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Cardinality of the sets of all bijections, injections and surjections*

Abstract. The results of Zarzycki for the cardinality of the sets of all bijections, surjections, and injections are generalized to the case when the domains and codomains are infinite and different. The elementary proofs the cardinality of the sets of bijections and surjections are given within the framework of the Zermelo-Fraenkel set theory with the axiom of choice. The case of the set of all injections is considered in detail and more explicit an expression is obtained when the Generalized Continuum Hypothesis is assumed.

1. Introduction

Students of Mathematics encountered the concepts of injection, bijection, and surjection already in the first year of study, during the „Introduction to Mathematics” course. They also became familiar with the cardinality of sets and cardinal numbers. Many examples of the cardinality determination can be found in the collections of problems related to the Introductory Mathematics or Set Theory. They are usually limited to the sets of functions that are defined on the set of natural or real numbers, and satisfy a certain condition. The determination of the cardinality of the sets of all injections, surjections and bijections is an interesting issue. For a function defined on the finite domain or codomain, the solution is simple and well known. The problem is more difficult for a set of functions $X \rightarrow Y$, where X i Y are infinite sets, and in the Set Theory textbooks the explicit expression for its cardinality is missing. For example, in Grell, 2006, the following theorem can be found:

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$$\beta \leq \alpha \wedge \alpha \geq \aleph_0 \Rightarrow \binom{\alpha}{\beta} = \alpha^\beta, \text{ where } \alpha, \beta \text{ are cardinal numbers,}$$

and the binomial coefficient is defined by:

$$\binom{\alpha}{\beta} = |\mathcal{P}_\beta(\alpha)| \text{ where } \mathcal{P}_\beta(\alpha) = \{Y \subset \alpha : |Y| = \beta\}.$$

Using the above theorem and some general facts about the cardinality of sets, it is possible to determine the cardinality of the set of all injections. But the problem is left to the reader.

Zarzycki, 1995 determined the cardinality of the set $\{f : f \in X^X, \aleph_0 \leq |X|\}$, where f is an injection, surjection or bijection, but only if f has the same domain and codomain. In this paper, I show the generalization of his results to the set of functions $f : X \rightarrow Y$ where f is a bijection, surjection, or injection, satisfying the condition that $\aleph_0 \leq |X| = |Y|$, $|X| \geq |Y| \geq \aleph_0$, and $1 \leq |X| \leq |Y|$, respectively.

The paper is organized as follows. In the first Section, I present elementary proofs using concepts that are understandable by the first-year students of Mathematics. The resulting theorem 2.2 states that the cardinality of the set of all bijections $X \rightarrow Y$ is equal to the cardinality of the set of all subsets of X , if the domain of f is infinite. Using this result, I will show in the second Section that in the case of the set of all surjections the result is the same. Finally, I will analyze the cardinality of the set of all injections in the third Section. More advanced mathematical concepts are required to analyse this case. The obtained $|Y|^{|X|}$ result is not very precise, but it can be made more specific if the Generalized Continuum Hypothesis (GCH) is assumed.

The proofs carried out in this paper are based on the ZFC (Zermelo–Fraenkel set theory with the axiom of choice) framework.

2. Cardinality of the set of all bijections $X \rightarrow Y$

The problem of the cardinality of the set of all bijections will be analysed, when X and Y are infinite. First, the limited case of bijections $X \rightarrow X$, is considered, for which the following lemma can be formulated:

LEMMA 2.1

Let X be an arbitrary infinite set. There exists a $X_1 \cup X_2$ partition of X , such that the X, X_1, X_2 sets have the same cardinality, so that

$$X_1, X_2 \subset X \wedge X \sim X_1 \sim X_2 \wedge X_1 \cup X_2 = X \wedge X_1 \cap X_2 = \emptyset.$$

Sketch of the proof.

It follows from the multiplication properties of cardinal numbers that if $|X| \geq \aleph_0$, then $|X| \cdot 2 = \max\{|X|, 2\} = |X|$, since $X \times \{1, 2\} \sim X$. Hence, there exists a bijection $h : X \times \{1, 2\} \rightarrow X$. Therefore:

$$h(X \times \{1\}) \sim X \times \{1\}, \quad h(X \times \{2\}) \sim X \times \{2\}, \quad (1)$$

and

$$X \times \{1\} \cup X \times \{2\} = X \times \{1, 2\}. \quad (2)$$

It is obvious that $X \times \{2\} \sim X$, so finally, by using (1),(2), and denoting $X_1 = h(X \times \{1\})$ and $X_2 = h(X \times \{2\})$, the lemma is obtained.

THEOREM 2.1

Let X be an arbitrary infinite set. The cardinality of the set of bijections $X \rightarrow X$ is equal to the cardinality of the set of all subsets of X :

$$\{f : f \in X^X \wedge |X| \geq \aleph_0, f \text{ is a bijection}\} \sim \mathcal{P}(X).$$

PROOF:¹

By Lemma 2.1 there exists such $X_1 \cup X_2$ partition of X , that the sets X_1, X_2 have the same cardinality. Let $g : X_1 \rightarrow X_2$ be a fixed arbitrary bijection and $A_i \in \mathcal{P}(X_1)$ an arbitrary subset of X_1 , $i \in I$, where I is the index set. It can be noted that for fixed $i_0 \in I$. the sets A_{i_0} and $g(A_{i_0})$ are disjoint. Let us define the following function:

$$f_{A_{i_0}}(x) := \begin{cases} x, & \text{for } x \in X \setminus (A_{i_0} \cup g(A_{i_0})), \\ g(x), & \text{for } x \in A_{i_0}, \\ g^{-1}(x), & \text{for } x \in g(A_{i_0}). \end{cases} \quad (3)$$

I am going to show that $f_{A_{i_0}}$ is one-to-one and onto. In order to do this, I will first show that $f_{A_{i_0}}$ is an injection, which means that $x_1 \neq x_2 \implies f_{A_{i_0}}(x_1) \neq f_{A_{i_0}}(x_2)$ for any $x_1, x_2 \in X$.

Let $x_1, x_2 \in X$. Note that X is the union of three sets² :

A_{i_0} , $g(A_{i_0})$ and $X \setminus (A_{i_0} \cup g(A_{i_0}))$, and the images of these sets:

$$f_{A_{i_0}}(A_{i_0}) = g(A_{i_0}) \subset X_2, \quad (4)$$

$$f_{A_{i_0}}(g(A_{i_0})) = g^{-1}(g(A_{i_0})) = A_{i_0} \subset X_1 \quad (5)$$

¹The proof of theorem 2.1 was inspired by the proof of the special case of bijection $\mathbb{R} \rightarrow \mathbb{R}$ that was presented by Paweł Wójcik. The sketch of the proof is following: Let $A \subset (0, +\infty)$, $f_A : \mathbb{R} \rightarrow \mathbb{R}$ is given by the formula:

$$f_A(x) = \begin{cases} 0, & \text{if } x = 0, \\ -x, & \text{if } x \in A \text{ or } x \in -A \text{ (where } -A := \{x : -x \in A\}), \\ x, & \text{in other cases.} \end{cases}$$

It is easy to verify that: 1) f_A is a bijection, 2) $A \neq B \implies f_A \neq f_B$, 3) the cardinality of the set of all bijections different bijections f_A is equal to the cardinality of the powerset $\mathcal{P}(\mathbb{R})$, because $|\mathcal{P}((0, +\infty))| = |\mathcal{P}(\mathbb{R})|$. This reasoning cannot be extended directly to the case of any infinite set X , because the fact that real numbers are elements of the ordered field is used here.

²The sum of these sets is X , they are pairwise disjoint, but they don't have to be a partition of the set X because one or two of them can be empty.

$$f_{A_{i_0}}\left(X \setminus (A_{i_0} \cup g(A_{i_0}))\right) = \text{id}\left(X \setminus (A_{i_0} \cup g(A_{i_0}))\right) = X \setminus (A_{i_0} \cup g(A_{i_0})), \quad (6)$$

are pairwise disjoint. It means that for both x_1, x_2 elements not belonging to any of the three sets simultaneously, $f_{A_{i_0}}(x_1) \neq f_{A_{i_0}}(x_2)$.

Three cases are remained to consider, when $x_1 \neq x_2$ and x_1, x_2 belong to one of the sets only:

1. $x_1, x_2 \in X \setminus (A_{i_0} \cup g(A_{i_0}))$. Then $f_{A_{i_0}}(x_1) = x_1$, $f_{A_{i_0}}(x_2) = x_2$, hence $f_{A_{i_0}}(x_1) \neq f_{A_{i_0}}(x_2)$.
2. $x_1, x_2 \in A_{i_0}$. Then $f_{A_{i_0}}(x_1) = g(x_1)$, $f_{A_{i_0}}(x_2) = g(x_2)$, hence $f_{A_{i_0}}(x_1) \neq f_{A_{i_0}}(x_2)$, because g is a bijection.
3. $x_1, x_2 \in g(A_{i_0})$. Then $f_{A_{i_0}}(x_1) = g^{-1}(x_1)$, $f_{A_{i_0}}(x_2) = g^{-1}(x_2)$, hence $f_{A_{i_0}}(x_1) \neq f_{A_{i_0}}(x_2)$, because g^{-1} is a bijection.

It can be concluded that $f_{A_{i_0}}$ is the injection.

In the next step I am going to show that the function $f_{A_{i_0}}$ is a surjection. It means, that $f_{A_{i_0}}(X) = X$.

Since $X = A_{i_0} \cup g(A_{i_0}) \cup (X \setminus (A_{i_0} \cup g(A_{i_0})))$ and by (4),(5),(6), one can conclude that

$$\begin{aligned} f_{A_{i_0}}(X) &= f_{A_{i_0}}\left(A_{i_0} \cup g(A_{i_0}) \cup (X \setminus (A_{i_0} \cup g(A_{i_0})))\right) = \\ &= f_{A_{i_0}}(A_{i_0}) \cup f_{A_{i_0}}(g(A_{i_0})) \cup f_{A_{i_0}}(X \setminus (A_{i_0} \cup g(A_{i_0}))) = \\ &= g(A_{i_0}) \cup A_{i_0} \cup (X \setminus (A_{i_0} \cup g(A_{i_0}))) = X, \end{aligned}$$

therefore the function $f_{A_{i_0}}$ is the surjection.

The proof that $f_{A_{i_0}}$ is the bijective function was independent of the choice of the set A_{i_0} , hence the function f_{A_i} is the bijection for all sets $A_i \in \mathcal{P}(X_1)$:³

$$f_{A_{i_0}}(X) = X.$$

Next I am going to show that $f_{A_i} \neq f_{A_j}$ for arbitrary $A_i, A_j \in \mathcal{P}(X_1)$, which satisfies the condition $A_i \neq A_j$, i.e. there exists $x \in X$ such that $f_{A_i}(x) \neq f_{A_j}(x)$. If $A_i \neq A_j$, the existence of x_0 such that $x_0 \in A_i$ and $x_0 \notin A_j$ can be assumed. Then $f_{A_i}(x_0) = g(x_0) \in X_2$, while $f_{A_j}(x_0) = g^{-1}(x_0) \in X_1$ or $f_{A_j}(x_0) = x_0$. In the first case $f_{A_i}(x_0) \neq f_{A_j}(x_0)$, since $X_2 \cap X_1 = \emptyset$. In the second case $f_{A_i}(x_0) = g(x_0) \neq x_0$, since $g: X_1 \rightarrow X_2$, therefore $f_{A_i}(x_0) \neq f_{A_j}(x_0)$.

In summary, if X is an infinite set and functions $f_{A_i}, f_{A_j} \in X^X$ are defined by (3), then they are bijections for all sets $A_i, A_j \subset X_1$. In addition, if $A_i \neq A_j$, then these bijections are different. Note that the number of all different bijective functions f_{A_i} is not smaller than the number of subsets X_1 . Since $|X_1| = |X|$, one has:

³Also for $A_i = \emptyset$, when f_\emptyset is the identity mapping.

$$|\mathcal{P}(X)| \leq \{f : f \in X^X \wedge |X| \geq \aleph_0, f \text{ is a bijection}\}. \quad (7)$$

In order to find the upper bound for the cardinality of the set of all bijections, one can notice that every bijection $X \rightarrow X$ is included in the set of all functions $X \rightarrow X$. Therefore:

$$|\{f : f \in X^X\}| = |X|^{|X|} = 2^{|X|}.$$

Here the following property was used: For any infinite set X (for any infinite cardinality α) one has $|X|^{|X|} = 2^{|X|}$ ($\alpha^\alpha = 2^\alpha$).⁴

Therefore:

$$|\{f : f \in X^X \wedge |X| \geq \aleph_0, f \text{ is a bijection}\}| \leq |\mathcal{P}(X)| \quad (8)$$

By (7),(8) one has:

$$|\mathcal{P}(X)| \leq |\{f : f \in X^X \wedge |X| \geq \aleph_0, f \text{ is a bijection}\}| \leq |\mathcal{P}(X)|,$$

which completes the proof.

Finally I am going to prove that the cardinality of the set of all bijections $f \in X^X$ is equal to the cardinality of the set of all bijections $g \in Y^X$, where⁵ $Y \sim X$. It is enough to observe that, if $g_0 \in Y^X$ is an arbitrary but fixed bijection, then the function $X^X \ni f \rightarrow g = g_0 \circ f \in Y^X$ is a bijection as well, hence both sets have the same cardinality. Therefore the following theorem can be formulated:

THEOREM 2.2

Let X, Y be the arbitrary infinite sets of the same cardinality. The cardinality of the set of all bijections $X \rightarrow Y$ is equal to the cardinality of the set of all subsets of X :

$$\{f : f \in Y^X \wedge |X| = |Y| \geq \aleph_0, f \text{ is a bijection}\} \sim \mathcal{P}(X).$$

3. Cardinality of the set of all surjections $X \rightarrow Y$

THEOREM 3.1

Let X, Y be the infinite sets and $|X| \geq |Y|$. The cardinality of the set of surjections $X \rightarrow Y$ is equal to the cardinality of the set of all subsets of X :

$$\{f : f \in Y^X \wedge |X| \geq |Y| \geq \aleph_0, f \text{ is a surjection}\} \sim \mathcal{P}(X).$$

⁴This property results from a more general theorem. If κ and λ are cardinal numbers such that λ is infinite and $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$. See Guzicki, Zakrzewski, 2005, theorem E 24, with proof.

⁵The condition that the sets X and Y have the same cardinality might seem unnecessary, but it is, because if $|X| \neq |Y|$, the set of all bijections $X \rightarrow Y$ is empty.

PROOF:

Let $|X| = |Y|$.

According to Theorem 2.2, the set of all bijections and $\mathcal{P}(X)$ have the same cardinality, and the first is contained in the set of all surjections. This in turn is contained in the set of all functions, which has the same cardinality as $\mathcal{P}(X)$. Hence, the set of all surjections and the power set $\mathcal{P}(X)$ have the same cardinality as well.

Let $|X| > |Y|$.

Let us fix $A \subset X$, such that $|A| = |Y|$, denote h to be an arbitrary bijective function from A to Y , and define the function $g_i : X \setminus A \rightarrow Y$, where $i \in I$, where I is the index set. Since the sets X and $X \setminus A$ are disjoint, it follows that $|X \setminus A| + |A| = |X|$. On the other hand, by properties of cardinal numbers, one has $|X \setminus A| + |A| = \max\{|X \setminus A|, |A|\} = |X \setminus A|$, since $|X \setminus A| = |X|$. Hence, the cardinality of the set of all functions g_i is equal to:

$$|Y|^{|X \setminus A|} = |Y|^{|X|} = 2^{|X|} = |\mathcal{P}(X)|. \quad (9)$$

Let us define the function $f_i : X \rightarrow Y$ by the following formula:

$$f_i(x) = \begin{cases} h(x), & \text{if } x \in A, \\ g_i(x), & \text{if } x \in X \setminus A. \end{cases}$$

Since $f_i(X) = h(A) \cup g_i(X \setminus A) = Y \cup g_i(X \setminus A) = Y$, the function f_i is a surjection. For $g_i \neq g_j$ one has $f_i \neq f_j$, and for any fixed h the cardinality of the set of surjections f_i (which is contained in the set of all surjections) is equal to the cardinality of the set of the functions g_i . This set and the power set $\mathcal{P}(X)$ have the same cardinality by (9). On the other hand, the cardinality of the set of all functions $X \rightarrow Y$ (which contains the set of all surjections) is equal to the cardinality of the $\mathcal{P}(X)$ by (9). Therefore it can be finally concluded that the set of all surjections and $\mathcal{P}(X)$ have the same cardinality.

The following Collorary can be deduced from the above Theorem:

COLLORARY 3.1

Let X, Y be the arbitrary infinite sets. The cardinality of the set of all surjections $X \rightarrow Y$ is equal to the cardinality of the set of all subsets of X , i.e.:

$$\{f : f \in Y^X \wedge |X| = |Y| \geq \aleph_0, f \text{ is a surjection}\} \sim \mathcal{P}(X).$$

4. Cardinality of the set of all injections $X \rightarrow Y$

In order to determine the cardinality of the set of injective mappings of X into an infinite set Y , some known facts and notions will be recalled. Moreover, the terminology and notation will be fixed.

$[Y]^\lambda$ will denote the family of all subsets Z of Y whose cardinality is λ :

$$[Y]^\lambda := \{Z \subset Y : |Z| = \lambda\}.$$

If $|X| = \lambda$, one can write $[Y]^\lambda = [Y]^{|X|}$.

THEOREM 4.1

Let Y be an infinite set and λ be cardinal such that $\lambda \leq |Y|$. Then:⁶

$$|[Y]^\lambda| = |Y|^\lambda$$

COLLORARY 4.1

The family of subsets of the infinite set Y , such that the cardinality of every subset is equal to the cardinal λ , has cardinality $|Y|^\lambda$.

COLLORARY 4.2

Let $X \subset Y$ and Y be an infinite set. The cardinality of the family of subsets of Y , such that every subset has cardinality $|X|$, is equal to the cardinality of the set of all function $\{f : f : X \rightarrow Y\}$. This can be written as:

$$|\{f : f \in Y^X\}| = |Y|^{|X|} = |[Y]^{|X|}|.$$

THEOREM 4.2

Let Y be an infinite set and $1 \leq |X| \leq |Y|$. The cardinality of the set $\{f : f \in Y^X, f \text{ is an injection}\}$ is equal to the cardinality of $\{f : f \in Y^X\}$, i.e.:

$$|\{f : f \in Y^X, f \text{ is an injection}\}| = |Y|^{|X|}$$

PROOF:

For an arbitrary set $Y_1 \subset Y$ satisfying the condition $Y_1 \in [Y]^{|X|}$ (i.e. $Y_1 \subset Y$, $|Y_1| = |X|$) there exists an injection f , such that $f(X) = Y_1$. Using the Corollary 4.2 one has:

$$|Y|^{|X|} = |[Y]^{|X|} \leq |\{f : f \in Y^X, f \text{ is an injection}\}|.$$

On the other hand, the set of injections is contained in the set of all functions from X into Y , therefore:

$$|\{f : f \in Y^X, f \text{ is an injection}\}| \leq |\{f : f \in Y^X\}| = |Y|^{|X|},$$

and

$$|\{f : f \in Y^X, f \text{ is an injection}\}| = |Y|^{|X|}.$$

COLLORARY 4.3

Let X, Y be the arbitrary infinite sets having the same cardinality. The cardinality of the set of all injections $X \rightarrow Y$ is equal to the cardinality of the set of all subsets of X , i.e.:

$$\{f : f \in Y^X \wedge |X| = |Y| \geq \aleph_0, f \text{ is an injection}\} \sim \mathcal{P}(X).$$

PROOF: I

The Corollary 4.3 follows directly from the Theorem 4.2.

PROOF: II

Here the proof of the Corollary 4.3 without the use of the Theorem 4.2 will be given.

⁶See Guzicki, Zakrzewski, 2005, Theorem E 25.

Under the assumptions used in the proof of the Theorem 4.2, the set of bijections is contained in the set of injections, which in turn is contained in the set of all functions:

$$\{f : f \in Y^X, f \text{ is a bijection}\} \subset \{f : f \in Y^X, f \text{ is an injection}\} \subset \{f : f \in Y^X\}.$$

By the Theorem 2.2, the cardinality of the set of all functions $|\{f : f \in Y^X\}|$ and of the set of all bijections $|\{f : f \in Y^X, f \text{ is a bijection}\}|$ is equal to $2^{|X|}$. Therefore: $2^{|X|} = |\{f : f \in Y^X, f \text{ is a bijection}\}| \leq |\{f : f \in Y^X, f \text{ is an injection}\}| \leq |\{f : f \in Y^X\}| = 2^{|X|}$.

Based on the Theorem 2.2, Corollary 3.1 and 4.3, an interesting observation can be made:

COLLORARY 4.4

Let X, Y be infinite sets of the same cardinality. The sets of bijections, injections and surjections from X to Y and the power set of X have the same cardinality.

Let us return to the Theorem 4.2 and denote $|X| = \lambda$, $|Y| = \kappa$. It was shown that if $\aleph_0 \leq \lambda \leq \kappa$, the cardinality of the set of injections is κ^λ . The following question arises: Can this result be made more precise? For example, the number of all injective sequences of real numbers is equal to $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. However, it is not always true that if $\kappa > \lambda$, then $\kappa^\lambda = \kappa$. In order to consider this problem in more detail, let us recall two known definitions and the Theorem concerning the exponentiation of infinite cardinality number.

DEFINITION 4.1

The successor κ^+ of the cardinal κ is the smallest cardinal that is greater than κ .

According to the Generalized Continuum Hypothesis (GCH), for an arbitrary infinite cardinal number κ one has:

$$\kappa^+ = 2^\kappa.$$

DEFINITION 4.2

The cofinality of an infinite cardinal κ , denoted $cf(\kappa)$, is the smallest cardinal μ such that κ is the supremum of μ smaller cardinals.

It was proved,⁷ that for an arbitrary infinite cardinal number κ

$$\kappa < \kappa^{cf(\kappa)},$$

so if

$$cf(\kappa) \leq \lambda \leq \kappa, \text{ then } \kappa < \kappa^\lambda \leq 2^\kappa.$$

Therefore for the case ($|X| = \lambda, |Y| = \kappa$), the cardinality of the set of injections $X \rightarrow Y$ is greater than the cardinality of the power set of Y , but not necessarily is equal to the cardinality of the set of all subsets of Y .

⁷See Guzik, Zakrzewski, 2005 Theorem E 29.

There are many problems concerning the exponentiation of cardinal numbers that are not provable with ZFC. However, if we add the GCH axiom ⁸, then we will be able to prove the theorem below. It describes the exponentiation properties of infinite cardinal numbers.

THEOREM 4.3

If λ and κ are the cardinals such that κ is infinite, and $\lambda > 0$, then⁹

$$\kappa^\lambda = \begin{cases} \kappa, & \text{if } \lambda < cf(\kappa), \\ \kappa^+, & \text{if } cf(\kappa) \leq \lambda < \kappa, \\ \lambda^+, & \text{if } \kappa \leq \lambda. \end{cases}$$

Using the above Theorem, one can state the following:

THEOREM 4.4

Let Y be an infinite set and $1 \leq |X| \leq |Y|$. Assuming GCH, the cardinality of the set of all injections from X into Y is equal to the cardinality of Y , and to $2^{|Y|}$, for $|X| < cf(|Y|)$, and $cf(|Y|) \leq |X|$, respectively.

Finally, let us consider the cardinality of the set of injections. According to the obtained results, when the cardinal numbers α, β satisfy the condition $\aleph_0 \leq \beta \leq \alpha$, then the cardinal exponent α^β is equal to the cardinality of the set of injections from β into α . Since α^β is in many cases ambiguously defined under ZFC, it is possible to take one of the admissible values of the cardinality of the set of injections as an additional axiom and examine the consequences of such theory.

References

- Błaszczczyk, A., Turek, S.: 2007, *Teoria mnogości*, PWN, Warszawa.
- Enderton, H.: 1977, *Elements of set theory*, Academic Press, San Diego.
- Grell, B.: 2006, *Wstęp do matematyki Zbiory struktury modele*, Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków.
- Guzicki, W., Zakrzewski, P.: 2005, *Wykłady ze wstępu do matematyki. Wprowadzenie do teorii mnogości*, PWN, Warszawa.
- Jech, T.: 1997, *Set Theory*, Springer, Berlin.
- Kuratowski, K.: 1973, *Wstęp do teorii mnogości i topologii*, PWN, Warszawa.
- Sierpiński, W.: 1965, *Cardinal and Ordinal Numbers*, Monografie Matematyczne 34, 2nd edn, PWN, Warszawa.
- Zakrzewski, P.: 2019, *Teoria mnogości, skrypt wykładu*, Uniwersytet Warszawski. Accessible at: <https://www.mimuw.edu.pl/~piotrzak/TM-skrypt.pdf> [dostęp: 2019-05-01].
- Zarzycki, P.: 1995, The cardinality of sets of functions, *Mathematics Magazine* **68**(3), 209–211. Accessible at: DOI:10.1080/0025570X.1995.11996315.

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⁸GCH is independent of ZFC, but Sierpiński, 1965 proved that ZF with GCH implies the axiom of choice (AC).

⁹See Guzicki, Zakrzewski, 2005, Theorem E 30.