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Different ways of solving quadratic equations*

Abstract. In this paper we explore different ways of solving quadratic equations. Our main goal is to review traditional textbooks methods and offer an alternative, often side-stepped method based on the area model. We conclude that whereas traditional methods offer effective algorithms that quickly lead to the desired results, alternative methods may enhance meaningful and joyful learning.

Many of the topics studied by our parents and grandparents will soon be only memories (if they are not now), and with them goes our appreciation of the difficulties experienced before the days of calculators and computers.

Hornsby, 1990

Working together on this paper we wanted to join our different experiences with and perspectives on quadratic equations. The first author has been conducting an in-depth analysis of different curricular topics, including quadratic equations, as part of the preparatory stage of a project aimed at investigating substantive and didactical competencies of pre-service teachers of mathematics.The second author has become familiar with various mathematical curricula from around the world and proposes his using an often side-stepped alternative approach to solving quadratic equations. We are aware of the fact that the methods of solving quadratic equations that we recall in this paper do not exhaust all the possible approaches. However, we wanted to stay as close to the school curriculum as possible, hence we decided not to consider some interesting, yet rarely used methods (e.g., the Carlyle's circle or von Staudt's method).

*2010 Mathematics Subject Classification: Primary: $11D09$; Secondary: $97D50$

Keywords and phrases: quadratic equations, the area model, alternative methods of solving mathematical problems

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1. Quadratic equations in the Polish curriculum and textbooks

The Polish educational system has undergone significant changes lately. Three stages of mandatory education, i.e. primary school (6 years), gymnasium (3 years) and secondary school (3-4 years depending on the type of the school) have been replaced by two stages: primary school (8 years) and secondary school (4-5 years), what restores the state from before the year 1999. These structural changes have been followed by the necessary changes in the core curricula, now revisited and adjusted to the new system. Depending on the profile of the class they attend, secondary school students learn mathematics at the basic or extended level (BL and EL respectively), and at each level they encounter quadratic equations. The scope of the material for the extended level differs significantly from that planned for the basic level. The table on the next page shows the scope of the material planned for each level before and after the curriculum reform in Poland.

The authors of the previous core curriculum clearly indicated that the student should solve quadratic equations with one unknown. The new core curriculum mentions quadratic equations in a very general manner. Time will show how this curriculum entry will be interpreted by the authors of the textbooks. According to the previous curriculum, the students were solving systems of equations leading to a quadratic equation only when engaged in mathematical education at the extended level. Currently, BL students will be dealing with this issue. The same applies to solving polynomial equations that can be reduced to quadratic equations. As a result of these changes, especially BL students will have to meet more demanding requirements.

School textbooks are said to be one of the most influential resources used both by the teachers and the students. They also mediate between the intended and implemented curriculum. Hence, we assume that what we find in the textbooks quite well indicates what might happen in the classrooms. Since the reform, regarding the secondary school mathematics, new textbooks have been approved only for the first grade. Textbooks for the remaining years are expected to be published within the next few years. For this reason, our analysis refers to the textbooks used currently by gymnasium graduates who follow the pre-reform teaching programs. We have analysed five, currently most popular series[1] of Polish secondary school mathematics textbooks in order to find out what different methods of solving quadratic equations are promoted by their authors. Below we present a brief summary review of the methods we have found. Since the methods are well known to the readers, we will only illustrate them with examples solved by the textbooks' authors, and then provide a few related comments.

[1] In the text we denote them with letters A,B,C,D and E. Full references are listed at the end of this paper.

Table 1. Table 1: Quadratic equations in the Polish curricula (BL–basic level, EL–extended level)

Polish former curriculum	**Polish current curriculum**
The student solves: • quadratic equations with one unknown (BL) The student solves: • systems of equations leading to quadratic equations with one unknown (EL) • quadratic equations with a parameter (EL) • polynomial equations that can be easily reduced to quadratic equations (EL) The student uses the Viete's formulas (EL)	The student solves: • quadratic equations (BL) • polynomial equations that lead to a quadratic equation, in particular biquadratic equations (BL) • (by substitution) the systems of equations, one of which is linear and the other is quadratic (BL) The student: • uses Viete's equations for quadratic equations (EL) • analyses quadratic equations with parameters, in particular determines the number of solutions depending on the parameters, gives the conditions under which the solutions have the desired property, and determines the solutions depending on the parameters (EL) • solves systems of quadratic equations (EL)

Quadratic equations in the form of $ax^2 + c = 0$ $(a \neq 0, c \neq 0)$

(Equations of the form $ax^2 = 0$ are considered straightforward to solve.)

Textbook[2] example 1

$$\tfrac{11}{3}x^2 - 7 = 0$$
$$\tfrac{11}{3}x^2 = 7$$
$$x = -\sqrt{\tfrac{21}{11}} \text{ or } x = \sqrt{\tfrac{21}{11}}$$

Textbook example 2

$$4x^2 - 9 = 0$$
$$(2x - 3)(2x + 3) = 0$$
$$x = \tfrac{3}{2} \text{ or } x = -\tfrac{3}{2}$$

Textbook example 3

$$2x^2 + 9 = 0$$

We may notice that for every real number x, the sum $2x^2 + 9$ takes values of 9 or higher, so it can never be equal 0. This equation has no real roots then.

Quadratic equations in the form of $ax^2 + bx = 0$ $(b \neq 0)$

Textbook example 4

$$4x^2 - 8x = 0$$

We factorise the expression on the left-hand side and get:

$$4x(x - 2) = 0$$

which is equivalent to

$$4x = 0 \text{ or } x - 2 = 0.$$

Hence:

$$x = 0 \text{ or } x = 2.$$

Quadratic equations in the form of $ax^2 + bx + c = 0$ $(a, b \neq 0)$

Textbook example 5

$$x^2 - 4x + 4 = 0$$

[2] Examples 1-6 are taken from textbook C.

We may use a short multiplication formula here, and express the left-hand side as $(x-2)^2$:

$$(x-2)^2 = 0$$
$$x = 2.$$

Textbook example 6

$$4x^2 - 20x + 23 = 0$$

In this case we may complete the left side to the square:

$$(2x-5)^2 - 25 + 23 = 0$$
$$(2x-5)^2 = 2$$
$$2x - 5 = \sqrt{2} \text{ or } 2x - 5 = -\sqrt{2}$$

which gives

$$x = \tfrac{5-\sqrt{2}}{2} \text{ or } x = \tfrac{5+\sqrt{2}}{2}.$$

Naturally, in case of any equation of the form $ax^2 + bx + c = 0, a \neq 0$ the well-known quadratic formula can be used. It is present in every textbook we looked into.

Quadratic equations in the form of $ax^{2n} + bx^n + c = 0$ $(n \in N_2)$

Such equations are typically solved by substituting $t = x^n$. Then, the students obtain an equation in one of the abovementioned forms and follow one of the procedures they know.

When presenting different methods of solving quadratic equations, textbooks authors give the students some suggestions that may guide them through the next examples they will try to solve on their own. In textbook A, for instance, the authors state that if $b \neq 0$ and $c \neq 0$, we solve the equation $ax^2+bx+c=0$ using the quadratic formula or Viete's formulas. The authors add that most often we solve equations $ax^2+c=0$ or $ax^2+bx=0$ by factorising their left sides. The same textbook also recommends using the quadratic formula as a convenient strategy whenever the coefficients are fractions or irrationals. Textbook C authors present different methods before they introduce the quadratic formula, but at the end they conclude that the formula is what we most often use. On the other hand, textbook D promotes completing the algebraic square (i.e. a square of some algebraic sum) and states that any quadratic equation can be solved with this method. In textbook E the authors mention, among other methods, finding a common factor, but they say that sometimes finding a common factor may not be easy. The authors give no hints and do not provide any strategy for dealing with more complex cases. This puts the quadratic formula in a position of a reliable and universal tool.

2. Viete's formula versus AC-method

When addressing quadratic equations, Polish secondary school mathematics textbooks also present Viete's formulas. What is worth noticing is that typically these formulas are introduced after the methods we mentioned earlier and are used mainly for solving problems involving parameters. In the analysed textbooks we have found only a few examples of easier tasks being solved with the Viete's formulas. The examples we have found, are listed below:

- Find the roots of the following trinomial $x^2 - 5x - 6$ (Textbook A) The coefficients are: $a = 1, b = -5$ and $c = -6$. From the Viete's formula we get $x_1 + x_2 = 5, x_1 \cdot x_2 = -6$. Then we decompose -6 into the product of two numbers whose sum is 5, and obtain $x_1 = -1, x_2 = 6$.

- Solve the following equation $x^2 - 10x + 9 = 0$ (Textbook B) The authors calculate the discriminant first and since it is positive, they conclude the equation has two roots. According to Viete's formulas $x_1+x_2 = 10, x_1 \cdot x_2 = 9$. The numbers that meet these conditions are 1 and 9.

- Find the zeros of function $f(x) = x^2 - px - 15$, if they are whole numbers and parameter p is a prime (Textbook E). The authors get $\Delta = p^2 + 60$, and infer from the sign of the discriminant that the given function has two zeros. Then they use the formulas and get: $x_1 + x_2 = p, x_1 \cdot x_2 = -15$. Now since we know that x_1 and x_2 are whole numbers, there are only four pairs of whole numbers that meet the conditions of the task: -1 and 15, 1 and -15, -3 and 5, and 3 and -5.

We were surprised to learn that the use of Viete's formula is mainly restricted to the case of quadratic equations with a parameter. For instance, in a textbook preparing for the International General Certificate of Secondary Education (BBlack, Ryan, Haese, Haese, Humphries, 2009), we find a sharply different approach. The learners who use this textbook become familiar with various methods of factorising trinomials in Chapter 1 and the methods of solving quadratic equations are shown two chapters later. When factorising trinomials $x^2 + bx + c$ (Chapter 1, p. 49) the authors state that in a general case:

$$x^2 + (\alpha + \beta)x + \alpha\beta = (x + \alpha)(x + \beta),$$

which means that the coefficient of x is the sum of two numbers whose product equals the constant term. In one of the given examples, the authors show how the numbers α and β can be found. When factoring trinomial $x^2 + 11x + 24$ they find pairs of factors of 24, i.e. 1 and 24, 2 and 12, 3 and 8, 4 and 6 and choose the numbers who sum up to 11, that is 3 and 8, and hence the factorisation: $x^2 + 11x + 24 = (x + 3)(x + 8)$.

One can easily relate this method to the use of Viete's formulas. Whereas according to the Cambridge textbook we factorise the trinomial seeking for p and q such that:

$$x^2 + bx + c = (x + p)(x + q)$$

$$p + q = b$$
$$pq = c,$$

while using Viete's formulas, we aim to find their negatives:

$$x^2 + bx + c = (x - x_1)(x - x_2)$$
$$x_1 + x_2 = -b$$
$$x_1 x_2 = c$$

It is clear that:

$$p + q = (-x_1) + (-x_2) = -(x_1 + x_2) = -(-b) = b$$
$$pq = (-x_1)(-x_2) = c$$

The Cambridge textbook authors also show how to factorise[3] a trinomial by splitting the middle term (see also e.g. Duttenhoeffer, 1979; Steinmetz, Cunningham, 1983; Erisman, 1986; Crowley, 1999). They state:

> The following procedure is recommended for factorising $ax^2 + bx + c$ by splitting the middle term:
>
> Step 1: Find ac.
>
> Step 2: Find the factors of ac which add to b.
>
> Step 3: If these factors are p and q, replace bx by $px + qx$.
>
> Step 4: Complete the factorisation.
>
> (Black, Ryan, Haese, Haese, Humphries, 2009, p. 53).

This procedure is illustrated with the following example:

$$8x^2 + 22x + 15.$$

When we want to factorise this trinomial by splitting the middle term, we need to find such two numbers whose product is equal to $8 \cdot 15$, and which add to 22. These numbers are 10 and 12, hence the factorisation:

$$8x^2 + 22x + 15 = 8x^2 + 10x + 12x + 15 =$$
$$= 2x(4x + 5) + 3(4x + 5) =$$
$$= (2x + 3)(4x + 5).$$

Whereas most textbooks state that finding appropriate numbers when using either Viete's formula or AC-method, can be done in one's head, they give no hint of how to deal with more challenging examples (nor explain why the AC-method is sure to work). Savage, 1989 provides a simple strategy for such cases, considering the following equation:

[3] Factoring trinomials is typically presented in English resources as the inverse of FOIL technique. It seems worth mentioning that the FOIL acronym has no Polish counterpart.

$$x^2 + 14x - 207 = 0$$

In this case want to find numbers p and q such that:

$$x^2 + 14x - 207 = (x + p)(x + q)$$
$$p + q = 14$$
$$pq = -207$$

Knowing the sum of two numbers, we are immediately given their arithmetic mean:

$$p + q = 14 \Rightarrow \tfrac{p+q}{2} = 7$$

On a number line the arithmetic mean of two numbers p and q lies in equal distance from both of these numbers, hence for some u>0:

$$p = 7 - u \text{ and } q = 7 + u$$

Now we can go back to the product of p and q:

$$pq = (7 - u)(7 + u) = -207$$
$$49 - u^2 = -207$$
$$u^2 = 256$$
$$u = \pm 16$$

Since we assumed $u > 0$, we get:

$$p = 7 - 16 = -9$$
$$q = 7 + 16 = 23$$
$$x^2 + 14x - 207 = (x - 9)(x + 23).$$

In another example taken from a workbook supplementary to the Polish textbook E, we show that the AC–method together with Savage's strategy can serve the students also in the case of more challenging equations. We want to solve the following equation:

$$3x^2 + 7\tfrac{1}{4}x = 2$$

If we prefer to avoid fractions and want to keep zero on the right side, we may transform this equation to get its equivalent:

$$12x^2 + 29x - 8 = 0$$

Now, we seek for p and q, such that:

$$p + q = 29 \Rightarrow \tfrac{p+q}{2} = \tfrac{29}{2}$$
$$pq = -96$$

We may express p and q adopting the same method as in the previous example and proceed with their product:

$$(\tfrac{29}{2} - u)(\tfrac{29}{2} + u) = -96, \text{ for some } u > 0$$

Again, if we do not want fractions to enter our calculations, we may multiply both sides by 4 and get:

$$(29 - 2u)(29 + 2u) = -384$$

Hence:

$$4u^2 = 1225 = 25 \cdot 49 = (5 \cdot 7)^2 = 35^2$$
$$(2u)^2 = 35^2$$
$$u = \tfrac{35}{2} \text{ (we need only consider } u > 0)$$
$$p = \tfrac{29}{2} - \tfrac{35}{2} = \tfrac{-6}{2} = -3$$
$$q = \tfrac{29}{2} + \tfrac{35}{2} = 32$$
$$12x^2 + 29x - 8 = 12x^2 - 3x + 32x - 8 = 3x(4x - 1) + 8(4x - 1) = (4x - 1)(3x + 8)$$

Oftentimes Polish textbooks convince students that if the discriminant of a quadratic equation is negative, the equation has no roots. As a consequence, students are also told that the Viete's formulas may be used if and only if the discriminant is non-negative, which is not true. In the case of negative discriminant, the relevant quadratic equation still has two roots, but they are to be found among complex, not real numbers. However, the Viete's formulas are still valid in such a case. This next equation considered bySavage, 1989 is a good illustrative example:

$$x^2 + x + 1 = 0$$
$$pq = 1$$
$$p + q = 1$$
$$\tfrac{p+q}{2} = \tfrac{1}{2}$$
$$p = \tfrac{1}{2} - u \text{ and } q = \tfrac{1}{2} + u, \text{ for } u > 0$$
$$(\tfrac{1}{2} - u)(\tfrac{1}{2} + u) = 1$$
$$\tfrac{1}{4} - u^2 = 1$$
$$u^2 = -\tfrac{3}{4}$$
$$u = \pm\tfrac{\sqrt{3}}{2}i$$
$$p = \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$
$$q = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$
$$x^2 + x + 1 = (x + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i)(x + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i)$$

We believe that mathematics is really about deep and true understanding. Once the students truly understand a particular topic, like for instance solving quadratic equations, they do not have to memorize any formulas. This is why we truly appreciate all the methods discussed thus far. In the next section we present a meaningful, coherent method of solving quadratic equations of any kind, developed by the second author of this paper.

3. Solving quadratic equations with the use of the area model

Much of the material presented in this section is closely based on the second author's informal writings available on-line and meant to serve all the interested learners.

3.1. Using the area model to the problems from arithmetic and algebra

We are going to start with some arithmetic problem, which in fact can be brought to a geometrical one.

Example I

$$\text{Compute } 37 \cdot 23$$

We may interpret this computational problem in terms of geometry, and then what we are being asked to compute is in fact the area of a 37-by-23 rectangle:

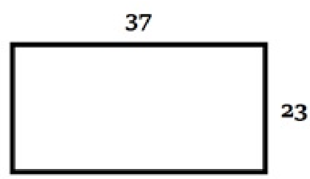

Fig. 1

Since the numbers that appear in that problem are not very friendly to the students, we may represent each one of them as a sum of numbers that will make our calculations easier. It seems natural to break each of the numbers 37 and 23 into friendlier numbers, say 30 and 7, and 20 and 3, and thus divide the rectangle into four pieces whose areas are simple to compute, as shown on a picture below:

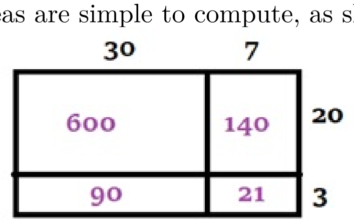

Fig. 2

Now, calculating the area of the rectangle becomes easy:

$$37 \cdot 23 = 600 + 140 + 90 + 21 = 851.$$

Area-thinking informs us of the general behaviour of numbers. For example, rotating a 37-by-23 rectangle 90-degrees gives a 23-by-37 rectangle. The area of the rectangle does not change in this process, so it must be the case that $37 \cdot 23$ gives exactly the same answer as $23 \cdot 37$. In general, rotating a picture of a rectangle 90-degrees shows that:

$$a \cdot b = b \cdot a$$

for all numbers that can be the side-lengths of rectangles. In the early grades, students know only the positive numbers, and the area model shows that $a \cdot b = b \cdot a$ for all positive counting numbers. But as our schooling progresses, we learn of other types of numbers. We can imagine rectangles with fractional side lengths, and so we come to believe that $a \cdot b = b \cdot a$ holds true for all fractions as well. And we can imagine rectangles with irrational number side-lengths, and so we believe that $a \cdot b = b \cdot a$ holds for irrational numbers as well. And this belief, by this point, feels so natural and so right that we believe, surely, that $a \cdot b = b \cdot a$ should hold for all numbers, even ones that extend beyond geometry, namely, negative numbers. Technically, one cannot have rectangles with negative side-lengths or with negative areas, but we like to believe that if we were to draw pictures of such rectangles, they would still speak truth about arithmetic, that $a \cdot b = b \cdot a$ holds too for negative numbers, for instance.

Example II

Compute $17 \cdot 18$

We are going to solve this arithmetical task in four different ways, adopting the area model each time. Below we present four diagrams, by which we extend the boundaries of geometry.

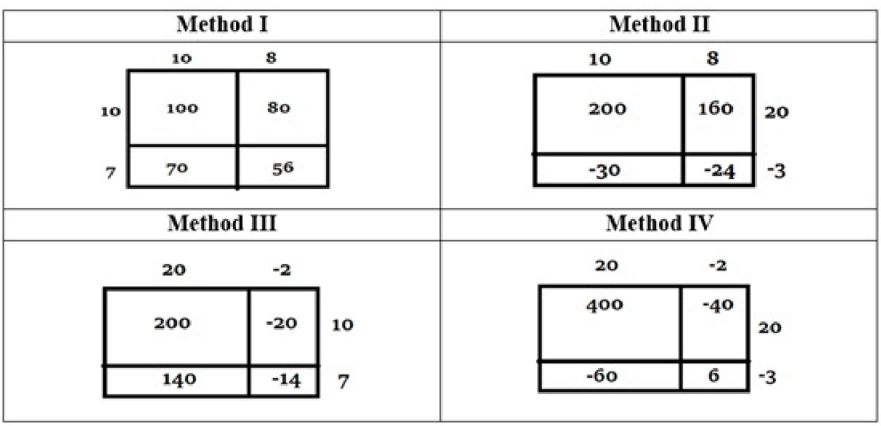

Fig. 3

The first method below is a direct application of what we have already done in the first example. The second method, however, is based on a different representation of 17, seen now as a sum of 20 and -3. This method shows that even if we push geometry to negative numbers, the diagram still speaks the arithmetical truth. The third method is similar to the second one: here we represent 18 as the sum of 20 and -2, and keep 17 represented with two natural numbers. In the fourth method, however, we use two negative numbers. Here we may rely on our knowledge that the product of two negative numbers is positive, or we can use this example to show that this product has to be positive, since we already know – from methods I-III – the area to be obtained.

This example can be adopted in the classroom in various ways. For instance, it may serve as an ice breaking activity or for a number talk. It shows that a mathematical problem can be solved in more than just one way and this may be a freeing experience to those students who are afraid of doing mathematics because they believe that they should remember "the rules" and "the formulas" and replicate particular procedures for certain types of problems.

The use of the are model can now be extended to algebra. If we are to multiply two algebraic sums, for instance, $2x + 3$ and $4x + 1$, we can do it applying the rectangle model as shown on the picture below:

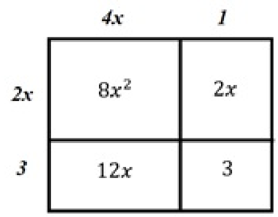

Fig. 4

Since the area model speaks truth for all numbers, we can use it for algebra. For instance, the area model shows that $(2x + 3) \cdot (4x + 1) = 8x^2 + 14x + 3$. And this is true no matter the value of x: positive or negative.

3.2. Completing the square–benefits from using symmetry

Scholars of some 2000 years ago realized that there are certain types of equations that can be readily solved by drawing pictures of squares. These problems became known as equations that can be solved by the "square method". The Latin word for square is *quadrus*, and as Latin took on the role of being the official scientific language in the West, these equations became known as the *quadrus equations* or, as we call them today, quadratic equations[4].

We are going to start with a very simple example of a quadrus equation, and then show a list of problems becoming more and more challenging. By doing so, we are going to show that each type of quadratic equation can in fact be solved by applying the method of completing the square.

[4]Due to space limitations we decided not to elaborate on the historical development of the methods of solving quadratic equations and the benefits of using historical sources in the classroom, however, we encourage the interested reader to see the works of some scholars who deserve recognition in these fields of expertise, e.g. Radford, Guerette, 2000; Katz, Barton, 2007; Clark, 2012; Blåsjö, 2016

Level I: $x^2 = 100$

The above equation can be interpreted as a question of the length of the side of a square whose area equals 100:

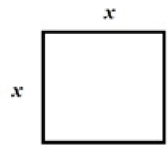

Fig. 5

If we literally treat this as a geometrical problem, we will obtain $x = 10$. But going beyond purely geometrical interpretation, we will find one more arithmetic solution, that is another number whose square gives us 100, which is -10. We can solve a whole class of problems of the same type, i.e., $x^2 = a, a > 0$ always obtaining one solution in geometry, and another one in arithmetic. If however $a = 0$ we will obtain only one solution, and if $a < 0$ we will not find any solution in the real numbers. We can extend this problem to examples such as for instance $36x^2 = 49$, which can be easily brought to the Level I problem.

Level II: $(x + 3)^2 = 25$

Solving this task we may rely on what has already been done at the previous stage. If we think about $x + 3$ as some number that squared gives us 25, we may easily figure out that this "something" necessarily has to be 5 or its negative counterpart. Hence we write:

$$x + 3 = 5 \text{ or } x + 3 = -5$$

which is equivalent to

$$x = 2 \text{ or } x = -8.$$

Level III: $x^2 + 6x + 9 = 25$

First encounter with such a quadratic equation may evoke some confusion in the students. Needless to say, this equation does not look like the two former examples. However, we may still refer to our earlier experiences. We see x^2, which is what we have already dealt with. We may use our area model again in order to represent this term as a square whose side length equals x.

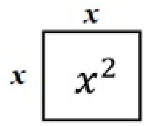

Fig. 6

We have just expressed some part of the above equation in a meaningful way. What may be scary to the students is the rest of that equation. But adding something to x^2 means adding some area to the area of the square we already drew. Since our goal is to complete a square, we need to proceed symmetrically. When adding the area of $6x$ we will add $3x$ on two sides of the square, so that the symmetry would be kept:

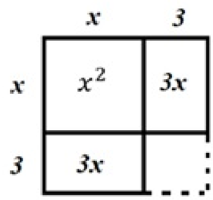

Fig. 7

What we need to do now is to complete the square. Since the length of the remaining side of the square is 3, the area of the small square has to be 9.

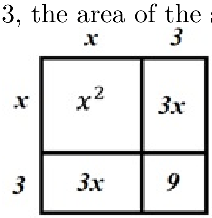

Fig. 8

Now we can easily see that the area of the whole square is exactly what we wanted: $x^2 + 6x + 9$ standing on the left-hand side of the equation, or $(x + 3)^2$ which may be inferred from the picture. And that, as we know, is said to be equal to 25. This is exactly the same equation as the one we considered at Level II. As an exercise, the students may solve next example:

$$x^2 - 8x + 16 = 17$$

Analogical reasoning will surely lead to the following sequence of diagrams:

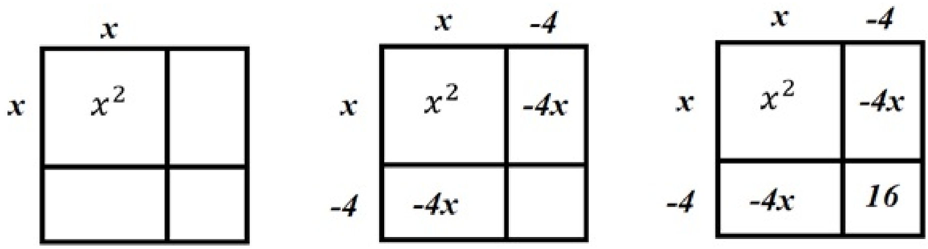

Fig. 9

Level IV: $x^2 - 4x + 3 = 15$

This problem varies from the previous one. After completing the diagram we may see that the square wants us to have the constant term equal 4, but in the original equation we have only 3.

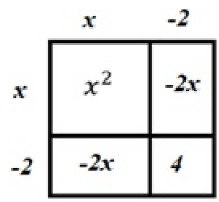

Fig. 10. Figure 10

We may solve this easily by adding one to the both sides of the equation and hence we get $x^2 - 4x + 3 + 1 = 15 + 1$, equivalent to $x^2 - 4x + 4 = 16$. But we know that the sum on the left-hand side of the equation gives us the area of a square whose side is $x - 2$, hence $(x - 2)^2 = 16$ which brings us back to the Level II problem.

Level V: $x^2 + 3x + 1 = 5$

At first, we may want to solve this equation following exactly the same steps as previously, but soon we will see that since the middle coefficient here does not split into two numbers nicely, we would have to deal with awkward fractions.

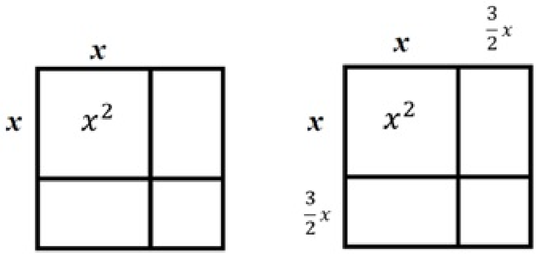

Fig. 11

Of course we could get the solution, but if we want to have nice numbers, we need the middle coefficient to be even. Hence we may think to multiply both sides of the equation by 2, and obtain:

$$2x^2 + 6x + 2 = 10$$

Unfortunately, trying to avoid fractions, we got irrational coefficients which might be even more discouraging to the students:

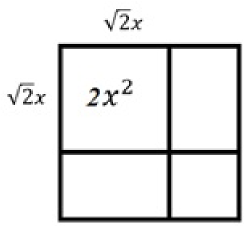

Fig. 12

Both methods are correct, yet we would not enjoy the calculations they involve. So instead of doubling everything in the original equation, we may try multiplying everything by four.

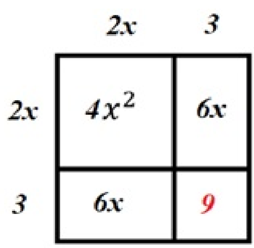

Fig. 13

This then gives us the equation:

$$4x^2 + 12x + 4 = 20.$$

As can be seen, the square wants the constant term to be 9, and we have got 4, but we were successful in dealing with the same problem earlier on, at level 4. Thus we may simply add 5 to both sides of the equation and write it as:

$$4x^2 + 12x + 9 = 25$$

But since the left side expresses the area of the square whose side length is 2x+3, we may write a level 2 form of our equation, which is:

$$(2x + 3)^2 = 25.$$

Level VI: $5x^2 - 3x + 2 = 4$

Here we have a problem that has every possible difficulty that can ever occur when solving a quadratic equation. The first issue is that we now have a number different from one in front of the x^2 term. All other levels avoided this. But in level 5 we did introduce a factor of four into our equations to work with $4x^2$, which we saw was a nice perfect square ($2x$ times $2x$). Here we have $5x^2$, which is not a nice perfect square. If we multiply the equation by 5, we will get a square that we want:

$$25x^2 - 15x + 10 = 20.$$

But then we still have an odd middle number, that is -15. However if we multiply this equation by 4, we will fix that without "ruining" a perfect square we already have. The equation that we will draw a diagram for, is the following:

$$100x^2 - 60x + 40 = 80$$

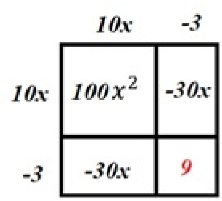

Fig. 14. Figure 14

And we can write it as:

$$(10x - 3)^2 = 49.$$

What can be inferred from the above journey through the different kinds of quadratic equations is that every such an equation is really just a level 2 question in disguise.

3.3. Obtaining the general quadratic formula

Almost every curriculum in the world insists that quadratic equations be written with some left side set equal to zero on the right. For example, most curricula would prefer to rewrite

$$5x^2 - 3x + 2 = 4$$

as

$$5x^2 - 3x - 2 = 0$$

We have never bothered to do that. With the square method we knew we were likely going to change the constant term in a quadratic expression, so we just waited until we could see what number would be best for it. Most school curricula insist, however, that you make a change right away and rewrite the equation so that it equals zero, even if you decide to change the number again later on. So let's follow the curriculum convention and assume we are solving a quadratic equation of the form

$$ax^2 + bx + c = 0$$

What we were doing in the earlier examples can now be applied to the general case. Let us multiply both sides of the equation by $4a$ to make sure that the first term is a perfect square and that the middle coefficient is even.

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax)^2 + 4abx + 4ac = 0$$

The diagram shows that we should have the constant term equal b^2 not $4ac$. Hence we subtract $4ac$ from both sides and add b^2 to them.

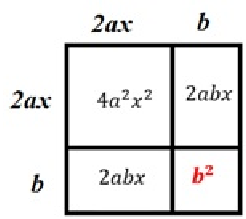

Fig. 15

$$(2ax)^2 + 4abx + b^2 = b^2 - 4ac$$

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Of course, this "quadrus method" is not new. Szurek, 2006, p. 17 gives exactly the same method and recalls Cardano who had been using it. When encountering this method for the first time, the students could ask how to figure out that we need to multiply both sides by $4a$ first. Szurek says that an honest response would be that we are completing the square, and hence the method. However, in his lecture, Szurek speaks about the square of some algebraic sum with no reference to geometry and no easy accessible visualisation. What we have shown thus far makes it clear that the students can discover the general method on their own, and that the area model makes all the transformation clear, reasonable and meaningful. In turn, the students do not have to memorize any formula, and also they no longer find the process a trick-based mathemagic.

Success in factoring relies on a lot of luck. As you attempt to factorise a quadratic expression you need to make a host of intelligent guesses and still cross your fingers that everything will work out. But the truth is that factoring rarely works! Most quadratic expressions will not factor with nice whole numbers and many will not factor at all in the real number system. But most school curricula want students to practice the art of making intelligent guesses and pushing through and so will present example after example of quadratics that just, by luck, happen to factor nicely. This often gives students the impression that factoring is a standard and fruitful solving technique and that luck will always be on their side. Unfortunately, factoring generally only works nicely for specially designed examples found in textbooks and exams. In the next section we give a brief discussion on applying the previous visual approach to factoring trinomials for those curious.

3.4. Completing the rectangle – anti-symmetrical solutions

We showed the benefits of using symmetry and we worked out the general method for solving any quadratic equation by completing a square. However, instead of a square we could consider a non-square rectangle. For instance, when we provided a diagram for the product of $2x + 3$ and $4x + 1$, we were not thinking about a square (at least as long as $x \neq 1$). As a product of these two algebraic sums we obtained some trinomial. Now we want to reverse this problem, and we are going to ask ourselves what are the sides of an unsymmetrical rectangle whose area is expressed by a given trinomial. We will now show on several examples how such problems can be solved.

Example I

$$\text{Factorise } x^2 + 9x + 20.$$

We can illustrate the first term of the left side as we did earlier, that is as a square whose side length is x. (Though, admittedly, if we are letting go of symmetry, this could also be represented by a $2x$ by $0.5x$ rectangle or some other rectangle.) With this choice, we can draw a full rectangle with two known dimensions p and q.

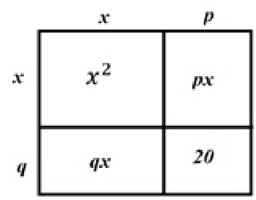

Fig. 16

The diagram shows we need numbers p and q such that

$$px + qx = 9x \text{ (which means that } p + q = 9)$$

and

$$pq = 20.$$

We see the classic techniques discussed earlier in this paper naturally arising. We can easily figure out that these numbers are 4 and 5, and hence write the given trinomial as

$$x^2 + 9x + 20 = (x + 4)(x + 5).$$

If the question went further and asked us to solve $x^2 + 9x + 20 = 0$, then we are being told that our pictured rectangle has area zero. It must be that one of its side lengths is zero and so either $x + 4 = 0$ or $x + 5 = 0$. We believe that this example brings a deeper meaning to the previously shown AC method.

Example II

$$2x^2 + 5x + 2 = 0$$

This example is more challenging than the former one, since now we have to deal with the first coefficient different from one. However, we can still make similar intelligent guesses and find a way to factorise the left side. We start as usual with the first 'box' in our diagram, but this time we will not draw a square, but a rectangle instead. Let us try to factorise the trinomial with the use of friendly coefficients, and interpret the first term as the area of a rectangle whose sides are x and $2x$. If we proceed as we did previously, we should complete our diagram to an unsymmetrical rectangle as presented on a picture below.

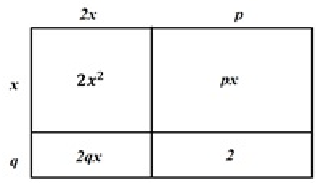

Fig. 17

We seek two numbers p and q, such that:

$$p + 2q = 5$$

and

$$pq = 2.$$

These numbers can be easily guessed to be:

$$p = 1 \text{ and } q = 2.$$

Rewriting the original equation, we get:

$$(2x + 1)(x + 2) = 0$$

The area of the rectangle is said to be zero and so either $2x + 1 = 0$ or $x + 2 = 0$.

Example III

$$3x^2 - 8x + 4 = 0$$

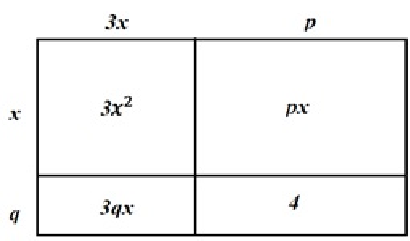

Fig. 18

$$p + 3q = -8$$

$$pq = 4$$

$$p = q = -2$$

$$(3x - 2)(x - 2) = 0$$

4. Summary

Most every quadratic equation presented to students in a curriculum uses "nice" numbers, but if we had to solve an awkward quadratic equation, like

$$1,3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0$$

by hand, the visual square method would be miserable. We would probably use the quadratic formula, though that would likely be miserable too. Many people prefer to use the quadratic formula because it is speedier. We need to be honest to ourselves and our students: if our goal really is just to get an answer and to do so as fast as possible for some reason, then ... the most intelligent thing is not to follow either approach, but simply use a free algebra system on the internet instead. We live in the 21st century after all! This explains why we really like the alternative methods such as completing the square or the rectangle, or using the AC-method. In these methods the path and the experience are more important than obtaining the final answer. These approaches are about thinking, meaning making and smart reasoning, not just mechanical application of algorithms.

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