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Application of Banach contraction principle to approximate the golden number*

Abstract. This work is devoted to the application of selected fixed point theorems in the problems of convergence of certain sequences to the golden number. It contains the theorem about the fixed point of so-called ψ -contraction specified on the closed interval $\langle a, b \rangle$ and the local version of Banach Contraction Principle as a conclusion. It will also be used to approximate the golden number.

1. Introduction

In paper (Barcz, 2019) one of the proofs of convergence of Fibonacci number quotients to the golden number is based on Edelstein's theorem about the fixed point of a mapping $f(x) = 1 + \frac{1}{x}$ of a certain interval. The presented work is a development of the issue of approximation of the golden number using other fixed point theorems, in particular the local version of Banach Contraction Principle. We get this version as a conclusion from the fixed point theorem of so-called ψ -contraction. In this work we will present the application of the local version of the Banach Principle to show the convergence of quotients: $\frac{f_{n+1}}{f_n}$ Fibonacci numbers f_n and $\frac{G_{n+1}}{G_n}$ of generalizations these numbers G_n .

It is worth mentioning that the classic proofs of the Banach principle uses a sequence of iterations (a sequence of successive approximations). The proof of the version of the Banach principle presented here, which applies to mapping of type " $|f(x) - f(y)| \leq \psi(|x - y|$ ", where the function ψ meets certain conditions, is different. We use the principle of descending intervals, and next we will obtain the Banach principle for a contraction f on a closed interval. Because a mapping

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$f, f(x) = 1 + \frac{1}{x}$, is not a contraction on the interval $\langle 1, 2 \rangle$, but it is a contraction on the interval $\langle 1 + \varepsilon, 2 \rangle, 0 < \varepsilon \leq 0,5$, we use Banach Contraction Principle for $f: \langle 1 + \varepsilon, 2 \rangle \rightarrow \langle 1 + \varepsilon, 2 \rangle$.

Moreover, the fixed point of the mapping f as a consequence of its existence for the mapping $f^2 = f \circ f : \langle 1, 2 \rangle \rightarrow \langle 1, 2 \rangle$ which is a contraction with the constant $q < 1$ is also used to show the convergence of the quotients of neighboring Fibonacci numbers f_n .

This work and work Barcz, 2019 may have didactic value. They can be used to highlight certain relationships connecting fixed points with the golden number, the golden number with fractals, and fractals with fixed points. In this works we did not discuss the relationship between the golden number and fractals, while the relationship between fractals and fixed points of contractions is visible, for example, in the result of the iterative procedure which is the Sierpinski triangle — one of the first fractals. The above mentioned relationships emphasize their numerous connections with the mathematics program in schools and colleges. They can be developed as part of other activities, for example mathematical circles.

Topics such as the above mentioned relationships are also important and useful, for example, in computer science and biology (where, for example, the golden number is the key to understanding the geometry of spirals in sunflower).

2. Some fixed point theorems and their applications

DEFINITION 1

A golden section of the segment of length d is called a division into smaller sections of lengths x and $d - x$, in which

$$\frac{d}{x} = \frac{x}{d - x}.$$

DEFINITION 2

For a given rectangle with side lengths in the ratio $1 : x$, we will call the golden proportion of the only ratio $1 : \varphi$ at which the original rectangle can be divided into a square and a new rectangle which has the same ratio of sides $1 : \varphi$.

DEFINITION 3

The golden rectangle is called a rectangle in which the ratio of the length of its sides is $1 : \varphi$.

DEFINITION 4

Fibonacci sequence is a sequence defined recursively as follows:

$$f_1 = f_2 = 1, f_{n+1} = f_{n-1} + f_n, n \geq 2$$

(sometimes formally accepted $f_0 = 0$ and then the recursive formula is valid for $n \geq 1$).

DEFINITION 5

Fibonacci numbers are called consecutive terms of the sequence (f_n) .

DEFINITION 6

A sequence (F_n) of the form $F_{n+1} = F_n + F_{n-1}, n \geq 2$, where F_1 and F_2 are given positive integers we call a Fibonacci type sequence.

DEFINITION 7

A generalized Fibonacci sequence is a sequence (G_n) defined recursively as follows: $G_{n+1} = G_n + G_{n-1}, n \geq 2$, with $G_1 = a$ and $G_2 = b, a, b > 0$.

THEOREM 1

(The principle of descending intervals) Let (Δ_n) be a sequence of closed intervals such that

- (i) $\Delta_{n+1} \subset \Delta_n$ for each n ;
- (ii) $\lim_{n \rightarrow \infty} d_n = 0$ where d_n is the length of interval Δ_n .

Then $\bigcap_{n=1}^{\infty} \Delta_n$ is a set consisting of exactly one point.

DEFINITION 8

We say that the mapping $f : \langle a, b \rangle \rightarrow \langle a, b \rangle$ is ψ -contraction if it meets the condition

$$|f(x) - f(x')| \leq \psi(|x - x'|)$$

for all $x, x' \in \langle a, b \rangle$, where $\psi : \langle 0, \infty \rangle \rightarrow \langle 0, \infty \rangle$ is any function such that

- (i) ψ is non-decreasing and right continuous;
- (ii) $\psi^n(t) \rightarrow 0$ for each $t > 0$.

LEMMA 1

(see Barcz, 1983) Let $\psi : \langle 0, \infty \rangle \rightarrow \langle 0, \infty \rangle$ be a non-decreasing function such that $\psi^n(t) \rightarrow 0$ for each $t > 0$. Then $\psi(t) < t$ for each $t > 0$.

THEOREM 2

(Bolzano) If f is a continuous and non-constant function defined on the closed interval $\langle a, b \rangle$, then its image $f(\langle a, b \rangle)$ is the interval $\langle a', b' \rangle$.

THEOREM 3

Each ψ -contractive mapping $f : \langle a, b \rangle \rightarrow \langle a, b \rangle$ has exactly one fixed point (i.e. such a unique point $u \in \langle a, b \rangle$ that $f(u) = u$), and $f^n(x) \rightarrow u$ for each $x \in \langle a, b \rangle$.

Proof. (Version 1)

It is easy to see that each ψ -contraction $f : \langle a, b \rangle \rightarrow \langle a, b \rangle$ is continuous. Under Theorem 2 there is interval $\langle a_1, b_1 \rangle \subset \langle a, b \rangle$ which is an image of $\langle a, b \rangle$, but the ends of $\langle a, b \rangle$ do not necessarily go into a_1 and b_1 . There are points x_1 and x_2 from $\langle a, b \rangle$ such that $f(x_1) = a_1, f(x_2) = b_1$. Because f is ψ -contractive mapping, so

$$|a_1 - b_1| = |f(x_1) - f(x_2)| \leq \psi(|x_1 - x_2|) \leq \psi(|a - b|).$$

Therefore $f(\langle a, b \rangle) = \langle a_1, b_1 \rangle \subset \langle a, b \rangle$ and $|a_1 - b_1| \leq \psi(|a - b|)$. Next f transforms $\langle a_1, b_1 \rangle$ into $\langle a_2, b_2 \rangle \subset \langle a_1, b_1 \rangle$ and $|a_2 - b_2| \leq \psi(|a_1 - b_1|)$. Let $\Delta_1, \Delta_2, \dots, \Delta_n, \dots$, where $\Delta_n = \langle a_n, b_n \rangle$ be a sequence of intervals obtained with $\langle a, b \rangle$ by applying f again. Then

$$|a_n - b_n| \leq \psi(|a_{n-1} - b_{n-1}|).$$

Now assuming that the inequality $|a_{n-1} - b_{n-1}| \leq \psi^{n-1}(|a - b|)$ is true for $n > 1$, we will show the inequality $|a_n - b_n| \leq \psi^n(|a - b|)$. We have

$$|a_n - b_n| \leq \psi(|a_{n-1} - b_{n-1}|) \leq \psi(\psi^{n-1}(|a - b|)) = \psi^n(|a - b|).$$

By induction we get

$$|a_n - b_n| \leq \psi^n(|a - b|)$$

for each natural n . Therefore the lengths $d_n = |a_n - b_n|$ of intervals $\langle a_n, b_n \rangle$ go to zero because $\psi^n(|a - b|) \rightarrow 0$.

We also get a sequence of intervals

$$\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \dots$$

because $\Delta_{n+1} = f(\Delta_n) \subset f(\Delta_{n-1}) = \Delta_n$ assuming that $\Delta_n \subset \Delta_{n-1}, n > 1$ (considering that $\Delta_2 \subset \Delta_1$).

Thus having regard the previous result $d_n \rightarrow 0$ ($n \rightarrow \infty$) by Theorem 1 we get exactly one point u belonging to all Δ_n , i.e. belonging to $\bigcap_{n=1}^{\infty} \Delta_n$. Because

$$f\left(\bigcap_{n=1}^{\infty} \Delta_n\right) \subset \bigcap_{n=1}^{\infty} f(\Delta_n) = \bigcap_{n=1}^{\infty} \Delta_n = \{u\},$$

so $f(u) = u$, i.e. u is the unique fixed point of f .

Let us now notice that $u = f(u) = f^2(u) = \dots$. Therefore, taking any $x \in \langle a, b \rangle$, for $n = 1, 2, \dots$ we have

$$\begin{aligned} |f^n(x) - u| &= |f^n(x) - f^n(u)| \leq \psi(|f^{n-1}(x) - f^{n-1}(u)|) \leq \\ &\leq \psi^2(|f^{n-2}(x) - f^{n-2}(u)|) \leq \dots \leq \psi^n(|x - u|). \end{aligned}$$

Because $\psi^n(|x - u|) \rightarrow 0$, so $f^n(x) \rightarrow u$.

(Version 2) Let $F(x) = f(x) - x$ and notice that $F(a) \geq 0, F(b) \leq 0$. It follows from Theorem 2 that $F(u) = 0$ for some $u \in \langle a, b \rangle$, i.e. $f(u) = u$. The fixed point u is unique. Indeed, assuming that $f(v) = v$ and $u \neq v$ we get a contradiction $|u - v| = |f(u) - f(v)| \leq \psi(|u - v|) < |u - v|$. Now taking any $x \in \langle a, b \rangle$, for $n = 1, 2, \dots$ we have

$$|f^n(x) - u| = |f^n(x) - f^n(u)| \leq \psi(|f^{n-1}(x) - f^{n-1}(u)|) \leq \dots \leq \psi^n(|x - u|) \rightarrow 0,$$

so $f^n(x) \rightarrow u$.

Before the formulation Banach Contraction Principle for contraction f (with constant $0 < q < 1$) specified on $\langle a, b \rangle$ we check whether the function $\psi(t) = qt$ meets the conditions (i), (ii) given in Definition 8. Checking condition (i) is not a problem. We will show condition (ii): $\psi^n(t) = q^n t \rightarrow 0$ for each $t > 0$. For this purpose it is enough to use the following fact, which (simple) proof we will present for the completeness of the argument.

FACT 1

$\lim_{n \rightarrow \infty} q^n = 0$ for $0 < q < 1$.

Proof. The sequence (q^n) meets the conditions:

1° $q^{n+1} < q^n, (n = 1, 2, \dots)$, which means that the sequence (q^n) is decreasing,

2° the sequence is bounded from the bottom because $q^n \geq 0$ for every natural n .

Therefore on the basis of one of the theorems (of analysis) this sequence has a limit $\lim_{n \rightarrow \infty} q^n = G$. Of course (q^{n+1}) as a subsequence of (q^n) also has a limit equal to G . Therefore

$$\lim_{n \rightarrow \infty} q^{n+1} = q \lim_{n \rightarrow \infty} q^n = qG,$$

i.e.

$$G = qG.$$

Because $q \neq 1$, so $G = 0$.

Therefore from Theorem 3 we get Theorem 4 which is Banach Contraction Principle on a contraction on an interval.

THEOREM 4

Each contraction $f : \langle a, b \rangle \rightarrow \langle a, b \rangle$ with a constant $0 < q < 1$ has a unique fixed point u . Further, for any $x \in \langle a, b \rangle$, the iterative sequence $(f^n(x))$ converges to u .

REMARK 1

Obviously, one can make a direct proof of Theorem 4 similar to the second proof of Theorem 3 (proof in version 2).

It is worth pointing out that Theorem 3 is due to J. Matkowski (see Matkowski, 1975, Theorem 1.2) and is valid in any complete metric space. Moreover, the assumption about right continuity of ψ is superfluous.

THEOREM 5

Let $f : \langle a, b \rangle \rightarrow \langle a, b \rangle$ be a map such that $f^N : \langle a, b \rangle \rightarrow \langle a, b \rangle$ is a contraction for some N (f need not be continuous). Then f has a unique fixed point u , and the sequence of iterates $f^n(x) \rightarrow u$ for each $x \in \langle a, b \rangle$ (comp. Goebel, 2005).

(We will present a proof for the completeness.)

Proof. Based on Theorem 4 f^N has a unique fixed point $u = f^N(u)$. However $f^N(f(u)) = f(f^N(u)) = f(u)$, therefore $f(u)$ is also a fixed point of f^N . Because the fixed point of f^N is only one, so $f(u) = u$. If for another point $v = f(v)$, then $v = f(v) = \dots = f^N(v)$, so $v = u$.

Now we want to show that for $x_0 \in \langle 1, 2 \rangle$ such that $x_0 = \frac{f_{i+1}}{f_i}$ for some i $f^n(x_0) \rightarrow \varphi$, where $f : \langle 1, 2 \rangle \rightarrow \mathbb{R}$ is defined by $f(x) = 1 + \frac{1}{x}$. It turns out that f is not a contraction of the interval $\langle 1, 2 \rangle$. Namely $|f(x) - f(x')| \leq \frac{|x-x'|}{xx'} \leq |x-x'|$ for $x, x' \in \langle 1, 2 \rangle$.

We have $f(\langle 1, 2 \rangle) \subset \langle 1, 2 \rangle$. Indeed, $f(1) = 1 + 1 = 2$, $f(2) = 1 + \frac{1}{2} > 1$. Because f is decreasing function, so $f(\langle 1, 2 \rangle) \subset \langle 1, 2 \rangle$. Therefore the second iteration of f $f^2 : \langle 1, 2 \rangle \rightarrow \langle 1, 2 \rangle$, because

$$f(f(\langle 1, 2 \rangle)) \subset f(\langle 1, 2 \rangle) \subset \langle 1, 2 \rangle.$$

We will now examine whether f^2 is a contraction. We have

$$|f^2(x) - f^2(x')| \leq \frac{|x-x'|}{1 + \frac{1}{x} + \frac{1}{x'} + \frac{1}{xx'}} \leq \frac{4}{9}|x-x'|$$

for $x, x' \in \langle 1, 2 \rangle$. Therefore f^2 is the contraction. By Theorem 5 f has in $\langle 1, 2 \rangle$ a unique fixed point u , and the sequence of iterates $f^n(x_0) \rightarrow u$ for each $x_0 \in \langle 1, 2 \rangle$. Take $x_0 = \frac{f_3}{f_2} = 2$, then $f^n(2) \rightarrow u$. Take now $x_0 = \frac{f_2}{f_1} = 1$, then $f^n(1) \rightarrow u = \varphi$ ($u = \varphi$ because $u = 1 + \frac{1}{u}$).

THEOREM 6

Let $C = \langle 1, 2 \rangle$. The mapping $f : C \rightarrow C$, $f(x) = 1 + \frac{1}{x}$, has a unique fixed point $u = \varphi$, and $\lim_{n \rightarrow \infty} f^n\left(\frac{f_2}{f_1}\right) = \varphi = \lim_{n \rightarrow \infty} f^n\left(\frac{f_3}{f_2}\right)$ (where $f_1 = f_2 = 1, f_3 = 2$).

The notion of compactness plays an important role in Edelstein's theorem (see Remark 2 below). In the case of metric space (X, d) we can say that set $A \subset X$ is compact, if we can choose a convergent subsequence from each sequence of elements in set A .

In the n -dimensional space \mathbb{R}^n we know the following criterion of compactness: the set $A \subset \mathbb{R}^n$ is compact if and only if, it is closed and bounded.

We will not use this criterion in the proof given below Bolzano-Weierstrass theorem instead we will use the principle of descending intervals to illustrate the application of this principle other than the one presented earlier (in proof Theorem 3). This proof is similar to other inductive proofs of this theorem.

THEOREM 7

(Bolzano-Weierstrass) Interval $\langle a, b \rangle \subset \mathbb{R}$ is a compact set.

Proof. We will prove that for any sequence (x_n) of points in this interval, we can choose the subsequence (x_{n_k}) converging to the point (x_0) of this interval. We have the sequence (x_n) , $x_n \in \langle a, b \rangle \subset \mathbb{R}$. Let's denote $\Delta_0 = \langle a, b \rangle$. Divide Δ_0 into two equal parts Δ'_0, Δ''_0 . Because at least one of them contains infinitely

many terms of the sequence (x_n) , so marking it by Δ_1 we choose the term x_{n_1} contained in this part and denote $y_1 = x_{n_1}$. We divide Δ_1 in half again and again from part Δ_2 containing an infinite number of the terms choose x_{n_2} and denote $y_2 = x_{n_2}$. By proceeding so we get a descending sequence of the following intervals: $\Delta_0, \Delta_1, \Delta_2, \dots$ with lengths going to zero (because the length d_k of Δ_k is equal to $\frac{b-a}{2^k}$). Each Δ_k contains the term $y_k = x_{n_k}$ of the sequence (x_n) . By the principle of descending intervals, there is exactly one point x_0 belonging to all $\Delta_k, (k = 0, 1, 2, \dots)$, and in particular to $\langle a, b \rangle$. The convergence y_k to x_0 is obvious.

This theorem will be useful in the following considerations in Remark 2 about the convergence of the sequence $\left(\frac{f_{n+1}}{f_n}\right)$ to the golden number.

REMARK 2

Work (Barcz, 2019) presents the theorem on the convergence of the sequence of quotients $\frac{f_{n+1}}{f_n}$ and it has been shown that $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \varphi$ using the Edelstein fixed point theorem which says that : if X is a compact metric space and $f : X \rightarrow X$ is a mapping such that

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y, \quad (*)$$

then f has exactly one fixed point, and the iterative sequence $(f^n(x))$ converges to the fixed point.

In work (Barcz, 2019), the interval $C = \langle 1, 2\varphi - 1 \rangle$ with the metric $d(x, y) = |x - y|$ was considered. On the basis of Bolzano-Weierstrass theorem C is a compact space. The mapping $f : C \rightarrow C, f(x) = 1 + \frac{1}{x}$, satisfies condition $(*)$ of Edelstein's theorem. Let's add that $f^n(x_0) \rightarrow \varphi$, where $x_0 = \frac{f_2}{f_1} = 1$ (see Barcz, 2019).

Let us now consider the problem of the existence of a limit of the sequence $\left(\frac{f_{n+1}}{f_n}\right)$ on the basis of Theorem 4.

Since $f(x) = 1 + \frac{1}{x}$ is not a contraction on the interval $\langle 1, 2 \rangle$, we will consider this mapping on the closed interval $C = \langle 1 + \varepsilon, 2 \rangle$, where $0 < \varepsilon \leq 0,5$. Mapping $f, f(x) = 1 + \frac{1}{x}$, meets conditions:

1° $f(C) \subset C$ because f is decreasing and $f(1 + \varepsilon) < 2, f(2) = 1,5 \geq 1 + \varepsilon$

2° for every $x, x' \in C$

$$|f(x) - f(x')| = \left| \frac{1}{x} - \frac{1}{x'} \right| = \frac{|x - x'|}{xx'} \leq \frac{|x - x'|}{(1 + \varepsilon)^2},$$

therefore

$$|f(x) - f(x')| \leq q|x - x'|, \text{ where } q = \frac{1}{(1 + \varepsilon)^2} < 1.$$

Based on Theorem 4 f has a unique fixed point $u \in \langle 1 + \varepsilon, 2 \rangle$. Because $u = 1 + \frac{1}{u}$, i.e. $u^2 - u - 1 = 0$, so $u = \varphi$, and the iteration sequence $f^n(x_0) \rightarrow u$, where $x_0 = \frac{f_3}{f_2} = 2$. Therefore we received the following

THEOREM 8

Let $C = \langle 1 + \varepsilon, 2 \rangle$. The mapping $f : C \rightarrow C$, $f(x) = 1 + \frac{1}{x}$, is a contraction, and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f^n(x_0) = \varphi, \text{ where } x_n = \frac{f_{n+1}}{f_n} = 1 + \frac{1}{x_{n-1}}, x_0 = \frac{f_3}{f_2} = 2.$$

Let us remind that the generalized Fibonacci sequence (G_n) is defined as follows

$$G_1 = a, G_2 = b \ (a, b > 0), G_{n+1} = G_n + G_{n-1} \ (n > 1).$$

Let $C = \langle 1 + \frac{a}{b}, 2 \rangle$ and $\frac{b}{a} \geq 2$.

Note that the mapping f specified on C by the formula $f(x) = 1 + \frac{1}{x}$ has an image $f(C) \subset C$. Indeed, $f(1 + \frac{a}{b}) = 1 + \frac{b}{a+b} < 2$, $f(2) = 1 + \frac{1}{2} \geq 1 + \frac{a}{b}$ (because $\frac{b}{a} \geq 2$) and f is decreasing. In addition, f is a contraction with the constant $q < 1$, because we get

$$|f(x) - f(x')| = \frac{|x - x'|}{xx'} \leq \frac{|x - x'|}{(1 + \frac{a}{b})^2} \text{ and } q = \frac{1}{(1 + \frac{a}{b})^2} < 1.$$

Therefore we get the convergence $f^n(x_0) \rightarrow \varphi$ for any $x_0 \in C$.

So for $x_0 = \frac{G_3}{G_2} = 1 + \frac{a}{b}$, i.e. for the beginning of the interval C we have the following

THEOREM 9

$f^n(1 + \frac{a}{b}) \rightarrow \varphi$, where $f(x) = 1 + \frac{1}{x}$, $x \in \langle 1 + \frac{a}{b}, 2 \rangle$, $\frac{b}{a} \geq 2$ and $a, b > 0$.

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