. 26, the third – on the middle one. Descartes' arguments are concise. He instructs a reader on drawing a figure represented on the accompanying diagram and states, with no further comments, that such and such line satisfies such and such equation.

In Fig. 26, $NL = \frac{1}{2}a$, $LM = b$. As for the first equation, Descartes simply states that

$$x = OM = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 + b^2}$$

is the root. Indeed, one can check it by calculations (in an ordered field), namely

$$\Big(\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 + b^2}\Big)^2 = a\Big(\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 + b^2}\Big) + b^2.$$

[26] See (Błaszczyk, 2021a), or (Błaszczyk, Mrówka, 2015).

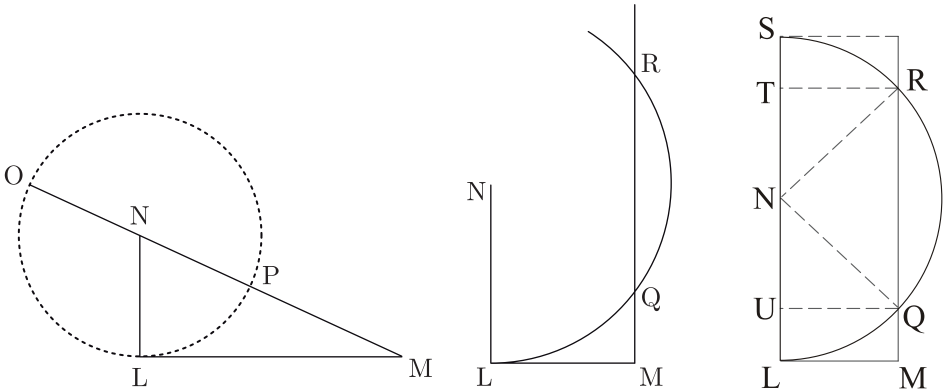


Figure 26: *La Géométrie*, p. 302, 303 (on the left and in the middle), modified Descartes' diagram (on the right).

Taking $x = PM$, one gets that the line

$$x = PM = -\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 + b^2}$$

satisfies the second equation.[27]

Comparing the diagram and its formal description, the formula $\sqrt{\frac{1}{4}a^2 + b^2}$ represents the hypotenuse $NM$, that is

$$NM = \sqrt{\tfrac{1}{4}a^2 + b^2}.$$

Descartes' arithmetic justifies this trick.[28]

Let us turn to the second diagram. It is designed to solve the third equation. That is how Descartes carries out the construction: "I make NL equal to $\frac{1}{2}a$ and LM equal to $b$ [...] draw MQR parallel to LN, and with N as a center, describe a circle through L cutting MQR in the points Q and R; [...] the line sought, is either MQ or MR" (Descartes, 2007, 14).

First, let us try to determine $MQ$ applying Euclidean technique. Given $LM \perp LN$, by III.36, the equality $LM^2 = RM.MQ$ obtains, or, by II.14, $LM^2 = RM.MQ = SU.UL$ (see Fig. 26, the diagram on the right).

[27]In Descartes's arithmetic, we can turn it into the equation $x^2 + ax = b^2$. Victor Blåsjö presents just discussed Descartes' solution as a special case of Euclid's VI.29; see (Blåsjö, 2016, p. 338). Van der Waerden provides virtually the same interpretation of VI.29. He finds it as a solution of the problem $xy = F$, $x - y = 2a$. It is reduced to finding the roots of the equation $x(2a + x) = b^2$, given $F = b^2$; see (Van der Waerden, 1961, p. 122). The equation $x^2 + ax = b^2$ can be rephrased in terms of areas. However, if Euclid had sought the roots of these equations, III.36 would do.

[28]See (Błaszczyk, 2021a) for further discussion.

Given $RM = a - x$, $MQ = x$, both propositions enable one to encode the problem $b^2 = (a - x)x$ not the solution, that is, a formula for $x$.[29]

To outline Descartes' solution, let additionally assume $b < \frac{1}{2}a$. We read off from the diagram the following equalities

$$QM = UL = \tfrac{1}{2}a - UN.$$

Now, Descartes states that these lines

$$x = MQ = \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}a^2 - b^2}, \quad x = MR = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 - b^2}$$

satisfy the equation

$$x^2 = ax - b^2.$$

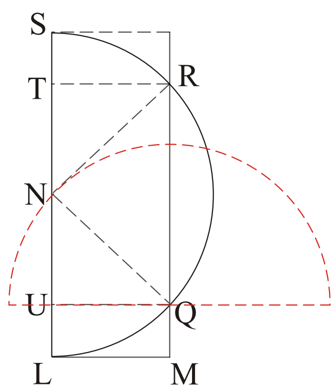


Figure 27: Solving the equation $x^2 = ax - b^2$ by II.14.

Let us focus on the first solution and the right angle triangle $NUQ$. Like in the previous cases, comparing the diagram and its formal description, the formula $\sqrt{\frac{1}{4}a^2 - b^2}$ represents the leg $NU$,

$$NU = \sqrt{\tfrac{1}{4}a^2 - b^2}.$$

In this sense, Descartes identifies the leg of a right-angled triangle by a difference of squares.

Euclid's II.14 enables one to determine the line $NU$. Yet, to this end, one needs a separate construction – for example, the one represented in Fig. 27 in red. Viewed in that context, $NU$ is the geometric mean, namely

$$NU = \sqrt{\left(\frac{a}{2} - UQ\right)\left(\frac{a}{2} + UQ\right)} = \sqrt{\left(\frac{a}{2} - b\right)\left(\frac{a}{2} + b\right)}.$$

[29] H. Bos (Bos, 2001, p. 305) claims that Euclid's III.36 suffice to solve Descartes' first equation, $x^2 = ax + b^2$. Yet, by III.36 he can only encode the problem, not determine its solution. While discussing Descartes' solution, he even does not explain how to transform $x(x - a) = b^2$ into $x^2 = ax + b^2$ within Euclid's framework.

The square on $NU$ is represented as a rectangle

$$NU^2 = \big(\frac{a}{2} - b\big)\big(\frac{a}{2} + b\big).$$

There are no means in Euclid's system to turn $\big(\frac{a}{2} - b\big)\big(\frac{a}{2} + b\big)$ into difference of squares $\frac{1}{4}a^2 - b^2$.

Modern mathematics, as well as school mathematics, adopts Descartes' solution.

## 7. Modern interpretations of Book II

### 7.1. History of mathematics

Leo Corry's (Corry, 2013) is a thorough study of Book II and its reception in Islamic and medieval mathematics. It also summarizes the debate over the so-called geometric algebra interpretation of Book II as follows: "In 1975 Sabetai Unguru published an article in which he emphatically criticized the geometric algebra interpretation. He claimed that Greek geometry is just that, geometry and that any algebraic rendering thereof is anachronistic and historically misguided. [...] Unguru's view became essentially a mainstream interpretation accepted by most historians. Unguru's criticism has since stood (at least tacitly) in the background of most of the serious historical research in the field" (Corry, 2013, p. 638). Recent papers by (Blåsjö, 2016), and (Katz, 2020) aim to undermine Unguru's position, yet, they base their arguments on semantic distinctions rather than new insights into the *Elements*.

In this section, we pay attention to the interpretation of Book II by a renowned historian of ancient mathematics, David Fowler. While the geometric algebra interpretation overestimates Euclid's technique, Fowler underestimates it. His (Fowler, 2003) reads: "With the exception of implicit uses of I47 and 45, Book II is virtually self-contained in the sense that it only uses straightforward manipulations of lines and squares of the kind assumed without comment by Socrates in the Meno. Moreover the only reference to I45, just quoted above, occurs in the last proposition, where it is tackled on to contribute extraneous generality, out of keeping with the style of the rest of the book. Also, it can be argued that a proof of the 'Phytagoras' theorem' [...] has been excised from between 8 and 9. This proof, which I shall call Proposition 8a, exploits the manipulation of gnomons, the basic technique of Book II, and with its obliquely placed square, is reminiscent of the successful third Meno; so the proof also conforms in style with the testified ingredients of early Greek mathematics" (Fowler, 2003, p. 70).

In Plato's *Meno*, the slave-boy is to guess a side of a square that is twice as big as a given square. Due to Socrates' hint – it is Socrates, who draws a diagonal, although Plato barely mentions it – the boy finds out that the diagonal is the desired solution. Since it cuts a square into equal triangles, four such triangles will make up the given figure; see Fig. 28. However, what is self-evident for Plato, Socrates, and the slave-boy, is a real problem for Euclid. In I.33, he shows that the diagonal cuts a parallelogram in halves. The proof relies on I.4, with its controversial, *ad* *hoc* rule that two straight lines can not encompass an area.[30]

In Book II, indeed, some arguments rely on simple dissections. We classify them as visual evidence. A diagonal cutting a square into halves is not of that kind.

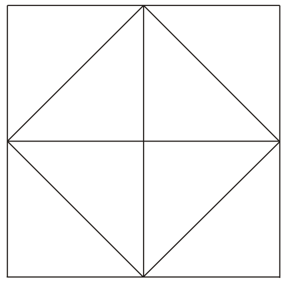

Figure 28: Meno's problem.

Also, Fowler seeks to portray Book II as an epitome of pre-Euclidean mathematics. However, schemes of its propositions presented in Appendix II reveal the application of the Pythagorean theorem and Common Notions, specifically CN3. The latter also plays a role in I.43, stating that *complements of a parallelogram about the diagonal* are equal. Let take this proposition on a Cartesian plane over a non-Archimedean, Euclidean field, where $\varepsilon$ is an infinitesimal, and $1/\varepsilon$, consequently, the infinitely large number.[31] Then, one can show that grey squares represented in Fig. 29 – *complements of the parallelogram* with vertexes $(0,0),(\varepsilon^{-1},0),(\varepsilon^{-1},1+\varepsilon)$, and $(0,1+\varepsilon)$ – are equal in Euclid's theory, but not by a dissection. Thus, Euclid's theory of equal figures is not as simple as Meno's dissections.

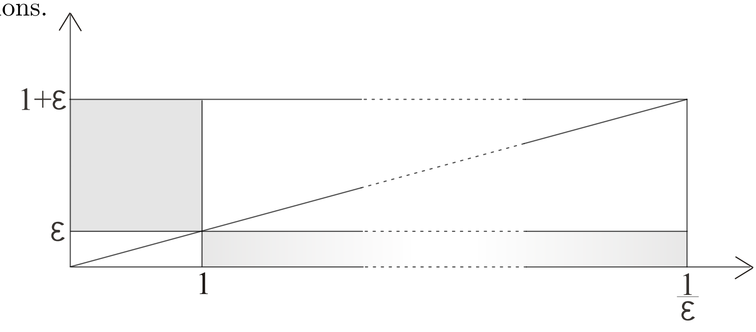


Figure 29: Euclid proposition I.43 on a non-Archimedean plane.

### 7.2. Foundations of geometry on difference of squares

The problem of differences of squares discussed in the paper finds a continuation in the modern foundations of geometry. Let us brief basic facts. Models of Euclid's plane are Cartesian planes over Euclidean fields, defined as closed under the square root operation, $E_+ \ni x \mapsto \sqrt{x} \in E$. Models of Hilbert's plane are

[30]In Hilbert system, I.4 is the axiom.

[31]It can be the field of hyperreals; see (Błaszczyk, 2016).

Cartesian planes over Pythagorean fields, defined as closed under the following operation $P \ni x \mapsto \sqrt{1+x^2} \in P$.[32]

The field of formal power series, $F$, is Pythagorean, yet it is not Euclidean. Although $x$ is an element of $F$ – given it is a formal series of the form $x = \sum_{-\infty}^{+\infty} a_i x^i$, with $a_i = 0$ for all indexes, except $a_1 = 1$ – the square root $\sqrt{x}$ has no representation in $F$. As a result, $\sqrt{(1+x)^2 - (1-x)^2}$, provided $0 < x < 1$, has no representation in $F$.

Moreover, let us note that the following condition provides an alternative characteristic of a Euclidean field

$$0 < x < 1,\ x \in E \Rightarrow \sqrt{1-x^2} \in E.$$

Indeed, it obviously follows from the standard definition of a Euclidean field. The reverse is due to the following observation

$$\sqrt{x} = \frac{1+x}{2}\sqrt{1 - \frac{(1-x)^2}{(1+x)^2}}.$$

From the logical perspective, thus, construction II.14 is equivalent to determining the difference of squares. Yet, while the axiom CN3 provides no constraints within the framework of an ordered field, it matters within the original system of Euclid's geometry.

With these observations let us proceed to Robin Hartshorne's (Hartshorne, 2000). Its chapter 5 develops Hilbert's theory of the content of figures. Motivating that part of his study, Hartshorne writes: "Looking at Euclid's theory of area in Books I–IV, Hilbert saw how to give it a solid logical foundation" (Hartshorne, 2000, p. 195). Furthermore, he claims that *Common Notions* 1–5 plus *Halves of equal figures are equal*, and *If squares are equal, than their sides are equal* constitute additional axioms. And continues "instead, following Hilbert, we will show that one can define a suitable notion of *equal area* and prove its properties, thus providing a new foundations for the theory of area" (Hartshorne, 2000, p. 196).

Viewed from the modern perspective, proposition II.14 constructs line segment $\sqrt{a}$ and enables one to square any polygon. That construction is in-feasible on a Hilbert plane. Or, to rephrase this claim in terms of constructions with Hilbert's tools: $\sqrt{a}$ can not be carried out by a ruler and transporter of line segments and transporter of angles.[33] Therefore, Hartshorne seeks to refine Euclid's theory of equal figures on a Euclidean plane.

The role of the so-called de Zolt's postulate, Z in short, is the most controversial issue of that project. This axiom reads: "If $Q$ is a figure contained in another figure $P$, and if $P - Q$ has nonempty interior, then $P$ and $Q$ do not have equal content" (Hartshorne, 2000, p. 201). Hartshorne finds it crucial in the reconstruction of Euclid's theory as he writes "I.39 uses *the whole is greater than the part* in its proof and so depends on (Z). Generally speaking, all of the results in which

[32] See (Hartshorne, 2000, p. 145).

[33] For a characteristic of Hilbert's construction tools see (Hartshorne, 2000, 102). Actually, these three devices produce the same results as a ruler and transporter of line segments.

Euclid shows that two figures are equal will be valid for the notion of equal content. However, when a hypothesis of equal content is used to conclude something involving congruence of segments or angels in a figure, then (Z) will be necessary. So I.40 also depends on (Z). In I.48 Euclid says that if squares have equal content, then their sides are equal, so this result also depends on (Z)" (Hartshorne, 2000, p. 203).

In the next section, we present an interpretation of the axiom *The whole is greater than the part*, and show that it enables to prove the supposed axiom *equal squares have equal sides.*

### 7.3. Common Notions 5 vs de Zolt's postulate

Euclid's Common Notions 5, CN5 in short, reads:

CN5 "And the whole (ὅλον) [is] greater than the part (μέρους)".

We argue that the following formula interprets this axiom, $(\forall x, y)(x + y > x)$. Proposition II.14 will play a key role in our argument.

We interpret CN5 in a broader context of Euclid's theory of magnitudes developed in Book V. Viewed in that context, it turns out to be equivalent to the axiom of real numbers called compatibility of order with sums, while usually, it is explained in terms of set theory.[34]

We formalize magnitudes of the same kind (line segments being of one kind, triangles being of another, etc.) as an additive semigroup with a total order, $(M, +, <)$, characterized by the following five axioms:

- E1 $(\forall x, y)(\exists n \in \mathbb{N})(nx > y)$,
- E2 $(\forall x, y)(\exists z)(x < y \Rightarrow x + z = y)$,
- E3 $(\forall x, y, z)(x < y \Rightarrow x + z < y + z)$,
- E4 $(\forall x)(\forall n \in \mathbb{N})(\exists y)(x = ny)$,
- E5 $(\forall x, y, z)(\exists v)(x : y :: z : v)$.

  The term $nx$ is defined by $nx = \underbrace{x + ... + x}_{n\ times}$.

Total order in ancient Greek mathematics means *greater-than* relation. It is primitive, i.e., non-defined, and characterized by transitivity and the law of trichotomy, that is, one and only one of the following conditions obtain [35]

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y.$$

Greater-than relation between, for example, triangles, rather than their measures, seem odd for a modern reader. However, that is what we find already in proposition I.6, where Euclid arrives at the conclusion that "the triangle DBC will be equal to the triangle ACB, the lesser to the greater. The very notion (is)

[34] See (Błaszczyk, 2021b).

[35] Some studies, e.g., (Beckmann, 1967), instead of E2 adopt iff version, $x < y \Leftrightarrow x + z = y$ and treat it as a definition of an order. That interpretation finds no textual corroboration.

absurd"; see Fig. 30. Here, contradiction consists of violation of the law of trichotomy: $\triangle DBC = \triangle ACB$ and $\triangle DBC < \triangle ACB$. The equality relies on I.4. The inequality seems as obvious that Euclid provides no arguments.

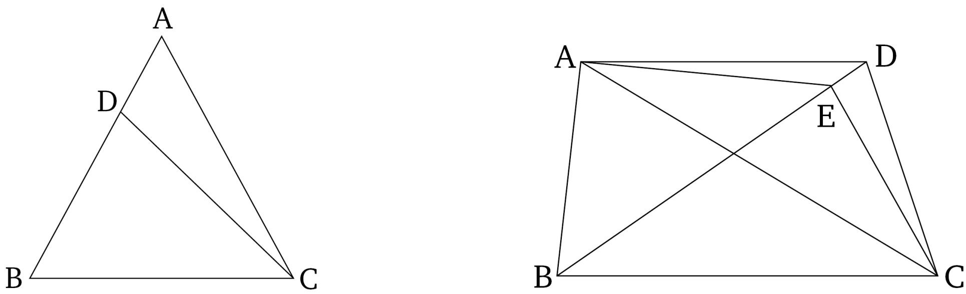


Figure 30: *Elements*, I.6 (on the left) and 39 (on the right).

In I.39, similarly, given "ABC is equal to triangle EBC", Euclid gets the conclusion "ABC is equal to DBC. Thus, DBC is also equal to EBC, the greater to the lesser. The very thing is impossible". Here, the equality $\triangle EBC = \triangle DBC$ follows from the transitivity of equality, while inequality $\triangle EBC < \triangle DBC$ seems self-evident for Euclid.

In both cases, a modern reader has to decide why a triangle is greater than another. Indeed, both cases exemplify scheme $x + y > x$. More to the point, in I.6 and I.39 respectively, the following equalities are based on visual evidence

$$\triangle ACB = \triangle DBC + \triangle DCA, \quad \triangle DBC = \triangle EBC + \triangle ECD.$$

Yet, the second parts of Euclid's arguments apply CN5, namely

$$\triangle DBC + \triangle DCA > \triangle DBC, \quad \triangle EBC + \triangle ECD > \triangle EBC. \tag{3}$$

Thus, arguments related to CN5 include a reference to diagrams: it is an equality based on simple dissection like $\triangle ACB = \triangle DBC + \triangle DCA$. Yet CN5 is a rule which does not rely on a diagram.[36]

To elaborate, we show that E3 is equivalent to formula $(\forall x, y)(x + y > x)$ relative to E1 and E2, that is

$$E1,\ E2,\ E3 \Leftrightarrow E1,\ E2,\ CN5.$$

Secondly, that this new form of E3 interprets Euclid's CN5.

Let start with the implication

$$E1, E2,\ CN5 \Rightarrow E1, E2, E3.$$

To show E3, suppose $x < y$. By E2, $x + v = y$, for some $v$, and consequently $(x+z)+v = y+z$, for each $z$. By CN5, $(x+z) < (x+z)+v$. Finally, $x+z < y+z$.

[36]Vincenzo De Risi views it differently. In his interpretation, CN5 is a rule of diagrammatic reasoning "The comparison of figures through CN4 and CN5 occurs by means of diagrammatic inferences" (De Risi, 2021, p. 316). In short, CN5 is to turn a subset relation ("larger in content") read off from a diagram into a logical relation greater-than.

Let pass to the other implication,

$$E1,\ E2,\ E3 \Rightarrow CN5.$$

To get a contradiction, suppose $x + y \leq x$, for some $x,\ y$. By E1, $ny > y$ for some $n$. By E3, it follows that $x + ny > x + y$. Since $x + y \leq x$, by the transitivity of greater-than relation, we get

$$x + 2y \leq x + y \leq x.$$

The $n$-th iteration of this move gives

$$x + ny \leq x + y.$$

In sum, $x + ny > x + y$ and $x + ny \leq x + y$. *The very thing is impossible.*

Note that, axioms E2, E3 and $\neg CN5$ are satisfied in the additive group of real numbers $(\mathbb{R}, +, <)$, which means E1 is necessary to prove the above implication. Note also that both E1 and CN5 exclude 0-magnitude, that is, a magnitude that satisfies the following equality $x + 0 = x$.

We can also buttress these logical arguments by showing that Euclid's terms *whole* and *part* are figures of the same kind, rather than a set and its subset – a circle inscribed in a triangle is not its part. In Book II, the term *whole* applies to rectangles or squares dissected on rectangles and squares. When *parts* result from a dissection of a *whole* like in I.6 or I.39, the symbol + interprets putting figures next to each other, like segments lying on a line sharing one end-point. By this, we treat figures like co-linear segments sharing one end-point.

I.47 enables more abstract interpretation. Its conclusion reads: "the whole square BDEC is equal to two squares GB, HC". Going back to its proof, "the whole square BDEC", i.e., the square on $BC$, is dissected into rectangles $BL$ and $LC$, which are equal to squares on $AB$, and $AC$ respectively (see Fig. 8). Thus,

$$BC^2 > AB^2, \quad BC^2 > AC^2.$$

Given $a, b$ are legs, $c$ the hypotenuse of a right-angled triangle, I.47 could be formalized as $a^2, b^2 = c^2$, or in a modern style, $a^2 + b^2 = c^2$. It follows, on our interpretation of CN5, that $c^2 > a^2$, although the square on $c$ is not dissected into squares $a^2$ and $b^2$.

Now, let proceed to that implicit rule applied throughout the *Elements*, *when squares are equal, their sides are equal*,

$$a^2 = b^2 \Rightarrow a = b.$$

To get a contradiction, suppose $a^2 = b^2$ and $a > b$. Then, by E2, $a = b + c$, for some $c$. By II.14, we find $h$ such that equality holds

$$(b + c)^2 = b^2 + h^2.$$

Applying the algebraic interpretation of CN5, we get $a^2 > b^2$. Thus, $a^2 = b^2$ and $a^2 > b^2$. *The very thing is impossible.*

The above prove also shows that comparing squares reduces to comparing their sides, that is

$$a > b \Rightarrow a^2 > b^2.$$

Since II.14 enables to square any polygon, sides of respective squares enable to compare polygons in terms of greater-than.[37]

## 8. Final remarks

This commentary on Book II proceeds on four levels: descriptive, speculative, mathematical, and historical. On the first, §1, Appendix II and III, we overview the content of each proposition. That part reveals a relation between visible and invisible figures and implicit rules that enable Euclid to start proofs with visible figures and reach conclusions regarding invisible ones. It also unveils the role of visual evidence in the deductive structure of Book II.

Visual evidence is a specification of a problem commonly known as the dependence of inferences on diagrams. Viewed from that perspective, propositions II.1–8 rely on diagrams. These are observations related to simple dissections of squares and rectangles.

The second level, section §4 and Appendix I, includes speculations on reducing the reliance on diagrams to one visual argument involved in I.47. Then, the chain of propositions crucial from the perspective of constructing the regular pentagon is as follows:

$$I.47 - II.9 - II.14 - III.35 - III.36 - II.11.$$

In these notes, we do not address the problem of replacing the concept of rectangle contained by. We only hint that Descartes' and Hilbert's arithmetic of line segments provides a possible solution.

On the third level, sections §2–3 start mathematical interpretation of Book II. We identify the main problem of Book II as consisting of finding a leg of a right-angled triangle, given hypotenuse $c$ and another leg, $a$, and relate it to the question of whether Greeks applied a difference of squares.

In the historical context, we trace this problem in Book III of the *Elements* and beyond. We identify the use of formula $\sqrt{c^2 - a^2}$ in the arithmetic context, in Heron's *Metrica*, and in a geometric context, in Descartes' *Geometry*.

In the mathematical context, §7.2, we identify echoes of the main problem in modern foundations of geometry. In section §7.3, we address Hartshorne's comments on Book II and prove one of the premises applied all through the *Elements*: if squares are equal, their sides are equal. Our proof builds on an interpretation of Common Notions 5 and proposition II.14. In section §7.1, we provide a counterexample to Fowler's argument relating Book II and pre-Euclidean geometry exposed in Plato's *Meno*.

[37] De Risi comes to a different conclusion regarding the role of CN5 in comparing figures, as he writes: "Through superposition, we may order the figures in terms of content, by saying that a figure is greater in content than another if it contains the other. In order to make such a comparison, we need CN4 to say that the smaller figure, which is superposed to a part of the larger figure, is equal in content to such a part. We then need CN5 to conclude that the larger figure is greater in content than its part and therefore greater than the other figure" (De Risi, 2021, p. 317).

# 9. Appendix I. Proposition II.14 by II.9

By II.9, we get (see Fig 31, diagram on the right)

$$BE^2, EF^2 = 2(BG^2, GE^2). \tag{4}$$

To simplify our argument, we set:

$$BG = b + a,\ GE = a,\ EF = CB = b.$$

With new names of line segments, the equation (4) takes the following form

$$(b + 2a)^2 + b^2 = 2(b + a)^2 + 2a^2.$$

Taking halves of the left and right side, we get

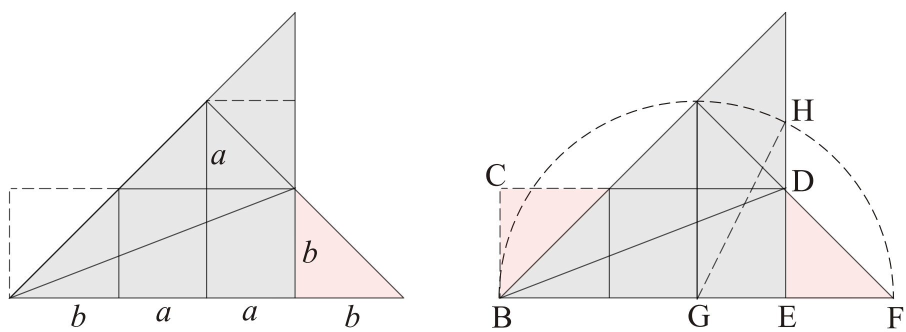


Figure 31: II.14 by II.9

$$\frac{(b + 2a)^2}{2} + \frac{b^2}{2} = (b + a)^2 + a^2. \tag{5}$$

Formulas $\frac{(b+2a)^2}{2}$, $\frac{b^2}{2}$ stand for triangles. By I.47, we get

$$(b + a)^2 = a^2 + HE^2 \tag{6}$$

From (5) and (6) it follows that

$$\frac{(b + 2a)^2}{2} + \frac{b^2}{2} = a^2 + HE^2 + a^2 \tag{7}$$

Triangles $\frac{(b+2a)^2}{2}$, $\frac{b^2}{2}$ are equal to $BCDE$ and two squares $a^2$, therefore

$$BCDE + 2a^2 = \frac{(b + 2a)^2 + b^2}{2}. \tag{8}$$

Now, the equation (7) takes the form

$$BCDE + 2a^2 = HE^2 + 2a^2$$

Applying CN3, we get the desired result, namely

$$BCDE = HE^2.$$

# 10. Appendix II. Rules relating visible and invisible figures

## 10.1. The structure of Euclid's proposition

The text of Euclidean proposition is a schematic composition made up of six parts: *protasis* (stating the relations among geometrical objects by means of abstract and technical terms), *ekthesis* (identifying objects of *protasis* with lettered objects), *diorisomos* (reformulating *protasis* in terms of lettered objects), *kataskeuē* (a construction part which introduces auxiliary lines exploited in the proof that follows, relevant new letters are introduced in the alphabetical order), *apodeixis* (proof, which usually proves the *diorisomos*' claim), *sumperasma* (reiterating *diorisomos*). References to axioms, definitions, and previous propositions are made *via* the technical terms and phrases applied in *prostasis*. Euclid's proof means the *apodeixis* part of a proposition.

In Appendix III, we present schemes of propositions II.1–14. They apply four colors that correspond to three groups of rules: visual evidence (red), renaming (blue), and substitutions: violet (substitutions to terms *contained by* or *square on*), and magenta (substitutions to equalities). Visual evidence and renaming, already discussed, seem quite simple and we treat them briefly. In this section, we focus on rules regarding substitution

## 10.2. Visual evidence and Common Notions 4

It is a standard of Euclid studies to differentiate two meanings of equality as applied to figures in the *Elements*. These are congruence and equality of non-congruent figures. The first links with the axiom *Common Notion* 4 introducing the idea of coinciding figures. Red statements in our schemes do not involve any other concepts except equality. If any justification is needed, CN4 would be a good choice. Yet visual evidence seems even more fundamental than the idea of coincidence covered in CN4, for it relates two aspects of the same object rather than two individual objects.

To elaborate, in proposition I.4, a triangle "is applied to" a triangle – in fact, the first one covers the other one due to translation. Then, on the ground of CN4 and an uncodified, *ad hoc* axiom[38], these triangles are to be equal. Visual evidence, on the contrary, is related to dissections of rectangles and squares and, in principle, does not involve translations. Nevertheless, it is not always the same, as it spans from straight dissections (II.1–6), through overlapping figures (II.7), to a dissection combined with a translation (II.8).

## 10.3. Renaming

Our schemes of Euclid's propositions expose the role of names of figures in analyzed arguments. Visible rectangles bear names of their vertices, diagonals, or they are *contained by* two line segments. Visible squares, similarly, are named by their vertices, diagonals, and as a *square on* a side. Invisible figures get only one name: it could be a *rectangle contained* by, or a *square* on. Thus, the most

[38] "[T]wo straight-lines will encompass an area. The very thing is impossible" (*Elements*, I.4).

important is that visible figures can also be named *rectangles contained by*, or *squares on*. Then, due to substitution rules, they can be related to invisible ones.

In a model example, in proposition II.2 (see Fig. 1), the rectangle $AF$ is represented on the diagram and gets the name *contained by* $DA, AC$. Segments $DA, AC$ are represented on the diagram and contain the right-angle. Then, Euclid states that "AF is contained by BA, AC", for "AD is equal to AB". However, the diagram does not represent the rectangle contained by $BA$, $AC$. Moreover, lines $BA$, $AC$ do not contain a right-angle. That is why, in our scheme, symbol $AF\,\pi\,DA.AC$ standing for the phrase "AF is contained by DA, AC" is in blue – it is simply a new name for a visible figure. Nevertheless, to turn $AF\,\pi\,DA.AC$ into $AF\,\pi\,BA.AC$ a substitution rule is needed, namely the rule (1) as explained in the next section.

### 10.4. Substitution to terms *contained by* and *square on*

There are two kinds of substitution rules applied throughout Book II: substitution to the term $X$ *is contained by* $Y, Z$, or $X$ *is square on* $Y$, and substitution to equality.

The first rule, denoted in violet in our schemes, is schematized by

$$X\,\pi\,Y.V \;\&\; Y = U \Rightarrow X\,\pi\,U.V. \tag{9}$$

Its substance is as follows: a diagram represents $X$ and $Y.V$, and $Y$, $V$ containing a right-angle, and $U$, but not $U.V$.

The following line from the scheme of proposition II.2 exemplifies the rule (1):

$$AF\,\pi\,AD.AC, \quad AD = AB \rightarrow AF\,\pi\,AB.AC.$$

A similar rule applies to the term *square on*, namely

$$X \textit{ is square on } Y \;\&\; Y = U \Rightarrow X \textit{ is square on } U. \tag{10}$$

Here is the substance of this rule: a diagram represents the square $X$ and its side $Y$, represents the side $U$ but not the square $U^2$. We already exemplified it by an argument from proposition II.4; nonetheless, it is worth of reiteration:[39]

$$HF \textit{ is square on } HG, \quad HG = AC \rightarrow HF \textit{ is square on } AC.$$

One may speculate whether the following rule, seemingly as evident as rule (1),

$$X\,\pi\,Y.V \;\&\; Y = U \;\&\; V = Z \Rightarrow X\,\pi\,U.Z$$

is also in use in Book II. Euclid does not apply it: in II.4, instead of the above rule, he refers to I.43.

[39] Propositions II.9, 10 exploit this rule to a great extent.

### 10.5. Substitution to equality

The substitution to equality, represented in our schemes by magenta, is formalized by:

$$X = Y \ \& \ X \,\pi\, U.W \Rightarrow U.W = Y. \tag{11}$$

Since the relation of equality is symmetric, by applying this rule, we can also get the following one

$$X = Y \ \& \ X \,\pi\, U.W \ \& \ Y \,\pi\, Z.V \Rightarrow U.W = Z.V. \tag{12}$$

Thus, in proposition II.1, the starting point is this

$$BH = BK, DL, EH.$$

Then, by the rule (1), we get the following results

$$BH \,\pi\, A.BC, \ BK \,\pi\, A.BD, \ DL \,\pi\, A.DE, \ EH \,\pi\, A.EC.$$

Finally, by rule (4), we reach the conclusion

$$A.BC = A.BD, \ A.DE, \ A.EC.$$

When an equality is combined with the term *square on*, the substitution rule takes the following form

$$X = Y \ \& \ U \textit{ is square on } X \Rightarrow U = Y^2. \tag{13}$$

That is, since the resulting term is equality, we interpret it as a substitution to equality, namely

$$X = Y \ \& \ U = X^2 \Rightarrow U = Y^2.$$

Thus, in II.5, 6, we find the equality $LG = CD^2$. We interpret it as a result of the rule (5) applied to an argument (skipped by Euclid):

$$LG \textit{ is square on } LH, \ LH = CD \to LG = CD^2.$$

In II.14, we find the following mutation of the substitution rule

$$BE.EF = HE^2, \ BD \pi \, BE.EF \to BD = HE^2.$$

More formally

$$U.W = X \ \& \ Y \pi \, U.W \Rightarrow X = Y. \tag{14}$$

As an obvious realization of the rule (5) we adopt yet another rule

$$X = Y \Rightarrow X^2 = Y^2. \tag{15}$$

It is implicitly used throughout the *Elements*.

## 11. Appendix III. Schemes of propositions II.1–14

Let us remind conventions we adopt:

| | |
|---|---|
| $CF.FA$ | interprets the phrase "rectangle contained by CF, FA" |
| $AF \,\pi\, DA.AC$ | "AF is contained by DA,AC" |
| $EF^2$ | interprets the phrase "square on [the line] EF" |
| $\rightarrow$ | stands for a conjunction, usually it is γάρ |
| $\xrightarrow[II6]{}$ | signals the explicit reference to proposition II.6 |
| $CN$ | stands for Common Notions |
| $\angle A = \pi/2$ | stands for "the angle at A (is) a right-angle". |

II.1

*Diorismos*

$$A.BC = A.BD,\ A.DE,\ A.EC.$$

*Apodeixis*

$$\begin{aligned} BH = BK, DL, EH & \\ BH \,\pi\, GB.BC,\ BG = A & \rightarrow BH \,\pi\, A.BC \\ BK \,\pi\, GB.BD,\ BG = A & \rightarrow BK \,\pi\, A.BD \\ DK = BG = A & \rightarrow DL \,\pi\, A.DE \\ & \rightarrow EH \,\pi\, A.EC \\ & \rightarrow A.BC = A.BD,\ A.DE,\ A.EC. \end{aligned}$$

II.2

*Diorismos*

$$AB.BC,\ BA.AC = AB^2.$$

*Apodeixis*

$$\begin{aligned} AE = AF,\ CE & \\ AE \text{ is } AB^2 & \\ AF \,\pi\, DA.AC,\ AD = AB & \rightarrow AF \,\pi\, BA.AC \\ BE = AB & \rightarrow CE \,\pi\, AB.BC \\ & \rightarrow BA.AC,\ AB.BC = AB^2. \end{aligned}$$

II.3

*Diorismos*

$$AB.BC = AC.CB,\ BC^2.$$

*Apodeixis*

$$\begin{aligned} AE = AD,\ CE & \\ AE \,\pi\, AB.BE,\ BE = BC & \rightarrow AE \,\pi\, AB.BC \\ DC = CB & \rightarrow AD \,\pi\, AC.CB \\ DB \text{ is } CB^2 & \rightarrow AB.BC = AC.CB,\ BC^2. \end{aligned}$$

II.4

*Diorismos*

$$AB^2 = AC^2,\ CB^2,\ 2AC.CB.$$

*Apodeixis*

$$\begin{array}{rcl}
CGKB\ is\ CB^2 & & \\
HF\ is\ HG^2, HF\ is\ AC^2 & \rightarrow & HF,\ KC\ are\ AC^2,\ CB^2 \\
GC = CB & \rightarrow & AG\,\pi\,AC.CB \\
AG = GE & \rightarrow & GE = AC.CB \\
 & \rightarrow & AG, GE = 2AC.CB \\
HF, CK\ are\ AC^2, CB^2 & \rightarrow & HF,\ CK,\ AG,\ GE = \\
 & & = AC^2,\ BC^2,\ 2AC.CB \\
HF,\ CK,\ AG,\ CE = ADEB, & & \\
ADEB\ is\ AB^2 & \rightarrow & AB^2 = AC^2,\ CB^2,\ 2AC.CB.
\end{array}$$

II.5

*Diorismos*

$$AD.DB, CD^2 = CB^2.$$

*Apodeixis*

$$\begin{array}{rcl}
 & \xrightarrow[I43]{} & CH = HF \\
 & \xrightarrow[CN2]{} & CH + DM = HF + DM \\
 & \rightarrow & CM = DF \\
AC = CB & \xrightarrow[I36]{} & CM = AL \\
 & \xrightarrow[CN1]{} & AL = DF \\
 & \xrightarrow[CN2]{} & AL + CH = DF + CH \\
 & \rightarrow & AH = NOP \\
DH = DB & \rightarrow & AH\,\pi\,AD.DB \\
 & \rightarrow & NOP = AD.DB \\
LG = CD^2 & \xrightarrow[CN2]{} & NOP + LG = AD.DB, +CD^2 \\
NOP, LG = CEFB & & \\
CEFB\ is\ CB^2 & \rightarrow & AD.DB, CD^2 = CB^2.
\end{array}$$

II.6

*Diorismos*

$$AD.DB, CB^2 = CD^2.$$

*Apodeixis*

$$\begin{array}{rcl}
AC = CB & \xrightarrow[I36]{} & AL = CH \\
 & \xrightarrow[I43]{} & CH = HF \\
AC = CB & \xrightarrow[CN1]{} & AL = HF \\
 & \rightarrow & AM = NOP \\
DM = DB & \rightarrow & AM\ \pi\ AD.DB \\
 & \rightarrow & NOP = AD.DB \\
LG = BC^2 & \rightarrow & AD.DB, BC^2 = NOP, BC^2 \\
NOL, LG^2 = CEFB & & \\
CEFD \text{ is } CD^2 & \rightarrow & AD.DB, BC^2 = CD^2.
\end{array}$$

II.7

*Diorismos*

$$AB^2, BC^2 = 2AB.BC, CA^2.$$

*Apodeixis*

$$\begin{array}{rcl}
AG = GE & \xrightarrow[CN2]{} & AG, CF = GE, CF \\
AF = CE & \rightarrow & AF, CE = 2AF \\
KLM, CF = AF, AG & \rightarrow & KLM, CF = 2AF \\
BF = CB & \rightarrow & 2AF = 2AB.BC \\
 & \rightarrow & KLM, CF = 2AB.BC \\
DG \text{ is } AC^2 & \rightarrow & KLM, BG, GD = 2AB.BC, AC^2 \\
 & \rightarrow & KLM, BG, GD = ADEB, CF \\
ADEB, CF = AB^2, BC^2 & \rightarrow & AB^2, BC^2 = 2AB.BC, AC^2.
\end{array}$$

II.8

*Diorismos*

$$4AB.BC,\ AC^2 = (AB + BC)^2$$

*Apodeixis*

$$\begin{array}{rcl}
\ldots & \rightarrow & DK = CK = GR = RN \\
DK = CK = GR = RN & \rightarrow & DK,\ CK,\ GR,\ RN = 4CK \\
\ldots & \rightarrow & AG = MQ = QL = RF \\
AG = MQ = QL = RF & \rightarrow & AG,\ MQ,\ QL,\ RF = 4AG \\
 & \rightarrow & DK,\ CK,\ GR,\ RN, AG,\ MQ,\ QL,\ RF = STU = 4AK \\
AK\pi AB.BD & \rightarrow & STU = 4AB.BD \\
OH = AC^2 & \xrightarrow[CN2]{} & STU + AC^2 = 4AB.BD + AC^2 \\
STU,\ AC^2 = AEFD & & \\
AEFD\, is\, AD^2 & \rightarrow & 4AB.BD,\ AC^2 = AD^2. \\
BC = BD & \rightarrow & 4AB.BC,\ AC^2 = (AB + BC)^2 \\
AD^2 = AB^2,\ BC^2 & \rightarrow & 4AB.BC,\ AC^2 = AB^2,\ BC^2.
\end{array}$$

In this diagram, we have omitted some tedious arguments which lead to the obvious conclusions. This is marked by three dots sign.

II.9

*Diorismos*

$$AD^2, DB^2 = 2(AC^2, CD^2)$$

*Apodeixis*

$$\begin{array}{rcl}
 & \vdots & \\
AC = CE & \rightarrow & AC^2 = CE^2 \\
 & \rightarrow & AC^2,\ CE^2 = 2AC^2 \\
 & \xrightarrow[I47]{} & AE^2 = AC^2,\ CE^2 = 2AC^2 \\
\ldots & \rightarrow & EG = GF \\
EG = GF & \rightarrow & EG^2 = GF^2 \\
 & \rightarrow & EG^2,\ GF^2 = 2GF^2 \\
 & \xrightarrow[I47]{} & EF^2 = EG^2,\ GF^2 = 2GF^2 \\
\ldots & \rightarrow & GF = CD \\
GF = CD & \rightarrow & EF^2 = 2CD^2 \\
 & \xrightarrow[CN2]{} & EA^2,\ EF^2 = 2(AC^2,\ CD^2) \\
 & \xrightarrow[I.47]{} & EA^2, EF^2 = AF^2 \\
 & \rightarrow & AF^2 = 2(AC^2, CD^2) \\
 & \xrightarrow[I47]{} & AF^2 = AD^2, DF^2 \\
 & \rightarrow & AD^2, DF^2 = 2(AC^2, CD^2) \\
\ldots & \rightarrow & DF = DB \\
DF = DB & \rightarrow & AD^2, DB^2 = 2(AC^2, CD^2).
\end{array}$$

II.10

*Diorismos*

$$AD^2, DB^2 = 2(AC^2, CD^2)$$

*Apodeixis*

$$\begin{array}{rcl}
\ldots & \rightarrow & EC = CA \\
EC = AC & \rightarrow & EC^2 = CA^2 \\
 & \rightarrow & EC^2,\ CA^2 = 2AC^2 \\
 & \xrightarrow[I47]{} & AE^2 = EC^2,\ CA^2 = 2AC^2 \\
FG = EF & \rightarrow & FG^2 = EF^2 \\
 & \rightarrow & FG^2,\ EF^2 = 2EF^2 \\
 & \xrightarrow[I47]{} & EG^2 = GF^2,\ EF^2 = 2EF^2 \\
EF = CD & \rightarrow & EG^2 = 2CD^2 \\
 & \xrightarrow[CN2]{} & AE^2,\ EG^2 = 2(AC^2,\ CD^2) \\
 & \xrightarrow[I47]{} & AG^2 = AE^2,\ EG^2 = 2(AC^2,\ CD^2) \\
 & \xrightarrow[I.47]{} & AG^2 = AD^2, DG^2 \\
 & \rightarrow & AD^2, DG^2 = 2(AC^2,\ CD^2) \\
DG = DB & \rightarrow & AD^2, DB^2 = 2(AC^2, CD^2).
\end{array}$$

II.11

*Diorismos* "[T]o cut AB such that the rectangle contained by the whole and one of the pieces is equal to the square on the remaining piece."

*Apodeixis*

$$\begin{array}{rcl}
\ldots & \xrightarrow[II.6]{} & CF.FA, AE^2 = EF^2 \\
EF = EB & \rightarrow & CF.FA, AE^2 = EB^2 \\
\angle A = \pi/2 & \xrightarrow[I.47]{} & AB^2, AE^2 = EB^2 \\
 & \rightarrow & CF.FA, AE^2 = AB^2, AE^2 \\
 & \xrightarrow[CN3]{} & CF.FA = AB^2 \\
AF = FG & \rightarrow & FK\,\pi\,CF.FA \\
AD\ is\ AB^2 & \rightarrow & FK = AD \\
 & \xrightarrow[CN3]{} & FK \setminus AK = AD \setminus AK \\
 & \rightarrow & FH = HD \\
AB = BD & \rightarrow & HD\,\pi\,AB.BH \\
FH\ is\ HA^2 & \rightarrow & AB.BH = HA^2.
\end{array}$$

Here, the implicit equalities $FK \setminus AK = FH$, and $AD \setminus AK = HD$ are based on visual evidence.

II.12

*Diorismos*

$$CB^2 = CA^2, AB^2, 2CA.AD.$$

*Apodeixis*

$$\begin{array}{rcl} & \xrightarrow[II.4]{} & DC^2 = AC^2, AD^2, 2CA.AD \\ & \xrightarrow[CN2]{} & DC^2 + DB^2 = AC^2,\ AD^2 + DB^2,\ 2CA.AD \\ \angle D = \pi/2 & \xrightarrow[I47]{} & CD^2, DB^2 = CB^2 \\ & & AD^2, DB^2 = AB^2 \\ & \rightarrow & CB^2 = CA^2, AB^2, 2CA.AD. \end{array}$$

II.13.

*Diorismos*

$$AC^2,\ 2CB.BD = CB^2,\ BA^2.$$

*Apodeixis*

$$\begin{array}{rcl} & \xrightarrow[II7]{} & CB^2, BD^2 = 2CB.BD,\ DC^2 \\ & \xrightarrow[CN2]{} & CB^2, BD^2 + DA^2 = 2CB.BD,\ DC^2 + DA^2 \\ \angle D = \pi/2 & \xrightarrow[I47]{} & AB^2 = BD^2, DA^2 \\ & & AC^2 = DC^2, DA^2 \\ & \rightarrow & CB^2, AB^2 = 2CB.BD,\ AC^2. \end{array}$$

II.14

*Diorismos* "[I]t is required to construct a square equal to the rectilinear figure A."

*Apodeixis*

$$\begin{array}{rcl} \ldots & \xrightarrow[II.5]{} & BE.EF, GE^2 = GF^2 \\ GF = GH & \rightarrow & BE.EF, GE^2 = GH^2 \\ HE^2, GE^2 = GH^2 & \rightarrow & BE.EF, GE^2 = HE^2, GE^2 \\ & \xrightarrow[CN3]{} & BE.EF = HE^2 \\ EF = ED & \rightarrow & BD\,\pi\,BE.EF \\ & \rightarrow & BD = HE^2 \\ BD = A & \rightarrow & A = EH^2. \end{array}$$

## References

Beckmann, F.: 1967, Neue geschichte zum 5. buch euklids, *Archive for the History of Exact Sciences* **IV**, 1–144.

Blåsjö, V.: 2016, In defence of geometrical algebra, *Archive for the History of Exact Sciences* **70**, 325–359.

Błaszczyk, P.: 2016, A purely algebraic proof of the fundamental theorem of algebra, *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* **8**, 5–21.

Błaszczyk, P.: 2018, From Euclid's Elements to the methodology of mathematics. Two ways of viewing mathematical theory, *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* **10**, 5–15.

Błaszczyk, P.: 2021a, Descartes' transformation of Greek notion of proportionality, in: B. Sriraman (ed.), *Handbook of the History and Philosophy of Mathematical Practice*, Springer.

Błaszczyk, P.: 2021b, Galileo's Paradox and Numerosities, *Zagdanienia Filozoficzne w Nauce* **70**.
`https://www.researchgate.net/publication/343065270_Galileo's_Paradox-_and_Numerosities`

Błaszczyk, P., Mrówka, K.: 2015, *Kartezjusz, Geometria. Tłumaczenie i komentarz*, Universitas, Kraków.

Błaszczyk, P., Petiurenko, A.: 2019, Euclid's theory of proportion revised, *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* **11**, 37–61.

Bos, H.: 2001, *Redefining Geometrical Exactness*, Springer, New York.

Corry, L.: 2013, Geometry and arithmetic in the medieval traditions of Euclid's Elements: a view from Book II, *Archive for History of Exact Science* **67**, 637–705.

De Risi, V.: 2021, Euclid's Common Notions and the Theory of Equivalence, *Foundations of Science* **26**, 301–301.

Descartes, R.: 1637, *La Géométrie*, Jan Maire, Lejda.

Descartes, R.: 2007, *Geometry of Rene Descartes. Translated from the French and Latin by D.E. Smith and L. M. Lethan*, CosimoCLassics, New York.

Fitzpatrick, R.: 2007, Euclid's Elements of Geometry translated by R. Fitzpatrick.
`http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf`

Fowler, D.: 2003, *The Mathematics of Plato's Academy*, Clarendon Press, Oxford.

Giovannini, E. N.: 2021, David Hilbert and the foundations of the theory of plane area, *Archive for the History of Exact Science* **75**(3).

Hartshorne, R.: 2000, *Geometry: Euclid and Beyond*, Springer, New York.

Heiberg, J.: 1883, *Euclidis Elementa*, Teubner, Lipsiae.

Katz, M.: 2020, Mathematical Conquerors, Unguru Polarity, and the Task of History, *Journal of Humanistic Mathematics* **10**(1), 475–515.

Mueller, I.: 2006, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, Dover, New York. (reprint of the first edition: MIT Press, Cambridge, Massachusetts, 1981).

Saito, K.: 2004, Book II of Euclid's Elements in the light of the theory of conic sections. *Historia Scientiarum* 28, 1985: 31–60. Reprinted in Jean Christianidis (ed.), *Classics in the History of Greek Mathematics*, Kluwer, Dordrecht-Boston.

Saito, K.: 2011, The Diagrams of Book II and III of the Elements in Greek Manuscripts. `https://www.greekmath.org/diagrams/diagrams_index.html`.

Schöne, H.: 1903, *Herons von Alexandria, Vermessungslehre und Dioptra, Opera quae supersunt omnia*, Tuebner, Leipzig.

Van der Waerden, B. L.: 1961, *Science Awaking*, Oxford University Press, New York.

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