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Creative mathematical mental activity of students while solving problems requiring use of Vieta's formulas II*

Abstract. This article is a continuation of the article (Chronowski, Powązka, 2016). In this article the next tasks regarding the use of Vieta's formulas for third degree polynomials (equations) with real coefficients are considered. The Olympic tasks in this article are an inspiration to pupils for creative mathematical activity. In this text, I limited myself to comments, hints and didactic suggestions regarding directly presented tasks and their solutions. The most of the general didactic considerations included in our article (Chronowski, Powązka, 2016) are also valid in this paper.

This article is a continuation of the article (Chronowski, Powązka, 2016). In the article (Chronowski, Powązka, 2016) the basic tasks concern the use of Vieta's formulas for third degree polynomials (equations) with real coefficients. In addition, the article includes didactic comments, hints and suggestions, which can be divided into two groups. In the first group they are of a general nature, and in the second group they are more detailed and refer directly to the included tasks.

In this article the next tasks regarding the use of Vieta's formulas for third degree polynomials (equations) with real coefficients are considered. The Olympic tasks in this article are an inspiration to pupils for creative mathematical activity. In this text, I limited myself to comments, hints and didactic suggestions regarding directly presented tasks and their solutions. The most of the general didactic considerations included in our article (Chronowski, Powązka, 2016) are also valid in this paper.

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The references in this article to the theorems, tasks, formulas contained in the article (Chronowski, Powązka, 2016) are marked with the symbol [T], e.g. Theorem 1[T] or Task 1[T]. The cited texts from the article (Chronowski, Powązka, 2016) marked with the symbol [T] are included in the English version in the appendix at the end of this article.

Because this text contains a quite large number of the examples of cubic equations, it is worth using an appropriate mathematical computer program (e.g. *Derive*, *Mathematica*) to solve them.

We will start with the task which according to the book (Pawłowski, 1994, p. 171, Task 5) comes from the XXXI Mathematical Olympiad.

Task 1. (Bryński, 1995, p. 13, Task 3.4; Pawłowski, 1994, p. 171, Task 5). Prove that if the polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where a and b are real numbers, has all positive roots, then they are equal.

S.: Let $f(x) = ax^3 - ax^2 + 9bx - b$, where $a, b \in \mathcal{R}$. Notice that $b \neq 0$. Indeed, if $b = 0$, then the polynomial $f(x)$ in this case has the root equal to 0, i.e. the assumptions of this task are not fulfilled. If $a = 0$ and $b \neq 0$, then $f(x) = 9bx - b$, and so the only root of this polynomial is the number $\frac{1}{9}$, and so the conditions of the task are satisfied. Next, we assume that $a, b \in \mathcal{R} \setminus \{0\}$. The polynomial $f(x)$ has three positive real roots, and so we can use the results of Task 7(iii)[T]. Since $a_3 = a$, $a_2 = -a$, $a_1 = 9b$, $a_0 = -b$, so $a_1a_2 = -9ab$ and $9a_0a_3 = -9ab$, i.e. $a_1a_2 = 9a_0a_3$. Therefore, all roots of the polynomial $f(x)$ are equal on the basis of Task 7(iii)[T].

By virtue of Task 1 we can formulate the following task.

Task 1.1. Prove that if the polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where a and b are real numbers, has all positive roots, then the polynomial $f(x)$ has the following form:
a) $f(x) = a(x - \frac{1}{3})^3$, if $a \neq 0$ and $b \neq 0$;
b) $f(x) = b(9x - 1)$, if $a = 0$ and $b \neq 0$.
If $b = 0$, then does not exist the polynomial $f(x)$ fulfilling the conditions of this task.

S.: ad a). Notice that in this case $f(x)$ is a third degree polynomial. Let x_1, x_2, x_3 be positive roots of the polynomial $f(x)$. The Vieta's formulas imply that

$$\begin{cases} x_1 + x_2 + x_3 = 1, \\ x_1x_2x_3 = \frac{b}{a}. \end{cases}$$

By the solution of Task 1 we have $x_1 = x_2 = x_3$, and so $3x_1 = 1$, that is $x_1 = x_2 = x_3 = \frac{1}{3}$. Then $\frac{b}{a} = \frac{1}{27}$, so $b = \frac{1}{27}a$. Hence, it follows that $f(x) = ax^3 - ax^2 + \frac{1}{3}ax - \frac{1}{27}a$. Therefore, $f(x) = a(x^3 - x^2 + \frac{1}{3}x - \frac{1}{27}) = a(x - \frac{1}{3})^3$.

Remark. Of course, using Task 1, the properties of polynomials and multiple roots, we can directly determine, in this case, the form of the polynomial $f(x)$.

ad b). In this case the form of the polynomial $f(x) = b(9x - 1)$ was given in the solution of Task 1.

The case, when $b = 0$ is included in the solution of Task 1.

From the solution of Task 1 it follows that it is necessary to consider the following cases: a) $a = 0$ or $b = 0$; b) $a, b \in \mathcal{R} \setminus \{0\}$. Pupils (and even mathematics students), who were accustomed during school teaching to schematic solving problems, without deepened reflection on the content of the tasks, the assumptions adopted in the tasks, as well as the results obtained after solving the tasks, often only consider the case b) by default, of course without explicitly indicating this assumption. In this paper, because it is a continuation of the article (Chronowski, Powązka, 2016), in solving Task 1 we used Task 7[T] from this article. We can of course solve Task 1 without referring to the article (Chronowski, Powązka, 2016). This type of solution can be found in the book (Bryński, 1995). The content of Task 1 in this book is formulated in Task 3.4, p. 13 (there is a typographical error: the world "different" should be replaced by the world "equal"), while the solution of this task is on pp. 81 - 82. Just in this solution there in no comment about the subject of case a), i.e. when $a = 0$ or $b = 0$.

The consequences of this are:
1) Certain expressions in this solution do not make sense (when $a = 0$).
2) The operation of division by sides of certain two expressions does not make sense (when $b = 0$).
3) The following statement is included: "Thus, the only polynomials $a(x - \frac{1}{3})^3$ fulfill the conditions of the task". This statement is not true for two reasons:
a) the polynomial $f(x) = a(x - \frac{1}{3})^3$, when $a = 0$ does not fulfill the conditions of the task;
b) the conditions of the task are also satisfied by the polynomials of the form $f(x) = b(9x - 1)$ for $b \in \mathcal{R} \setminus \{0\}$, and so the infinite (uncountable) number of such polynomials was omitted.

According to the content of Task 1, the above solution included in this article, should be considered as complete. The content of the task does not require determining whether polynomials of this form exist. Bryński Bryński, 1995) also does not deal with the existence of such polynomials in the main solution of this task. In the comments after completing the solution of the task, it is additionally stated that the polynomials $a(x - \frac{1}{3})^3$ satisfy the conditions of the task. Is this formulation of Task 1 at the olympiad (XXXI Mathematical Olympiad) correct enough? It can be disputable. The problem may arise evaluating this task: can the solution of the task without discussion (which is not required in the content of the task) about the existence of such polynomials be considered fully correct? It is highly probable that there will be correct solutions of this task including a discussion about the existence of such polynomials. Should the scoring of two such solution be the same? On the other hand, I believe that the above formulation of Task 1 is inspirational when we have the opportunity (e.g. during extracurricular activities or lessons) to discuss with the pupils the role of the coeffitiens a and b in these polynomials, as well as the problem of the existence of such polynomials.

The problem of the existence of mathematical objects considered in tasks will be very clearly visible in the next task. I think that a valid pretext to formulate the following task is the content of Task 7(iii)[T].

Task 1.2. If the polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where a and b are real numbers, has all negative roots, then they are equal.

The solution of this task can be carried out according to the pattern of the solution of Task 1. If a pupil considers cases $a = 0$ or $b = 0$, he will receive a signal that then there are no polynomials fulfilling the conditions of this task.

However, when $a \neq 0$ and $b \neq 0$, using the results of Task 7(iii)[T], the pupil will receive the result that the polynomials of the form specified in Task 1.2 have equal roots. The problem is that there are no polynomials fulfilling the requirements of Task 1.2. Indeed, suppose that there is a polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where $a, b \in \mathcal{R} \setminus \{0\}$, which has all negative real roots x_1, x_2, x_3. We know that $x_1 = x_2 = x_3$ in virtue of Task 7(iii)[T]. Then Vieta's formulas show that $3x_1 = 1$, and so $x_1 = x_2 = x_3 = \frac{1}{3}$. We received a contradiction. Therefore, there is no such polynomial that satisfies the above assumptions. The reasoning scheme that we use to apply Task 7(iii)[T] is as follows: $\alpha \Longrightarrow (\beta \Longleftrightarrow \gamma)$. Because the predecessor of this implication is false, then the given implication is true. Thus, Task 1.2 may be a trap for pupils who prove some properties of a mathematical object that does not exist.

Remark. The formulation of Task 1.2 can be a good and intriguing impulse for pupils to make them aware of the need for deep analysis of the mathematical problems being considered. We can afford to formulate this task when we have opportunity to discuss with pupils the solution of this task. This type of tasks should not take place during exams, competitions or tests.

This task can be formulated in a less tricky way as follows:

Is there a polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where a and b are real numbers, that has all equal negative roots?

In the following task we will solve the problem of the existence of the roots of the third degree polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where $a, b \in \mathcal{R}$ and $a \neq 0$.

Task 1.3. Given the polynomial $f(x) = ax^3 - ax^2 + 9bx - b$, where $a, b \in \mathcal{R}$ and $a \neq 0$. Prove, that the polynomial $f(x)$ has:
a) three different real roots if and only if $ab < 0$ and $b \neq \frac{1}{27}a$;
b) all real roots, where one of them is multiple if and only if $b = 0$ or $b = \frac{1}{27}a$;
c) exactly one real root, and the other two are complex conjugates if and only if $ab > 0$ and $b \neq \frac{1}{27}a$;
In case b) give the form of the polynomial $f(x)$. In cases a) and c) provide examples of appropriate polynomials.

S.: In solving the task we use the notation in accordance with formulas (4)[T], (5)[T], (6)[T], (7)[T]. To write the polynomial $f(x)$ in the form $f(x) = x^3 + px + q$ we determine p and q. After making the appropriate calculations we get:

$$\begin{cases} p = \frac{27b - a}{3a}, \\ q = \frac{54b - 2a}{27a}. \end{cases}$$

We know that

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}.$$

Then

$$\Delta = \frac{b(a - 27b)^2}{27a^3}.$$

Therefore, using Theorems 8[T], 9[T], 10[T] we receive directly conditions a), b), c).

Consider case b). If $b = 0$, then $f(x) = ax^2(x-1)$, and so in this case the polynomial $f(x)$ has the double root 0 and the root equal to 1. According Corollary 1[T], the graph of the polynomial $f(x) = ax^2(x-1)$ at the point $x_0 = 0$ is tangent to the axis of abscissae x and it has the local extremum at the point $x_0 = 0$, moreover, the maximum if $a > 0$, and the minimum if $a < 0$. If $b = \frac{1}{27}a$, then $f(x) = a(x^3 - x^2 + \frac{1}{3}x - \frac{1}{27}) = a(x - \frac{1}{3})^3$, and so in this case the polynomial $f(x)$ has the triple root equal to $\frac{1}{3}$. According to Corollary 2[T] the graph of the polynomial $f(x) = a(x-\frac{1}{3})^3$ at the point $x_0 = \frac{1}{3}$ is tangent to the axis of abscissae x and it has the inflexion point at $x_0 = \frac{1}{3}$.

The example concerning case a).

The polynomial

$$f(x) = 17x^3 - 17x^2 - 36x + 4$$

has the following roots:

$$\begin{cases} x_1 = 2, \\ x_2 = -\frac{1}{2} + \frac{5\sqrt{17}}{34}, \\ x_3 = -\frac{1}{2} - \frac{5\sqrt{17}}{34}. \end{cases}$$

The example concerning case c).

The polynomial

$$f(x) = 3x^3 - 3x^2 + 9x - 1$$

has the following roots:

$$\begin{cases} x_1 = \frac{1}{3}(-2\sqrt[3]{4} + 2\sqrt[3]{2} + 1), \\ x_2 = \frac{1}{3}(\sqrt[3]{4} - \sqrt[3]{2} + 1) + \frac{\sqrt{3}}{3}(\sqrt[3]{4} + \sqrt[3]{2})i, \\ x_3 = \frac{1}{3}(\sqrt[3]{4} - \sqrt[3]{2} + 1) - \frac{\sqrt{3}}{3}(\sqrt[3]{4} + \sqrt[3]{2})i. \end{cases}$$

The following Tasks 2, 2.1, 2.2, 2.3 can be a good material for pupils to consolidate the knowledge, skills and methods used in Tasks 1, 1.1, 1.2, 1.3.

Task 2. Prove that if the polynomial $f(x) = ax^3 + ax^2 + 9bx + b$, where a and b are real numbers, has all negative roots, then they are equal.

Task 2.1. Prove that if the polynomial $f(x) = ax^3 + ax^2 + 9bx + b$, where a and b are real numbers, has all negative roots, then the polynomial $f(x)$ has the following form:
a) $f(x) = a(x + \frac{1}{3})^3$, if $a \neq 0$ and $b \neq 0$;
b) $f(x) = b(9x + 1)$, if $a = 0$ and $b \neq 0$.
If $b = 0$, then does not exist the polynomial $f(x)$ fulfilling the conditions of this task.

Task 2.2. If the polynomial $f(x) = ax^3 + ax^2 + 9bx + b$, where a and b are real numbers, has all positive roots, then they are equal.

Task 2.3. Given the polynomial $f(x) = ax^3 + ax^2 + 9bx + b$, where $a, b \in \mathcal{R}$ and $a \neq 0$. Prove, that the polynomial $f(x)$ has:
a) three different real roots if and only if $ab < 0$ and $b \neq \frac{1}{27}a$;
b) all real roots, where one of them is multiple if and only if $b = 0$ or $b = \frac{1}{27}a$;
c) exactly one real root, and the other two are complex conjugates if and only if $ab > 0$ and $b \neq \frac{1}{27}a$.

In case b) give the form of the polynomial $f(x)$. In cases a) and c) provide examples of appropriate polynomials.

In Tasks 3 and 4 we will present inequalities that will be used to solve Task 7.

Task 3. Prove that for any real numbers x_1, x_2, x_3 the following inequality is satisfied:

$$(x_1 + x_2 + x_3)^2 \geq x_1x_2 + x_2x_3 + x_3x_1. \tag{1}$$

We leave an easy proof of this inequality as an exercise.

Task 4. Prove that for any real numbers x_1, x_2, x_3 the following inequality is satisfied:

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 \geq (x_1x_2x_3)(x_1 + x_2 + x_3). \tag{2}$$

S.: Let $x_1, x_2, x_3 \in \mathcal{R}$. We have: $(x_1x_2 + x_2x_3 + x_3x_1)^2 - [(x_1x_2x_3)(x_1 + x_2 + x_3)] = (x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2 + 2x_1x_2^2x_3 + 2x_1^2x_2x_3 + 2x_1x_2x_3^2 - x_1^2x_2x_3 - x_1x_2^2x_3 - x_1x_2x_3^2 = (x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2 + x_1x_2^2x_3 + x_2x_3^2x_1 + x_3x_1^2x_2 = \frac{1}{2}[2(x_1x_2)^2 + 2(x_2x_3)^2 + 2(x_3x_1)^2 + 2x_1x_2^2x_3 + 2x_2x_3^2x_1 + 2x_3x_1^2x_2] = \frac{1}{2}[((x_1x_2)^2 + 2(x_1x_2)(x_2x_3) + (x_2x_3)^2) + ((x_2x_3)^2 + 2(x_2x_3)(x_3x_1) + (x_3x_1)^2) + ((x_3x_1)^2 + 2(x_3x_1)(x_1x_2) + (x_1x_2)^2)] = \frac{1}{2}[(x_1x_2 + x_2x_3)^2 + (x_2x_3 + x_3x_1)^2 + (x_3x_1 + x_1x_2)^2] \geq 0$. Therefore, inequality (2) is satisfied.

In Tasks 5 and 6 we will prove inequalities that are a strengthening of inequalities occurring in Tasks 3 and 4. The results obtained in Tasks 5 and 6 will be used in the solution of Task 8.

Task 5. Prove that for any real numbers x_1, x_2, x_3 the following inequality is satisfied:

$$(x_1 + x_2 + x_3)^2 \geq 3(x_1x_2 + x_2x_3 + x_3x_1). \tag{3}$$

Moreover,

$$(x_1 + x_2 + x_3)^2 = 3(x_1x_2 + x_2x_3 + x_3x_1) \Longleftrightarrow x_1 = x_2 = x_3. \tag{4}$$

S.: Let $x_1, x_2, x_3 \in \mathcal{R}$. We have: $(x_1 + x_2 + x_3)^2 - 3(x_1x_2 + x_2x_3 + x_3x_1) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1 - 3x_1x_2 - 3x_2x_3 - 3x_3x_1 = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 = \frac{1}{2}[2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1] = \frac{1}{2}[(x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + (x_3^2 - 2x_3x_1 + x_1^2)] = \frac{1}{2}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] \geq 0$. Therefore, inequality (3) is satisfied.

If $x_1 = x_2 = x_3$, then $(x_1 + x_2 + x_3)^2 = (3x_1)^2 = 9x_1^2$, $3(x_1x_2 + x_2x_3 + x_3x_1) = 3 \cdot 3x_1^2 = 9x_1^2$. Thus, the equality $(x_1 + x_2 + x_3)^2 = 3(x_1x_2 + x_2x_3 + x_3x_1)$ is satisfied. Conversely, let us assume that the equality $(x_1 + x_2 + x_3)^2 = 3(x_1x_2 + x_2x_3 + x_3x_1)$ is satisfied. Conducting analogous transformations to those used in the proof of inequality (3) we get:

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 = 0.$$

Thus $x_1 = x_2 = x_3$. Therefore, condition (4) is satisfied.

The following corollary follows from conditions (3) and (4).

Corollary 1. For any real numbers x_1, x_2, x_3 the following condition is satisfied:

$$(x_1 + x_2 + x_3)^2 > 3(x_1x_2 + x_2x_3 + x_3x_1) \Longleftrightarrow (x_1 \neq x_2 \vee x_2 \neq x_3 \vee x_3 \neq x_1).$$

Task 6. Prove that for any real numbers x_1, x_2, x_3 the following inequality is satisfied:

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 \geq 3(x_1x_2x_3)(x_1 + x_2 + x_3). \tag{5}$$

If $x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$, then

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 = 3(x_1x_2x_3)(x_1 + x_2 + x_3) \Longleftrightarrow x_1 = x_2 = x_3. \tag{6}$$

If $x_1 = x_2 = x_3$, then

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 = 3(x_1x_2x_3)(x_1 + x_2 + x_3). \tag{7}$$

S.: Let $x_1, x_2, x_3 \in \mathcal{R}$. We have: $(x_1x_2 + x_2x_3 + x_3x_1)^2 - 3[(x_1x_2x_3)(x_1 + x_2 + x_3)] = (x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2 + 2x_1x_2^2x_3 + 2x_1^2x_2x_3 + 2x_1x_2x_3^2 - 3x_1^2x_2x_3 - 3x_1x_2^2x_3 - 3x_1x_2x_3^2 = (x_1x_2)^2 + (x_2x_3)^2 + (x_3x_1)^2 - x_1x_2^2x_3 - x_2x_3^2x_1 - x_3x_1^2x_2 = \frac{1}{2}[2(x_1x_2)^2 + 2(x_2x_3)^2 + 2(x_3x_1)^2 - 2x_1x_2^2x_3 - 2x_2x_3^2x_1 - 2x_3x_1^2x_2] = \frac{1}{2}[((x_1x_2)^2 - 2(x_1x_2)(x_2x_3) + (x_2x_3)^2) + ((x_2x_3)^2 - 2(x_2x_3)(x_3x_1) + (x_3x_1)^2) + ((x_3x_1)^2 - 2(x_3x_1)(x_1x_2) + (x_1x_2)^2)] = \frac{1}{2}[(x_1x_2 - x_2x_3)^2 + (x_2x_3 - x_3x_1)^2 + (x_3x_1 - x_1x_2)^2] \geq 0$. Thus, inequality (5) is satisfied.

Next we will prove condition (6). Assume that $x_1 = x_2 = x_3$. Then $(x_1x_2 + x_2x_3 + x_3x_1)^2 = (3x_1^2)^2 = 9x_1^4$, $3(x_1x_2x_3)(x_1 + x_2 + x_3) = 3x_1^3 \cdot 3x_1 = 9x_1^4$. Thus, the equality $(x_1x_2 + x_2x_3 + x_3x_1)^2 = 3(x_1x_2x_3)(x_1 + x_2 + x_3)$ is satisfied. Conversely, let us assume that the equality $(x_1x_2 + x_2x_3 + x_3x_1)^2 = 3(x_1x_2x_3)(x_1 + x_2 + x_3)$ is satisfied. Conducting analogous transformations to those used in the proof of inequality (5) we get:

$$(x_1x_2 - x_2x_3)^2 + (x_2x_3 - x_3x_1)^2 + (x_3x_1 - x_1x_2)^2 = 0.$$

Therefore,

$$\begin{cases} x_1x_2 - x_2x_3 = 0, \\ x_2x_3 - x_3x_1 = 0, \\ x_3x_1 - x_1x_2 = 0. \end{cases}$$

Hence, it follows that $x_1 = x_2 = x_3$.

The proof of condition (7) is obvious.

Remark. If $x_1 = 0$ or $x_2 = 0$, or $x_3 = 0$, then equality (7) need not imply equality $x_1 = x_2 = x_3$. If $x_1 = 0$ and the equality (7) is satisfied, then $x_2 = 0$ or $x_3 = 0$. If $x_1 = 0$ and $x_2 = 0$, then equality (7) is satisfied for any real number x_3. Similarly, when $x_1 = 0$ and $x_3 = 0$. Analogous reasoning can be carried out for $x_2 = 0$ or $x_3 = 0$.

The following conclusion follows from conditions (5) and (6).

Corollary 2. For any real numbers $x_1, x_2, x_3 \in \mathcal{R} \setminus \{0\}$ the following condition is satisfied:

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 > 3(x_1x_2x_3)(x_1 + x_2 + x_3) \Longleftrightarrow (x_1 \neq x_2 \vee x_2 \neq x_3 \vee x_3 \neq x_1).$$

It is easy to check that the following conclusion is satisfied:

Corollary 3. Let x_1, x_2, x_3 be real numbers. Then the following conditions are satisfied:

a) $[(x_1 = 0 \wedge x_2 \neq 0 \wedge x_3 \neq 0) \vee (x_2 = 0 \wedge x_1 \neq 0 \wedge x_3 \neq 0) \vee (x_3 = 0 \wedge x_1 \neq 0 \wedge x_2 \neq 0)] \Longrightarrow [(x_1x_2 + x_2x_3 + x_3x_1)^2 > 3(x_1x_2x_3)(x_1 + x_2 + x_3)]$;

b) $[(x_1x_2 + x_2x_3 + x_3x_1)^2 > 3(x_1x_2x_3)(x_1 + x_2 + x_3) \wedge x_1 = 0] \Longrightarrow [x_2 \neq 0 \wedge x_3 \neq 0]$;

c) $[(x_1x_2+x_2x_3+x_3x_1)^2 > 3(x_1x_2x_3)(x_1+x_2+x_3) \wedge x_2 = 0] \Longrightarrow [x_1 \neq 0 \wedge x_3 \neq 0]$;

d) $[(x_1x_2 + x_2x_3 + x_3x_1)^2 > 3(x_1x_2x_3)(x_1 + x_2 + x_3) \wedge x_3 = 0] \Longrightarrow [x_1 \neq 0 \wedge x_2 \neq 0]$.

Corollary 4.

a) Inequality (1) results from inequality (3).

b) Inequality (2) results from inequality (5).

Proof. ad a). Indeed, if $x_1x_2+x_2x_3+x_3x_1 < 0$, then of course inequality (1) is satisfied. If $x_1x_2+x_2x_3+x_3x_1 \geq 0$, then $3(x_1x_2+x_2x_3+x_3x_1) \geq x_1x_2+x_2x_3+x_3x_1$, and so inequality (1) is satisfied.

ad b). The proof is similar to that one in corollary a).

Corollary 5.

a) The number 3 is the largest number for which inequality (3) is satisfied for any numbers $x_1, x_2, x_3 \in \mathcal{R}$.

b) The number 3 is the largest number for which inequality (5) is satisfied for any numbers $x_1, x_2, x_3 \in \mathcal{R}$.

Proof. ad a). Let us suppose that there exists the number $a > 3$ such that $(x_1 + x_2 + x_3)^2 \geq a(x_1x_2 + x_2x_3 + x_3x_1)$ for any $x_1, x_2, x_3 \in \mathcal{R}$. Let $x_1 = x_2 = x_3 = 1$. Then $a \leq 3$, and we have a contradiction.

ad b). The proof is similar to that one in corollary a).

Task 7. (Pawłowski, 1994, p. 172, Task 10). Prove that if the equation

$$ax^3 + bx^2 + cx + d = 0 \tag{8}$$

has three real roots, then

$$b^2 \geq ac \text{ i } c^2 \geq bd. \tag{9}$$

S.: From the assumption that equation (8) has three roots, it follows that $a \neq 0$. Let $x_1, x_2, x_3 \in \mathcal{R}$ be roots of equation (8). From Vieta's formulas it follows:

$$\begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a}, \\ x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}, \\ x_1x_2x_3 = -\frac{d}{a}. \end{cases} \tag{10}$$

By virtue of inequality (1) we have:

$$(-\frac{b}{a})^2 \geq \frac{c}{a},$$

$$\frac{b^2}{a^2} \geq \frac{c}{a},$$

$$b^2 \geq ac.$$

Applying inequality (2) we have:

$$(\frac{c}{a})^2 \geq (-\frac{d}{a}) \cdot (-\frac{b}{a}),$$

$$\frac{c^2}{a^2} \geq \frac{bd}{a^2},$$

$$c^2 \geq bd.$$

Inequalities (9) are trivial, if $ac \leq 0$ and $bd \leq 0$.

We will give the examples of cube equations having three real roots, for which:
a) inequalities (9) are trivial,
b) inequalities (9) are not trivial.

ad a). For equation $(x-1)^2(x+1) = x^3 - x^2 - x + 1 = 0$ we have:
$a = 1$, $b = -1$, $c = -1$, $d = 1$, $ac = -1 < 0$, $bd = -1 < 0$.

ad b). For equation $(x+1)^2(x+3) = x^3 + 5x^2 + 7x + 3 = 0$ we have:
$a = 1$, $b = 5$, $c = 7$, $d = 3$, $ac = 7 > 0$, $bd = 15 > 0$.

We will give the examples of cube equations, for which inequalities (9) will allow us to conclude that they do not have three real roots.

Consider the equation:

$$x^3 + 2x^2 + 5x + 4 = 0. \tag{11}$$

We have: $a = 1$, $b = 2$, $c = 5$, $d = 4$, $b^2 = 4$, $ac = 5$, $b^2 < ac$, $c^2 = 25$, $bd = 8$, $c^2 > bd$. Then equation (11) does not have three real roots.

Consider the equation:

$$x^3 + 2x^2 + 2x + 4 = 0. \tag{12}$$

We have: $a = 1$, $b = 2$, $c = 2$, $d = 4$, $b^2 = 4$, $ac = 2$, $b^2 > ac$, $c^2 = 4$, $bd = 8$, $c^2 < bd$. Then equation (12) does not have three real roots.

Consider the equation:

$$x^3 - x^2 + 2x - 8 = 0. \tag{13}$$

We have: $a = 1$, $b = -1$, $c = 2$, $d = -8$, $b^2 = 1$, $ac = 2$, $b^2 < ac$, $c^2 = 4$, $bd = 8$, $c^2 < bd$. Then equation (13) does not have three real roots.

Task 8. Let us assume that the equation

$$ax^3 + bx^2 + cx + d = 0, \tag{14}$$

where $a, b, c, d \in \mathcal{R}$, has three real roots.

Prove that:
a) The following inequalities are satisfied:

$$b^2 \geq 3ac \text{ and } c^2 \geq 3bd. \tag{15}$$

b) Equation (14) has a triple root if and only if

$$b^2 = 3ac. \tag{16}$$

c) Equation (14) has two different roots if and only if

$$b^2 > 3ac. \tag{17}$$

d) If equation (14) has a triple root, then

$$c^2 = 3bd. \tag{18}$$

e) If $d \neq 0$, then equation (14) has a triple root if and only if

$$c^2 = 3bd. \tag{19}$$

f) Let $d \neq 0$. Equation (14) has at least two different roots if and only if

$$c^2 > 3bd. \tag{20}$$

S.: From assumption that equation (14) has three roots it follows that $a \neq 0$. Let x_1, x_2, x_3 be the real roots of equation (14).

ad a). The proof of inequality (15) is analogous to the proof of inequality (9), when applying inequalities (3) and (5).

ad b). Applying condition (4) and Vieta's formulas (10) we have: $x_1 = x_2 = x_3 \Longleftrightarrow (x_1 + x_2 + x_3)^2 = 3(x_1x_2 + x_2x_3 + x_3x_1) \Longleftrightarrow (-\frac{b}{a})^2 = 3\frac{c}{a} \Longleftrightarrow \frac{b^2}{a^2} = 3\frac{c}{a} \Longleftrightarrow b^2 = 3ac$. We proved condition b).

ad c). Condition c) results directly from conditions a) and b) in this task.

ad d). Using condition (7) we get condition d).

ad e). If $d \neq 0$, then equation (14) has no zero roots, i.e. $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 \neq 0$. Applying condition (6) and Vieta's formulas (10) we have: $x_1 = x_2 = x_3 \Longleftrightarrow (x_1x_2 + x_2x_3 + x_3x_1)^2 = 3(x_1x_2x_3)(x_1 + x_2 + x_3) \Longleftrightarrow (\frac{c}{a})^2 = 3(-\frac{d}{a}) \cdot (-\frac{b}{a}) \Longleftrightarrow \frac{c^2}{a^2} = 3\frac{bd}{a^2} \Longleftrightarrow c^2 = 3bd$. We proved condition e).

ad f). Condition f) results directly from conditions a) and e) in this task.

To emphasize the significance of inequality (15) in relation to inequality (9), consider the following example.

Based on inequalities (9) and (15) determine whether the equation

$$x^3 + 2x^2 + 2x + 1 = 0 \tag{21}$$

has three real roots.

We have: $a = 1$, $b = 2$, $c = 2$, $d = 1$.

For inequality (9):
$b^2 = 4$, $ac = 2$, that is $b^2 > ac$;
$c^2 = 4$, $bd = 2$, that is $c^2 > bd$.
By means of inequality (9) we do not determine whether equation (21) has three real roots.

For inequality (15):
$b^2 = 4$, $3ac = 6$, that is $b^2 < 3ac$;
$c^2 = 4$, $3bd = 6$, that is $c^2 < 3bd$.
Therefore, based on inequality (15) we conclude that equation (21) does not have three real roots.

We will give some examples regarding the equalities and inequalities occuring in Task 8, to draw pupils' attention to the significance of the assumptions established in this task, as well as to a thorough analysis of the results obtained in this task.

ad a). For the equation $x^3+5x^2+4x+1=0$ we have: $a=1$, $b=5$, $c=4$, $d=1$. It is easy to check that $b^2>3ac$ and $c^2>3bd$. The equation under consideration has three different roots, but two of them are complex, so in general inequalities (15) do not imply the existence of three real roots for the equations of form (14).

ad b). For the equation $x^3+3x^2+3x+4=0$ we have: $a=1$, $b=3$, $c=3$, $d=4$. Notice that $b^2=3ac$. The equation under consideration has three different roots, where two of them are complex. Thus, in general equality (16), without taking into account the assumption, that the equation of form (14) has three real roots, does not imply the fact that equation (14) has a triple root.

ad c). For the equation $x^3+2x^2+x+1=0$ we have: $a=1$, $b=2$, $c=1$, $d=1$. Of course $b^2>3ac$. The equation under consideration has three different roots, where two of them are complex. Thus, in general inequality (17), without taking into account the assumption, that the equation of form (14) has three real roots, does not imply the fact that equation (14) has at least two different real roots.

ad d). For the equation $x^3-x^2=0$ we have: $a=1$, $b=-1$, $c=0$, $d=0$. Of course $c^2=3bd$. The equation under consideration has the double root equal to 0 and the root equal to 1. Thus, in general equality (18) does not imply the fact that equation (14) has a triple root.

ad e). For the equation $x^3+x^2+3x+3=0$ we have: $a=1$, $b=1$, $c=3$, $d=3$. We get $c^2=3bd$. The equation under consideration has three different roots, where two of them are complex. Thus, in general equality (19), without taking into account the assumption, that the equation of form (14) has three real roots, does not imply the fact that equation (14) has a triple root, although $d\neq 0$.

ad f). For the equation $x^3+x^2+2x+1=0$ we have: $a=1$, $b=1$, $c=2$, $d=1$. Of course $c^2>3bd$. The equation under consideration has three different roots, where two of them are complex. Thus, in general inequality (20), without taking into account the assumption, that the equation of form (14) has three real roots, does not imply the fact that equation (14) has at least two different real roots.

In virtue of Task 8 a), b) we get the following corollaries:

Corollary 6. If the equation

$$x^3+px+q=0,$$

where $p,q\in\mathcal{R}\setminus\{0\}$, has three real roots, then $p<0$.

Corollary 7. If the equation

$$x^3+px^2+q=0,$$

where $p,q\in\mathcal{R}\setminus\{0\}$, has three real roots, then $pq<0$.

Moreover, it is easy to notice that the following corollary is true.

Corollary 8. The equation

$$x^3+px^2+qx=0,$$

where $p,q\in\mathcal{R}\setminus\{0\}$, has three real roots, if and only if $p^2\geq 4q$.

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Appendix

Task 7[T]:

Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_3 \neq 0$, be a polynomial with real coefficients. Prove that:

(i) If the polynomial $f(x)$ has three positive real roots, then

$$a_1a_2 \leq 9a_0a_3.$$

(ii) If the polynomial $f(x)$ has three negative real roots, then

$$a_1a_2 \geq 9a_0a_3.$$

(iii) If the polynomial $f(x)$ has three positive or three negative real roots, then

$$a_1a_2 = 9a_0a_3 \Longleftrightarrow x_1 = x_2 = x_3.$$

Formulas:

a) (4)[T]:

$$ax^3 + bx^2 + cx + d = 0, \quad a, b, c, d \in \mathcal{R}, a \neq 0,$$

b) (5)[T]:

$$x^3 + px + q = 0,$$

c) (6)[T]:

$$p = \frac{-b^2 + 3ac}{3a^2}, \quad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3},$$

d) (7)[T]:

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}.$$

Theorems:

Theorem 8[T]. Equation 5[T] with real coefficients p and q has three different real roots if and only if $\Delta < 0$.

Theorem 9[T]. Equation 5[T] with real coefficients p and q has all real roots, where one of them is multiple if and only if $\Delta = 0$.

Theorem 10[T]. Equation 5[T] with real coefficients p and q has exactly one real roots, and the other two are complex conjugates if and only if $\Delta > 0$.

Corollaries:

Corollary 1[T]. Let a real number x_0 be a double root of the third degree polynomial $f(x)$ with real coefficients. Then the graph of the polynomial $f(x)$ is tangent to the axis of abscissae x at the point x_0 and the polynomial $f(x)$ has a local exstremum at the point x_0.

Corollary 2[T]. Let a real number x_0 be a triple root of the third degree polynomial $f(x)$ with real coefficients. Then the graph of the polynomial $f(x)$ is tangent to the axis of abscissae x at the point x_0 and the graph has the inflexion point at x_0.

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