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Remarks on the Principle of Permanence of Forms*

Abstract. We discuss the role of a heuristic principle known as the *Principle of Permanence of Forms* in the development of mathematics, especially in abstract algebra. We try to find some analogies in the development of modern formal logic. Finally, we add a few remarks on the use of the principle in question in mathematical education.

1. Introductory remarks

Many design features of modern mathematics are rooted directly in the 19th century. We dare to say that at that time mathematics underwent revolutionary changes. This can be seen, among others, in: the development of abstract algebra, the emergence of new systems of geometry, efforts devoted to arithmetization of analysis on (thus abolishing intuitions related to kinematics and geometry), and the characterizations of the most important number systems (works by Dedekind, Grassmann, Peano, Weber, Cantor, and others). Below we focus our attention on certain facts from the history of abstract algebra only.

It should be noticed that up to the beginning of the 19th century mathematicians had certain reservations as far as the full acceptance of negative and complex numbers was concerned. When these numbers entered calculations they were treated as useful *tools* and not as fully legitimate mathematical objects. For instance, Francis Maseres and William Frend formulated arguments against the legitimacy of these numbers in the 17th century. Counterarguments were formulated for example by John Playfair, William Greenfield, Robert Woodhouse, and Adrien-Quentin Buée.

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Robert Woodhouse and Charles Babbage belong to the first authors who expressed the idea that algebraic expressions might not be interpreted as relating to concrete numbers only (Woodhouse, 1803), (Babbage, 1821). Surely, there are many further authors whose works are relevant in this respect but in this short note we are not going to present the history of abstract algebra in general – we limit ourselves to the role of one heuristic principle.

2. The Principle of Permanence of Forms: George Peacock and Hermann Hankel

2.1. George Peacock

George Peacock (1791–1858) published in 1830 *A Treatise on Algebra* (second edition: Peacock 1845). He presented in 1834 *Report on recent progress and present state of certain branches of analysis* (Peacock, 1834) to mathematical community at Cambridge. These works can be seen as the first which propose to treat algebra as a symbolical science based on deductive grounds. According to Peacock's opinion algebra should be a science of combinations of arbitrary signs governed by definite (though also arbitrary) rules.

Peacock distinguished *arithmetical algebra* from *symbolical algebra*. The first was based on truths about natural numbers. The second, in turn, should contain cognitively valuable assumptions and conventions. The arithmetical algebra was a *suggesting science* for the symbolical algebra. The first begins with definitions which determine the meaning of algebraic operations, the second begins with conditions or laws concerning combinations of signs:

In arithmetical algebra, the definitions of the operations determine the rules; in symbolical algebra, the rules determine the meanings of the operations, or more properly speaking, they furnish the means of interpreting them. . . . We call those rules . . . *assumptions*, in as much as they are not deducible as conclusions from any previous knowledge of those operations which have corresponding names: and we might call them *arbitrary* assumptions, in as much they are *arbitrarily* imposed upon a science of symbols and their combinations, which might be adapted to any other assumed system of consistent rules. (Peacock, 1834; Detlefsen, 2005))

Peacock's *Principle of Permanence of Equivalent Forms* was formulated shortly as follows:

Whatever algebraic forms are equivalent when the symbols are general in form, but specific in value, will be equivalent likewise when the symbols are general in value as well as in form. (Peacock, 1845)

The next table gives examples of laws which the principle in question takes into account:

commutativity	$a + b = b + a, a \cdot b = b \cdot a$
associativity	$(a + b) + c = a + (b + c), (a \cdot b) \cdot c = a \cdot (b \cdot c)$
distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$
exponentiation rules	$a^{b+c} = a^b \cdot a^c, (a^b)^c = a^{b \cdot c}, (a \cdot b)^c = a^c \cdot b^c$

Notice that all these laws are valid for real and complex numbers.

The development of algebraic symbolism is visible in the works of such mathematicians from the epoch in question as: George Boole, Augustus De Morgan, Duncan Gregory, William Hamilton, Arthur Cayley, John Graves, Hermann Grassmann, William Clifford, Benjamin Pierce, Ernst Schröder. They knew Peacock's proposals and each of them added original ideas to the development of algebra which is described in details in many places, for instance Kline, 1972, Kolmogorov, Yushkevich, (Eds.), 2001, Corry, 2004. The origins of abstract algebra with special emphasis put on the Peacock's proposals is interestingly presented in Pycior 1981.

2.2. Hermann Hankel

A few decades after Peacock Hermann Hankel (1839–1873) also formulated a similar principle (*das Princip der Permanenz der formalen Gesetze*):

Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedrückte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfache Grössen zu bezeichnen, und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen. (Hankel, 1867)

It should be stressed that Hankel's principle was formulated in different circumstances as compared to those contemporary to Peacock's principle. Besides such "standard" numbers as the integers, the rational and real numbers as well as already quite well "domesticated" complex numbers also certain hypercomplex numbers entered the stage, namely Hamilton's quaternions (Hamilton, 1861).

Hankel wrote in the work cited above that he knew Peacock's principle from the report Peacock 1934 and admitted that he did not read either Peacock's *Treatise on Algebra* (Peacock, 1845) or De Morgan's *On the Foundations of Algebra* (De Morgan, 1842). He referred to the short paper by Duncan Gregory *On the Real Nature of Symbolical Algebra* (Gregory, 1840). Hankel criticized Cauchy's remarks concerning complex numbers and highly appreciated two works by Hermann Grassmann: *Die lineale Ausdehnungslehre* (Grassmann, 1844) and *Lehrbuch der Arithmetik für höhöre Lehranstalten* (Grassmann, 1861). Hankel discussed also the works by Hamilton devoted to complex numbers and quaternions.

After a concise description of properties of arithmetical operations on integers (including the iterated exponentiation: $a^a, a^{(a^a)}, a^{(a^{(a^a)})}$, and so on) Hankel provided his definition of the number concept:

Die Zahl ist der begriffliche Ausdruck der gegenseitigen Beziehung zweier Objekte, soweit dieselbe quantitativen Messungen zugänglich ist. (Hankel, 1867)

Hankel declared further in the text that after *geometrical* interpretation of numbers of the form $a + b\sqrt{-1}$ and arithmetical operations on such numbers they should no longer be considered as *impossible*: they obtained the same reality as positive and negative integers.

Hankel considered his principle as fundamental in his algebraic considerations called by him *Formenlehre* (the study of forms):

Es hat die Formenlehre nicht allein den engen Zweck, die gewöhnliche arithmetica universalis mit ihren ganzen, gebrochenen, irrationalen, negativen und imäginaren Grössen zu erläutern und streng zu deduciren, sondern sie erweist sich mit ihrem Principe der Permanenz zugleich als eminent fruchtbar für den ganzen Organismus der Mathematik. (Hankel, 1867)

In addition to this declaration Hankel wrote that his *Formenlehre* could be related not only to numbers but also to spatial objects (points, segments, surfaces, solids) as well as to mechanical phenomena (forces, moments).

An important result established by Hankel was the characterization of the complex field \mathbb{C} as the only field which can be obtained by addition of roots of polynomials with coefficients from \mathbb{C} . Any generalization (expansion) of \mathbb{C} must therefore result in certain conflict with the principle of permanence of forms, that is some well recognized laws (commutativity or associativity of multiplication, for example) should be rejected. Hankel's result is thus a statement of a certain *maximality* property of the field \mathbb{C} . The complex field is the maximal field (among many-dimensional number structures) which preserves the maximum amount of standard laws concerning numbers (with the exception of ordering, of course).

2.3. Reception and critical remarks

It seems that the principle of permanence of forms was considered as a (heuristic) rule warranting that the development of mathematics would be secured from potentially nonsensical attempts at generalizations which could introduce objects with "strange" properties too much different from standard (normal, natural) properties.

But the principle was also criticized by mathematicians and philosophers at that time. Giuseppe Peano wrote critically about Hermann Schubert's work *Grundlagen der Arithmetik* in which the principle in question took a bizarre form demanding that all the rules concerning new numbers should be the same as the rules valid for the already known numbers. According to Peano, the rules should be only conservative as much as possible.

William Hamilton wrote that $\sqrt{-1}$ is an *absurd* object in the context of single (one-dimensional) numbers but it is entirely sensible in the context of complex

(two-dimensional) numbers. As we remember, Hamilton proposed an algebraic definition of complex numbers as pairs of real numbers with suitably defined arithmetical operations on them. According to Helena Pycior (Pycior, 1981) Hamilton did not accept Peacock's proposals concerning symbolical algebra as a science of arbitrary symbols governed by arbitrary laws because he claimed (as did also many of his contemporaries) that algebra should deal with symbols connected with *meaning* and the rules governing them should be based on intuition.

In a recent paper by Toader (Toader, 2019) the author compares the ideas of Edmund Husserl and Ernst Mach and talks about the principle of permanence of forms as: the principle of theoretical rationality (necessary for the authentic development of science), the principle of practical rationality (related to the principle of least effort in intellectual work), a metaphysical principle connected with mathematical intuition, and a semantic principle expressing relations between a formal theory and its interpretation.

The paper Šikić, 1984 shows that the definition of exponentiation in the field \mathbb{C} is, in a sense, forced by the principle of permanence of forms (see the laws from the table above) when we also take into account the standard laws of differentiation.

Meir Buzaglo investigates logical aspects of the process of expansion of concepts in Buzaglo, 2002. He considers many examples from the history of mathematics showing, among others, changes in meaning of the number concept, expansion of applicability of arithmetical operations, and paradoxes in set theory. He points to the role of the principle of permanence of forms in these processes:

During the nineteenth century, when rigor gradually resumed a place of importance in mathematics, there was a systematic attempt to conceptualize the idea of expansions, as presented in Peacock's "principle of permanence of equivalent forms." This attempt transferred the issue from the products of the expansion to the *process* of expansion itself. Peacock claimed that the symbolic algebra obtained from the expansion of arithmetic is logically independent of arithmetic, yet suggested by it. How an expansion of a realm can be "suggested" by the existing realm has not, however, been analyzed properly. Apparently this lack is due to the fact that discussions in logic are generally centered on deduction, which involves closed realms, thus marginalizing the issue of the expansion of concepts. But the most cursory survey shows that there is an abundance of logical, mathematical, and philosophical material that is continually raising the idea of expansions as logical and philosophical issue which naturally invites a more comprehensive discussion. (Buzaglo, 2002)

3. David Hilbert: The Axiom of Solvability

The Principle of Permanence of Forms was widely accepted by mathematicians in the second half of the 19th century as a *heuristic* principle behind the expansion of the concept of number. New number systems were introduced either genetically (by expanding "old" universe and adding new operations) or in an axiomatic way. The second possibility became prevailing later than the first. Hermann Grassmann,

Richard Dedekind, Giuseppe Peano characterized natural numbers (Grassmann, 1861, Dedekind, 1888, Peano, 1889). Among many constructions of real numbers those proposed by Georg Cantor and Richard Dedekind (Cantor, 1872, Dedekind, 1872) were most commonly known. David Hilbert proposed an axiomatic characterization of real numbers already in 1900 (Hilbert, 1900). We shall say later a few words about hypercomplex numbers introduced in the 19th century.

The principle of permanence of forms should be considered in connection with those factors which are responsible for mathematical discoveries, creation of new mathematical objects, and acceptance of new rules. One of such factors is Hilbert's *axiom of solvability*. David Hilbert claimed that there is no *ignorabimus* in mathematics: each properly formulated mathematical problem can be solved (or it can be proved that under accepted assumptions there is no solution to the problem). Hilbert expressed this idea in Hilbert, 1901. His epistemological optimism was a reaction to pessimistic claims of some of his contemporaries (notably Emil Du Bois-Reymond, Paul Du Bois-Reymond, Rudolph Virchow) concerning alleged limitations of human knowledge and cognition and known as the so-called *Ignorabimusstreit*. For instance, Paul Du Bois-Reymond claimed that there exist objective limitations in our understanding of the continuum. Hilbert's epistemological optimism, in turn, is clearly visible in his critical remarks of the views of Leopold Kronecker and Luitzen Brouwer. The famous limitative theorems proved later in the 20th century threw more light on possibilities and limitations of the deductive method.

Michael Detlefsen in the paper *Formalism* (Detlefsen, 2005) writes about the interconnections between the axiom of solvability and the principle of permanence of forms:

Together, the Axiom of Solvability and the Principle of Permanence guided the progressive extension of the number-concept. The Axiom of Solvability expressed the mathematician's goal to solve problems. The Principle of Permanence acted as a constraint upon the applicability of this axiom. It required that newly introduced numbers preserve the basic laws of arithmetic. More precisely, it required that the laws governing new numbers be *consistent with* the laws governing the old ones. (Detlefsen, 2005)

In the famous work *Über das Unendliche* (Hilbert, 1926) the author discusses, among others, the role played by *ideal elements* in several mathematical domains. Introduction of *points in infinity* in projective geometry brings nice symmetries between points and straight lines in the system. Introduction of complex numbers gives a natural simplification of theorems about the existence and number of roots of equations. Kummer's ideal numbers secure the uniqueness of decomposition into prime factors and similar role is played by Dedekind's ideals. All these cases are related to Hilbert's axiom of solvability.

Hilbert formulated also *das schöpferische Princip*, a principle which proclaimed the freedom of mathematical activities in the domain of concept formation and introduction of rules of inference. The symbolic methods of mathematics should be, according to Hilbert, consistent and fruitful. The axiomatic method is fundamental

in Hilbert's formalistic approach to mathematics. It is interesting that in a short note Hilbert, 1900 the author uses the verb *denken* (think) rather than *anschauen* (imagine, intuit) when he is proposing an axiom system for real numbers. Finally, Hilbert stresses the fact that an important property of certain mathematical domains characterized axiomatically is their *completeness* understood as a unique characterization of the domain in question. This completeness may take a form of *categoricity* (existence of exactly one model, up to isomorphism) or *semantic completeness* (all models of a theory are elementarily equivalent) or the so-called *hilbertian completeness*, that is a property of being a maximal or minimal model of the theory. The last property is related to Hilbert's *axiom of completeness* from the first edition of *Grundlagen der Geometrie* (Hilbert, 1899). This axiom was replaced in the later editions by the axiom of *line completeness*. The axiom system of geometry presented in Borsuk, Szmielew, 1975 includes instead the standard axiom of completeness for real numbers. As we remember, the field of real numbers is the only (up to isomorphism) completely ordered field. It is also the maximal Archimedean field.

4. Digression: hypercomplex numbers

Domestication of negative and complex numbers took a rather long time. The situation was different in the case of hypercomplex numbers introduced during the 19th century. We assume that the reader is familiar with fundamental facts from the history of complex numbers. It seems that the final domestication of these numbers took place when they were given algebraic definition (Hamilton) and geometric representation (Wessel, Argand, Gauss) and when the fundamental theorem of algebra was proved by Gauss.

The important step in the generalization of the concept of number was the acceptance of the fact that numbers may be more than one-dimensional. First, we have two-dimensional numbers:

1. *Complex numbers*. They are understood as pairs of real numbers with addition and multiplication given by the conditions:

$$(a, b) \oplus (c, d) = (a + b, c + d) \quad (a, b) \otimes (c, d) = (ac - bd, ad + bc).$$

In particular, $(0, 1) \otimes (0, 1) = -1$. Complex numbers form a field \mathbb{C} . This field is characterized (up to isomorphism) as the only algebraically closed field of characteristic zero whose transcendence degree over the field of rational numbers equals continuum. Complex numbers have a matrix representation.

2. *Dual numbers*. They are understood as pairs of real numbers with addition and multiplication given by the conditions:

$$(a, b) \oplus (c, d) = (a + b, c + d) \quad (a, b) \otimes (c, d) = (ac, ad + bc).$$

In particular, $(0, 1) \otimes (0, 1) = (0, 0)$. Dual numbers form a commutative ring with unity and with zero divisors. They have a matrix representation.

3. *Double numbers*. They are understood as pairs of real numbers with addition and multiplication given by the conditions:

$$(a, b) \oplus (c, d) = (a + b, c + d) \quad (a, b) \otimes (c, d) = (ac + bd, ad + bc).$$

In particular, $(0, 1) \otimes (0, 1) = (1, 0)$. Double numbers form a commutative ring with unity and zero divisors. They have a matrix representation.

Besides complex numbers, the most popular hypercomplex numbers are *quaternions* introduced by William Hamilton. They are understood as quadruples (a, b, c, d) of real numbers usually written in the form known from the language of vector spaces as $a + bi + cj + dk$, where i, j, k are imaginary elements (basis vectors). We understand quaternions \mathbb{H} as a four dimensional vector space over the field of real numbers, where addition and multiplication by scalars are defined in the usual way and multiplication is uniquely characterized by the conditions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Alternatively, multiplication can be characterized by the following table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

As we see the multiplication of quaternions is not commutative. Victor Katz writes about the significance of this fact for investigations in symbolical algebra as follows:

Hamilton's system was the first significant system of "quantities" that did not obey all the standard laws that Peacock and De Morgan had set down. As such, its creation broke a barrier to the consideration of systems violating these laws, and soon the freedom of creation advocated by Peacock became a reality. (Katz, 2018)

Thus passing from the real numbers to complex numbers we keep the arithmetical properties from the table from section 2.1. but we lose ordering; passing from complex numbers to quaternions we keep all mentioned properties with the exception of commutativity of multiplication. Some further kinds of hypercomplex numbers include:

1. *Octonions*. The octonions \mathbb{O} form an eight-dimensional algebra with division over the field of real numbers. Multiplication of octonions is neither commutative nor associative. They do not have a matrix representation.
2. *Sedenions*. The sedenions \mathbb{S} form a sixteen-dimensional algebra over the field of real numbers. Multiplication of sedenions is neither commutative nor associative. It does not satisfy the condition of alternativity:

$$x \otimes (x \otimes y) = (x \otimes x) \otimes y \quad (y \otimes x) \otimes x = y \otimes (x \otimes x).$$

However, the power associativity law holds: $x^{m+n} = x^m \otimes x^n$. There are zero divisors in \mathbb{S} .

3. *Cayley-Dickson construction.* This general construction gives an infinite sequence of algebras over the field of real numbers. The first five elements of this sequence are: \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , and \mathbb{S} . Complex numbers in this construction are pairs of real numbers, quaternions are pairs of complex numbers, and so on.
4. *Clifford algebras.* They are algebras isomorphic with the ring of matrices over \mathbb{R} , \mathbb{C} , \mathbb{H} , or direct unions of such algebras.

There are still many other types of hypercomplex numbers. Properties of numbers mentioned above concerning dimension, ordering and multiplication are presented in the following table:

	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}	\mathbb{S}	
dimension	1	2	4	8	16	> 16
ordering	yes	no	no	no	no	no
commutativity	yes	yes	no	no	no	no
associativity	yes	yes	yes	no	no	no
alternativity	yes	yes	yes	yes	no	no
power associativity	yes	yes	yes	yes	yes	yes
zero divisors	no	no	no	no	yes	yes

Several *isomorphism theorems* characterize certain of the number systems mentioned in this section. For instance, Frobenius theorem states that any associative algebra with division over the field of real numbers is isomorphic with \mathbb{R} , \mathbb{C} , or \mathbb{H} . Ostrowski theorem, in turn, says that each field which is complete with respect to the metric determined by the Archimedean norm is isomorphic with either \mathbb{R} or \mathbb{C} and the norm is equivalent with the standard absolute value.

Hypercomplex numbers are not the only generalizations natural, rational, and real numbers. Another types of generalizations include, among others, p -adic numbers, hyperreal numbers, or surreal numbers.

Still another story concerns the arithmetic of ordinal and cardinal numbers. Properties of the operations on these numbers differ significantly from those typical for standard natural numbers. This is of course not surprising, because ordinal and cardinal numbers are defined for all sets, including the infinite sets. It should be perhaps added that there are several different ways to develop arithmetic of ordinal and cardinal numbers, for instance that proposed by Cantor and that proposed by Hessenberg.

5. The Principle of Permanence of Forms in logic

Could the Principle of Permanence of Forms be applied to formal logic? We think that certain analogies between extensions (or generalizations) of the algebraic concepts and extensions (or generalizations) of logical systems could be drawn.

New logical systems are introduced on the basis of several motivations, coming from mathematics, linguistics, philosophy, information science, and even empirical sciences. In most cases the starting point is the system of classical propositional logic or the system of classical first-order logic.

We think that in the following situations in logic one could possibly see an analogy with the algebraic principle of permanence of forms:

1. *Finitary language.* The assumption that expressions of formal languages are finite syntactic objects is considered to be perfectly natural. Expressions are finite strings of symbols from a countable list and each expressions has a syntactic representation in a form of a finite tree. But we can consistently consider also languages with expressions of infinite length: in languages $L_{\alpha\beta}$ (where α and β are infinite ordinal numbers) we admit conjunctions and disjunctions of length less than α and quantifier strings of length less than β . Such languages have a well-developed model theory and they find serious applications also in mathematics itself. Syntax of these languages presupposes a certain portion of set theory. Infinitary languages have greater expressive strength than finitary first-order logic but they lack some “good” (natural) deductive properties of the latter.
2. *Finitary consequence.* Also the assumption that the rules of inference are finitary (that is, we infer a conclusion from a finite set of assumptions) is considered as very natural. In systems over finitary language and with finitary rules of inference all proofs also are finite syntactic objects (represented as strings or trees). If we allow infinitary rules of inference (as for instance the ω -rule), then proofs are no longer finite, of course. Observe that proofs based on the standard mathematical induction are finitary, because they are based on the induction axiom in arithmetic (or on an instance of the induction axiom schema). It is debatable whether infinitary rules of inference reflect adequately human inferences but we are obviously free to consider logical systems with such rules.
3. *Logical constants.* Classical logical constants include: propositional connectives and quantifiers (sometimes also the identity predicate). Usually, logical constants are given by explicit list, but there are also attempts at a general characterization of logical constants, for instance in terms of rules of inference or in semantic terms (see for example Tarski, 1986). In certain systems logical constants are mutually definable (as in the classical propositional logic), in other are mutually independent (as in intuitionistic logic). Theorem proved by Andrzej Grzegorzczak says that predicates, function symbols, and individual constants are neutral (that is, they are indistinguishable from the point of first-order logic). Necessity and possibility are treated as logical constants in modal logic, but some logicians claim that modal logics are only *theories* of these notions. Addition of new logical constants may significantly improve the expressive power of logic, as the case of generalized quantifiers (characterized semantically) clearly shows.

4. *Consistency.* This property was always a methodological ideal in logic and mathematics. In the latter it was even considered as a criterion of existence. Inconsistent systems (in which everything can be proved) are epistemologically useless. However, systems which tolerate (local) inconsistencies and in which not everything can be proved are relevant as adequate representations of systems of belief.
5. *Non-Fregean logic.* In the Fregean approach there are only two denotations of sentences: Truth and Falsity. More subtle approach was proposed by Roman Suszko: in his non-Fregean logic the denotations of sentences are *situations* (described by the sentences). Non-Fregean logic is two-valued and fully extensional (it does not admit intensional constructions). The ontology of this logic is much richer than that of classical logic.
6. *Truth and proof.* Theorems about soundness and completeness of deductive systems are doubtfully the most important logical metatheorems: they establish a correspondence between proof and truth. Not all systems of logic satisfy the completeness theorem (for instance second-order logic with the standard semantics). In intuitionistic logic existence of proof is the sole criterion of validity. Semantical notions are in general much more complicated from the computational point of view. Limitative theorems proved in the 20th century show the possibilities and limitations of the deductive method as far as the relation between proof and truth is concerned.
7. *First-order thesis.* The problem which system of logic is the right one (for instance from the point of view of applications) is vividly discussed in the philosophy of logic (see for instance Woleński, 2004). The *First-order thesis* says that first-order logic (FOL) is *the* logic. Among arguments supporting this thesis are:
 - (a) FOL has “good” deductive properties: it is consistent, sound, complete, compact, satisfies the deduction theorem, and so on.
 - (b) FOL does not distinguish infinite powers (Löwenheim-Skolem-Tarski theorem).
 - (c) FOL is universal in applications. Many important mathematical theories are formalized within FOL (notably Zermelo-Fraenkel set theory).
 - (d) FOL is maximal logic satisfying the compactness theorem and the Löwenheim-Skolem theorem (Lindström theorem).

FOL has at the same time very poor expressive power: such important mathematical notions as for instance infinity, continuity, set of measure zero cannot be expressed (by a single formula) in FOL. Mathematicians seem to evaluate expressive power rather highly and thus they may not adhere to the first-order thesis:

But if you think of logic as the mathematicians in the street, then the logic in a given concept is what it is, and if there is no set

of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with. (Barwise, Feferman, 1985)

8. *Paradoxes*. Certain properties of classical logic are considered as paradoxical (for instance paradoxes of material implication). There are many logical systems which omit these paradoxes preserving the “good” properties of classical logic as much as possible. In particular, several logical systems are constructed in order to model adequately specific intensional phenomena occurring in inferences conducted in natural language.
9. *Points of view*. Harvey Friedman expressed the opinion that many incompleteness phenomena are caused by the fact that we admit completely arbitrary elements instead of limiting ourselves to investigation of well-recognized objects (Friedman, 1992). For example, the continuum hypothesis which is independent from the axioms of Zermelo-Fraenkel set theory is nevertheless true, if we restrict ourselves to Borel sets only. Friedman discusses in the cited paper several points of view (for instance: Borel, predicative, constructive) and shows how they influence the incompleteness phenomena.

Notice that in each of these cases we deal with *permanence* of some properties and at the same time with certain *creative* aspects determining the original features of the generalizations in question. Generalizations in logic are not linear: rather, they form a star-like structure with classical logic at the center and axes corresponding to the main ideas motivating particular generalizations.

Arguments based on analogy are risky. It may happen that what we see as an analogy is only a coincidental resemblance. However, we think that the search for invariants in the development of logical systems suggested by the factors listed above (and possibly further ones) may elucidate the mechanisms responsible for that development. We have only sketched the idea here and we hope to return to this issue in another paper.

6. The Principle of Permanence of Forms in teaching mathematics

Finally, let us say a few words about the Principle of Permanence of Forms in mathematical education. We think that the principle in question is, at least implicitly, present in the introduction of number systems via genetic method. Numbers which are new to the pupils are shown as obeying the laws valid for the numbers already known.

Another implicit occurrence of the discussed principle is related to these situations in which it is useful to pay attention to the *form* of algebraic expressions before tedious calculations take place. Experts in mathematical education often stress that pupils hurry with “blind” calculations without notice that they can be avoided, if one carefully looks at the form of expressions. The paper Menghini, 1994 reports on the following activities of sixteen years old pupils:

Find the value of the following expression:

$$\frac{z^4}{c^2} \cdot (b - a) + \frac{z^4}{c^2} \cdot (a - b)$$

For this exercise most of the students carried out very detailed calculations.

For which values of x is the following inequality true?

$$x^2 + x < x^2 + x + 1$$

In the best answers students carried out a comparison of the two parabolas represented by the two parts of the inequality.

Express each of a , b , c in terms of the other two values:

$$a - 4c^2 = \frac{1}{b}$$

Only 50% of the students gave three correct answers.

If m , n , p , q are natural numbers, the following equalities are true (always, never, sometimes):

$$(2n + 1)(2m + 3) = 2p$$

$$(2n + 5)(2m + a) = 2p + 3$$

$$(2n + 1)(2m + 1) = 4pq$$

This last exercise was set for university students (in the Mathematics Department, of course!) and very few answered correctly, most of them saying things like: “The first equality is true when $2p$ is an odd number”. (Menghini, 1994)

The principle of permanence of forms can be useful in explanations of certain mathematical concepts which often cause hesitation on the part of a student. This is the case of exponentiation with exponent zero, if exponentiation was formerly explained as iterated multiplication. Support for the validity of $a^0 = 1$ can be seen from the law $a^{b+c} = a^b \cdot a^c$, because $a^n = a^{0+n} = a^0 \cdot a^n$, and therefore $a^0 = 1$.

Similarly, explanation of the fact that *negative times negative is positive* which causes troubles for many students may be supported by the law of distributivity of multiplication over addition in the form $(a + b) \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$. Specifying the values, say $a = 3$, $b = -1$, $c = 5$, $d = -2$, we then ask the students to calculate the values of both sides of the equation and to notice that $(-1) \cdot (-3)$ must be equal 3 for the equation to hold true.

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