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## Euclid's theory of proportion revised\*

**Abstract.** The theory of similar figures, as developed in school mathematics, emulates the theses of Euclid's propositions included in book VI of the *Elements*. It does not, however, represent Euclid's proof technique, i.e. proportions. The theory is usually developed within a metric space, with line segments having lengths, figures having areas, fractions simulating proportions, and the similarity scale being a real number. In like manner, in (Hilbert, 1902, ch. III), David Hilbert develops his own proportion theory to prove Euclid's propositions VI.2 and VI.4. Yet, Hilbert applies proportion only to line segments, while applying similarity only to triangles. Thus far, no one has managed to develop it further to get Euclid's proposition VI.31, which crowns the ancient Greek theory of similar figures. Although Robin Hartshorne, in (Hartshorne, 2000, ch. 5), suggests that Hilbert's project to reinterpret book VI can be completed by applying a concept of the content of a figure, contents of figures are not considered as terms of Hilbert's proportions.

In this paper, we apply the area method as introduced in (Chou, Gao, Zhang, 1994) to reconstruct Euclid's theory of similar figures, both his propositions, and the proof technique. Our interest is focused on proposition VI.1. It plays a crucial role in Euclid's system, and yet, it is the most controversial proposition of book VI when viewed from the modern perspective. As the only proposition in book VI, it relies on comparing figures in terms of greater-lesser. Since Euclid's system does not provide any criteria on how to decide whether one figure is greater than another, this relation relies on diagrammatic evidence rather than explicit mathematical rules. To bypass any reference to the greater-lesser relation, we adopt an axiomatic account of the area method introduced in (Janjic, Narboux, Quaresma, 2012), since it includes proposition VI.1 as an axiom.

The plan of this paper is as follows: first, we introduce axioms of the area method, next, we present a model for these axioms. Then, we reconstruct exemplary propositions of book VI within the framework of the area method. Finally, we compare Euclid's proposition VI.1 with the fundamental theorem of the area method, the so-called Co-side theorem.

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## 1. Area Method Axiomatically

(Chou, Gao, Zhang, 1994, ch. 1) provides an intuitive introduction into the area method based on the standard formula for the area of a triangle: one-half base times height. This formula enables to derive the fundamental tool of this method, namely the Co-side theorem. We will show that Euclid's proposition VI.1 is a special case of this theorem.

(Chou, Gao, Zhang, 1994, ch. 2) sketches an axiomatic development of the theory, which includes Euclid's proposition VI.1 as an axiom. (Janicic, Narboux, Quaresma, 2012) provides a revised version of that system of axioms, where Euclid's VI.1 is also included as an axiom. In section § 1.2 below, we present the later axioms.

With axiom A13, (Janicic, Narboux, Quaresma, 2012) goes back to the formula for the area of a triangle. For the sake of completeness, we keep this axiom in our presentation, nevertheless, we do not rely on it in our development. We seek to develop a theory that does not apply any formula for an area of a triangle. It is to be like Euclid's system in this respect.

### 1.1. Primitive Notions and Basic Definitions

There are three primitive notions in the area method: point, length of a directed segment, and signed area of a triangle. They are characterized by axioms A1 to A13 presented in the next section.

In what follows, capital letters  $A, B, C$  etc. stand for points;  $\mathbb{P}$  stands for the set of all points. An ordered pair of points is a segment, an ordered triple of points is a triangle.

Let  $(\mathbb{F}, +, \cdot, 0, 1, <)$  be an ordered field closed under the square root operation. The length of a segment  $AB$ ,  $\overline{AB}$ , in short, is an element of the set  $\mathbb{F}$ . Hence, we assume that there is a map which assigns to an ordered pair of points a number:

$$\mathbb{P}^2 \ni \langle A, B \rangle \mapsto \overline{AB} \in \mathbb{F}.$$

The number  $\overline{AB}$  can be positive, negative or zero.

The signed area of a triangle  $ABC$ ,  $S_{ABC}$ , in short, is an element of  $\mathbb{F}$ . Then, we assume there is a map  $S$ , which assigns to the ordered triple of points a number (positive, negative or zero):

$$\mathbb{P}^3 \ni \langle A, B, C \rangle \mapsto S_{ABC} \in \mathbb{F}.$$

To model a Euclidean scenery, (Chou, Gao, Zhang, 1994) provides definitions of basic geometric relations, namely, the parallel and perpendicular segments. These are as follows.

#### DEFINITION 1

*Points  $A, B, C$  are co-linear iff  $S_{ABC} = 0$ .*

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## DEFINITION 2

Two segments  $AD$  and  $BC$ , where  $A \neq D$  and  $B \neq C$ , are parallel, iff  $S_{ABC} = S_{DBC}$ . For this relation, we adopt the standard symbol  $AD \parallel BC$ .

## DEFINITION 3

For three points  $A$ ,  $B$  and  $C$ , the Pythagorean difference, denoted by  $P_{ABC}$ , is defined by

$$P_{ABC} = \overline{AB}^2 + \overline{BC}^2 - \overline{AC}^2.$$

## DEFINITION 4

Two segments  $DB$  and  $CA$ , where  $D \neq B$  and  $C \neq A$ , are perpendicular iff  $P_{DCA} = P_{BCA}$ . This relation is denoted by  $DB \perp CA$ .

One may object that while we provide definitions of parallel or perpendicular line segments, Euclid considers (infinite) lines instead. In fact, the infinite straight line is the 18th century interpretation of the *Elements*. Euclid's term *straight-line*, εὐθεῖα γραμμή, refers to what we would call a closed line-segment (a segment with its end-points). Indeed, his definition of parallel lines (definition 23 of Book I) refers to line-segments that "being produced to infinity in each direction, meet with one another in neither".<sup>1</sup> Note that an infinite straight line cannot be "produced to infinity".<sup>2</sup>

## 1.2. Axioms for the Area Method

Here are the axioms for the area method.

A1.  $\overline{AB} = 0$  iff  $A$  and  $B$  are identical.

A2.  $S_{ABC} = S_{CAB}$ .

A3.  $S_{ABC} = -S_{BAC}$ .

A4. If  $S_{ABC} = 0$ , then  $\overline{AB} + \overline{BC} = \overline{AC}$  (Chasles' axiom).

A5. There are points  $A$ ,  $B$  and  $C$  such that  $S_{ABC} \neq 0$  (not all points are collinear).

A6.  $S_{ABC} = S_{DBC} + S_{ADC} + S_{ABD}$  (all points are in the same plane).<sup>3</sup>

A7. For each element  $r$  of  $F$ , there exists a point  $P$ , such that  $S_{ABP} = 0$  and  $\overline{AP} = r\overline{AB}$  (construction of a point on a line).

A8. If  $A \neq B$ ,  $S_{ABP} = 0$ ,  $\overline{AP} = r\overline{AB}$ ,  $S_{ABP'} = 0$  and  $\overline{AP'} = r\overline{AB}$ , then  $P = P'$ .

A9. If  $PQ \parallel CD$  and  $\frac{\overline{PQ}}{\overline{CD}} = 1$ , then  $DQ \parallel PC$  (parallelogram).

A10. If  $S_{PAC} \neq 0$  and  $S_{ABC} = 0$ , then  $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$  (Euclid's proposition VI.1).

A11. If  $C \neq D$  and  $AB \perp CD$  and  $EF \perp CD$ , then  $AB \parallel EF$ .

A12. If  $A \neq B$ ,  $AB \perp CD$  and  $AB \parallel EF$ , then  $EF \perp CD$ .

A13. If  $FA \perp BC$  and  $S_{FBC} = 0$ , then  $4 \cdot S_{ABC}^2 = \overline{AF}^2 \overline{BC}^2$  (formula for the area of a triangle).

<sup>1</sup>All English translations of the *Elements* after (Fitzpatrick, Heiberg, 2007).

<sup>2</sup>(Błaszczyk, Mrówka, 2013, pp. 181–210) provides more evidence that Euclid's straight-line means line segment with its end-points.

<sup>3</sup>The idea of signed area originates from Hilbert's *Foundations of Geometry*. In (Hilbert, 1970, §20) he proves the theorem that is a counterpart of axiom A6.

### 1.3. Introducing Euclidean context

In proposition I.38, Euclid states that  $\triangle ABC = \triangle DBC$ , given  $AD \parallel BC$  (see Fig. 1), while the equality of triangles means equality of areas, as explained in (Błaszczuk, 2018). Euclid’s I.38 is covered by the first part of Definition 2, that is, if  $AD \parallel BC$ , then  $S_{ABC} = S_{DBC}$ . Definition 2 also mirrors its reverse, namely I.39:

“Let ABC and DBC be equal triangles which are on the same base BC, and on the same side. I say that they are also between the same parallels”.

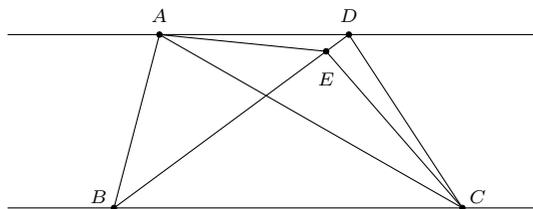


Fig. 1. *Elements*, I.39.

In the area method, Euclid’s stipulation “on the same side” is rendered by the fact that the area of a triangle agrees with the order of vertexes in a way encoded by axiom A3. Moreover, in Definition 2, instead of the equality of triangles  $\triangle ABC = \triangle DBC$ , signed areas  $S_{ABC}$ ,  $S_{DBC}$  are involved.

Similarly, the first part of Definition 4 includes modern version of the Pythagorean theorem, namely: in Fig. 2a, take  $A = B$ , then the triangle  $CDB$  turns into a right triangle  $CDA$ . If  $DA \perp CA$ , then  $P_{DCA} = P_{BCA}$ . In this special case, the equality  $P_{DCA} = P_{BCA}$  means  $\overline{DC}^2 = \overline{DA}^2 + \overline{AC}^2$ . Yet, Definition 4 mirrors also the reverse of the Pythagorean theorem, that is, if  $\overline{DC}^2 = \overline{DA}^2 + \overline{AC}^2$ , then  $DA \perp CA$ . In the *Elements*, the Pythagorean theorem is in proposition I.47. Its reverse, proposition I.48, reads:

“For let the square on one of the sides, BC, of triangle ABC be equal to the squares on the sides BA, AC. I say that angle BAC is a right-angle”.

Below, we present two simple theorems. In Euclidean context, they are more than obvious.<sup>4</sup> Here, we provide proofs based on the area method axioms system.<sup>5</sup>

THEOREM 1

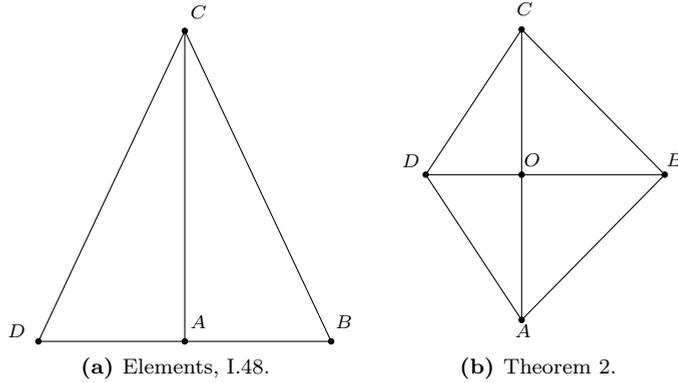
If  $AD \parallel BC$ , then  $BC \parallel AD$  (**Fig. 1**)

Proof. Let  $AD \parallel BC$ . By Definition 2,  $S_{ABC} = S_{DBC}$ . Then, the following equalities obtain

$$S_{BAD} \stackrel{A6}{=} S_{CAD} + S_{BCD} + S_{BAC} \stackrel{A2}{=} S_{CAD} + S_{DBC} + S_{BAC} \stackrel{A3}{=} S_{CAD} + S_{DBC} - S_{ABC} = S_{CAD}$$

<sup>4</sup>Those and many other lemmas about parallelism and perpendicularity have been already proved and machine checked in (Narboux, 2004).

<sup>5</sup>Symbol  $\stackrel{A6}{=}$  represents a reference to axiom A6.



**Fig. 2.** Reverse of the Pythagorean theorem.

Again, by definition of parallel lines, we get: since  $S_{BAD} = S_{CAD}$ , then  $BC \parallel AD$ . □

**THEOREM 2**

If  $DB \perp CA$ , then  $CA \perp DB$  (**Fig. 2b**)

Proof. Since  $\overline{DC} = -\overline{CD}$ , then  $\overline{DC}^2 = \overline{CD}^2$ . By Definition 4, if  $DB \perp CA$ , then  $P_{DCA} = P_{BCA}$ .

From the definition of the Pythagorean difference, equalities follow

$$P_{DCA} = \overline{DC}^2 + \overline{CA}^2 - \overline{DA}^2,$$

$$P_{BCA} = \overline{BC}^2 + \overline{CA}^2 - \overline{BA}^2.$$

Hence,  $P_{DCA} = P_{BCA}$  gives

$$\overline{DC}^2 + \overline{CA}^2 - \overline{DA}^2 = \overline{BC}^2 + \overline{CA}^2 - \overline{BA}^2.$$

Or, in an equivalent form

$$\overline{DC}^2 - \overline{DA}^2 = \overline{BC}^2 - \overline{BA}^2,$$

$$\overline{DC}^2 - \overline{BC}^2 = \overline{DA}^2 - \overline{BA}^2.$$

Due to the last equality, we have

$$\begin{aligned} P_{ADB} &= \overline{AD}^2 + \overline{DB}^2 - \overline{AB}^2 = \overline{DA}^2 - \overline{BA}^2 + \overline{DB}^2 \stackrel{(1)}{=} \overline{DC}^2 - \overline{BC}^2 + \overline{DB}^2 = \\ &= \overline{CD}^2 + \overline{DB}^2 - \overline{CB}^2 \stackrel{def}{=} P_{CDB}. \end{aligned}$$

Finally, since  $P_{CDB} = P_{ADB}$ , by Definition 4,  $CA \perp DB$ . □

## 2. Model for the area method

### 2.1. Interpreting primitive notions

In this section, we provide a model for axioms A1–A13.<sup>6</sup> We interpret the set of points as elements of the Cartesian plane  $\mathbb{R} \times \mathbb{R}$ . Thus, a point  $A$  is an ordered pair of real numbers. In this chapter, we assume that  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$ ,  $C = (x_3, y_3)$ ,  $D = (x_4, y_4)$ .

The lexicographical order on the plane  $\mathbb{R} \times \mathbb{R}$  is defined as follows:

DEFINITION 5

$$A \preceq B \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leq y_2),$$

where  $x_1 < x_2$  is the inequality of real numbers.

DEFINITION 6

The length of a directed segment  $AB$  is the number  $\overline{AB}$  defined by

$$\overline{AB} = \begin{cases} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, & \text{when } A \preceq B, \\ -\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, & \text{when } B \preceq A. \end{cases}$$

DEFINITION 7

The signed area for a triangle  $ABC$  is the number  $S_{ABC}$  defined by

$$S_{ABC} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Note that in analytic geometry, the number  $|S_{ABC}|$  stands for the area of a triangle with vertexes  $A, B, C$ . In our interpretation, the absolute value is omitted, hence the number  $S_{ABC}$  can be positive or negative.

### 2.2. Co-linearity, parallelism, perpendicularity

We show that in our model, Definitions 1–4 introduce the standard meaning of (a) co-linearity of points, as well as (b) parallel and (c) perpendicular lines.

(Ad a) First, we show that if points  $A, B, C$  are co-linear, then  $S_{ABC} = 0$ .

Let then  $A, B, C$  lie down on a line given by the equation  $y = ax + b$ . Then we have

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b, \quad y_3 = ax_3 + b.$$

Hence,

$$S_{ABC} = \frac{1}{2} \begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_2 & ax_2 + b & 1 \\ x_3 & ax_3 + b & 1 \end{vmatrix} = 0.$$

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<sup>6</sup>Julien Narboux has pointed out that a combination of [https://github.com/GeoCoq/GeoCoq/blob/master/Tarski\\_dev/Ch16\\_coordinates\\_with\\_functions.v](https://github.com/GeoCoq/GeoCoq/blob/master/Tarski_dev/Ch16_coordinates_with_functions.v) and [https://github.com/GeoCoq/GeoCoq/blob/master/Meta\\_theory/Models/POF\\_to\\_Tarski.v](https://github.com/GeoCoq/GeoCoq/blob/master/Meta_theory/Models/POF_to_Tarski.v) gives a machine checked proof of results in this section.

Suppose now that  $S_{ABC} = 0$  and  $A, B$  lie on the line  $y = ax + b$ . This means that the equalities hold

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b.$$

From the assumption

$$S_{ABC} = \begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_2 & ax_2 + b & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

it follows that

$$x_1(ax_2 + b) + x_2y_3 + x_3(ax_1 + b) - x_3(ax_2 + b) - x_1y_3 - x_2(ax_1 + b) = 0.$$

As a result, we get  $y_3 = ax_3 + b$ . This means that  $A, B, C$  lie on the line  $y = ax + b$ .

(Ad b) Let

$$S_{ACD} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}, \quad S_{BCD} = \frac{1}{2} \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}.$$

If  $S_{ACD} = S_{BCD}$ , then

$$\frac{y_1 - y_2}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}.$$

This means that the slopes of the lines on which points  $A, B$ , on the one hand, and  $C, D$ , on the other, lie, are equal. As a result, lines running through points  $A, B$  and  $C, D$ , respectively, are parallel.

Finally, suppose  $A, B$  and  $C, D$  lie on parallel lines, that is

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b \quad \text{and} \quad y_3 = ax_3 + c, \quad y_4 = ax_4 + c.$$

Hence,

$$S_{ACD} = \frac{1}{2} \begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_3 & ax_3 + c & 1 \\ x_4 & ax_4 + c & 1 \end{vmatrix} = \frac{1}{2}(x_4 - x_3)(b - c),$$

$$S_{BCD} = \frac{1}{2} \begin{vmatrix} x_2 & ax_2 + b & 1 \\ x_3 & ax_3 + c & 1 \\ x_4 & ax_4 + c & 1 \end{vmatrix} = \frac{1}{2}(x_4 - x_3)(b - c).$$

It means

$$S_{ACD} = S_{BCD}.$$

We can conclude that in our model, the relationship  $\parallel$ , as given by Definition 2 is the same as parallelity in analytic geometry.

(Ad c) Suppose segments  $DB$  and  $CA$  are perpendicular in the sense of Definition 2, i.e.,  $P_{DCA} = P_{BCA}$

From the definition of perpendicularity we get:

$$\begin{aligned} P_{DCA} &= \overline{DC}^2 + \overline{CA}^2 - \overline{DA}^2, \\ P_{BCA} &= \overline{BC}^2 + \overline{CA}^2 - \overline{BA}^2, \\ \overline{DC}^2 - \overline{DA}^2 &= \overline{BC}^2 - \overline{BA}^2, \\ \frac{y_4 - y_2}{x_1 - x_3} &= \frac{x_4 - x_2}{y_1 - y_3}. \end{aligned}$$

This means that the slopes of the lines  $AB$  and  $CD$  are in inverse proportion to each other and have opposite signs. Thus, we can conclude that the relationship  $\perp$  in the defined model means perpendicularity in analytical geometry.

Suppose  $D, B$  and  $C, A$  lie on perpendicular lines, therefore the scalar product of  $\overline{DB}$  and  $\overline{CA}$  is zero;

$$(\overline{DB} \cdot \overline{CA}) = (x_2 - x_4)(x_1 - x_3) + (y_2 - y_4)(y_1 - y_3) = 0,$$

$$x_2x_1 - x_2x_3 - x_4x_1 + x_4x_3 + y_2y_1 - y_2y_3 - y_4y_1 + y_4y_3 = 0.$$

Now,

$$P_{DCA} = 2x_3^2 - 2x_3x_4 - 2x_1x_3 + 2x_1x_4 + 2y_3^2 - 2y_3y_4 - 2y_1y_3 + 2y_1y_4,$$

$$P_{BCA} = 2x_3^2 - 2x_3x_2 - 2x_1x_3 + 2x_1x_2 + 2y_3^2 - 2y_3y_2 - 2y_1y_3 + 2y_1y_2,$$

Obviously,

$$P_{DCA} = P_{BCA} \iff P_{DCA} - P_{BCA} = 0,$$

Hence, the difference  $P_{DCA} - P_{BCA}$  equals

$$-2(x_2x_1 - x_2x_3 - x_4x_1 + x_4x_3 + y_2y_1 - y_2y_3 - y_4y_1 + y_4y_3).$$

Therefore, when  $\overline{DB}$  and  $\overline{CA}$  are perpendicular, then  $P_{DCA} = P_{BCA}$ .

### 2.3. Interpreting axioms

In this section, we show that under the interpretation of section 2.1, all axioms of the area method are theorems of analytic geometry on the Cartesian plane  $\mathbb{R} \times \mathbb{R}$  with lexicographic order. Since parallelism and perpendicularity in our interpretation mean the same as parallelism and perpendicularity in analytic geometry, then axioms A9, A11, A12, A13 are well-known theorems.

Ad A1. Given  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$ ,  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0$  if and only if  $x_1 = x_2, y_1 = y_2$ .

A2 and A3 follow from properties of a determinant.

Ad A2. Indeed, the equality  $S_{ABC} = S_{CAB}$  obtains, because

$$S_{CAB} = \frac{1}{2} \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = -\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = S_{ABC}.$$

Ad A3. Similarly, the equality  $S_{ABC} = -S_{BAC}$  is the result of the two interchanging rows of the determinant rule.

Ad A4. If  $S_{ABC} = 0$ , then  $\overline{AB} + \overline{BC} = \overline{AC}$ .

Let us consider different cases of the location of these points according to lexicographic order. Let,  $C \preceq A \preceq B$ .

$$\overline{AB} = |AB|, \overline{BC} = -|BC|, \overline{AC} = -|AC|.$$

Hence

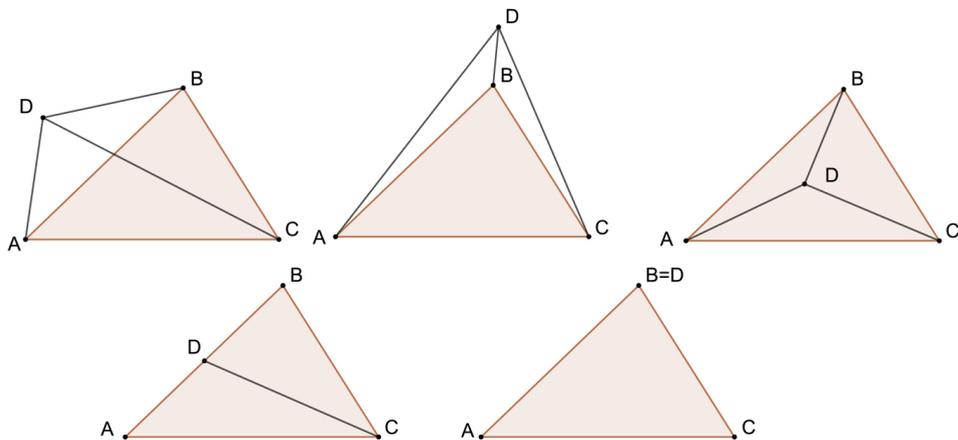
$$\overline{AB} + \overline{BC} = |AB| - |BC| = -|AC| = \overline{AC}.$$

It can be shown analogously that the equality stated in axiom A4 is met for all possible cases.

Ad A5. Here are points for which the signed area is not equal to 0.

$$S_{BAC} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}.$$

Ad A6. The equality  $S_{ABC} = S_{DBC} + S_{ADC} + S_{ABD}$  follows from simple calculations, which do not depend on a location of point  $D$  relative to  $A$ ,  $B$  and  $C$ , namely



**Fig. 3.**  $S_{ABC} = S_{DBC} + S_{ADC} + S_{ABD}$ .

$$S_{ABC} = \frac{1}{2}(x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_2y_1).$$

$$S_{DBC} = \frac{1}{2}(x_4y_2 + x_3y_4 + x_2y_3 - x_3y_2 - x_4y_3 - x_2y_4).$$

$$S_{ADC} = \frac{1}{2}(x_1y_4 + x_3y_1 + x_4y_3 - x_3y_4 - x_1y_3 - x_4y_1).$$

$$S_{ABD} = \frac{1}{2}(x_1y_2 + x_4y_1 + x_2y_4 - x_4y_2 - x_1y_4 - x_2y_1).$$

$$S_{DBC} + S_{ADC} + S_{ABD} = S_{ABC}.$$

Ad A7. Let  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$  and  $P = (x, y)$  lie on a line  $y = ax + b$  passing through  $A, B$ . Let us define the coordinates of  $P$  as follows

$$x = \frac{1}{r}(x_2 - x_1) + x_1, \quad y = \frac{1}{r}(y_2 - y_1) + y_1.$$

Then

$$r\overline{AP} = \pm r\sqrt{(x - x_1)^2 + (y - y_1)^2},$$

where  $\pm$  depends on whether  $A \prec P$  or  $P \prec A$ . Since

$$|r\overline{AP}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |\overline{AB}|,$$

by putting  $A \prec P$  or  $P \prec A$ , we get

$$r\overline{AP} = \overline{AB}.$$

Ad A8.  $S_{ABP} = 0 = S_{ABP'}$ , then points  $A, B, P, P'$  are co-linear. Since

$$\overline{AP'} = r\overline{AB} = \overline{AP},$$

$P, P'$  must be identical.

Ad A10. When  $S_{ABC} = 0$ , then  $A, B, C$  lie on a line  $y = ax + b$ . Let  $A = (x_1, ax_1 + b)$ ,  $B = (x_2, ax_2 + b)$ ,  $C = (x_3, ax_3 + b)$ ,  $P = (x_4, y_4)$ . According to Definition 6, we have

$$\overline{AB} = \pm\sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = \pm|x_2 - x_1|\sqrt{a^2 - 1},$$

$$\overline{AC} = \pm\sqrt{(x_3 - x_1)^2 + (ax_3 - ax_1)^2} = \pm|x_3 - x_1|\sqrt{a^2 - 1},$$

where  $\pm$  depends on relative location of the points  $A, B$  and  $A, C$  respectively according to Definition 6. By considering possible locations of points  $A, B, C$  on the line  $y = ax + b$ , we get the following equality

$$\frac{\overline{AB}}{\overline{AC}} = \frac{x_2 - x_1}{x_3 - x_1}.$$

Signed areas of triangles are as follows

$$S_{PAB} = \frac{1}{2}(x_1 - x_2)(ax_4 - y_4 + b),$$

$$S_{PAC} = \frac{1}{2}(x_1 - x_3)(ax_4 - y_4 + b).$$

Finally, we get

$$\frac{S_{PAB}}{S_{PAC}} = \frac{(x_1 - x_2)(ax_4 - y_4 + b)}{(x_1 - x_3)(ax_4 - y_4 + b)} = \frac{(x_2 - x_1)}{(x_3 - x_1)} = \frac{\overline{AB}}{\overline{AC}}.$$

### 3. Euclid's proportions by the area method

From our perspective, the crux of the axiomatic development of the area method concerns the fact that the area of a triangle is not reduced to any formula, be it  $S_{ABC} = \frac{1}{2}ah$ ,  $S_{ABC} = \frac{1}{2}ab\sin\alpha$ , or any other. Indeed,  $S_{ABC}$  is a primitive concept, it is an entity of its own. At the same time, it is also subject to the arithmetic of fractions. Regarding terms such as (1)  $S_{PAB} = S_{PAC}$ , or (2)  $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$ , equality means equality in an ordered field, and it should be introduced by logical axioms. In mathematics, such axioms are usually implicit. Still, terms such as this  $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$  mimic ancient proportions, in a way we explain below.

(Ad 1) In Book VI of the *Elements*, triangles, along with other figures, are entities of their own; they are also subject to some fraction-like operations, namely proportions, as introduced in Book V. Within Euclid's framework, equality refers to figures; it is a mathematical rather than a logical concept. In fact, a group of propositions from Book I and II form the theory of the area (theory of equal figures), as explained in (Błaszczuk, 2018). It builds on procedures justified by Common Notions and straight-edge and compass constructions. Currently, we focus on the equality of triangles. Thus, terms such as  $S_{PAB} = S_{PAC}$  do not refer to identity or congruence, but rather to Euclid's theory of equal figures.

What is really new, it is the role of an order of points. Within Euclid's system, an order of points, be it end-points of a line segment or vertexes of a triangle, plays no role. In the area method, the role of the order of vertexes, as encoded in axiom A3, is crucial.

(Ad 2) Euclid's proportions can be interpreted in an Archimedean field as follows

$$a : b :: c : d \Leftrightarrow a \cdot b^{-1} = c \cdot d^{-1}.$$

In the 17th century, proportions were replaced by fractions. Yet, the arithmetic of fractions and proportions is not the same. The arithmetic of fractions provides us with more mathematical capabilities. For instance, while the square of fraction,  $(\frac{a}{b})^2$ , is a legitimate object, the square of ratio,  $(a : b)^2$ , is not. Moreover, the ratio  $a : b$  itself is not a legitimate object, since it has a mathematical meaning only in a proportion, say  $a : b :: c : d$ . Still, when the area method explores fractions in terms such as  $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$  or  $\frac{\overline{AB}}{\overline{AC}} = r$ , and transforms fractions according to specific rules, it strongly mimics ancient proportions. To clarify this claim, we require some basics of Euclid's Book V.

#### 3.1. *Elements*, Book V

The theory of proportions, as developed in Book V of Euclid's *Elements*, is founded on definitions 5 and 7. Viewed from the perspective of these definitions, a proportion is a relation between two pairs of geometric figures (magnitudes) of the same kind, triangles being of one kind, line segments of another kind *etc.* Magnitudes of the same kind form an ordered additive semi-group  $\mathfrak{M} = (M, +, <)$  characterized by the five axioms given below.

$$(E1) (\forall a, b \in M)(\exists n \in \mathbb{N})(na > b).$$

$$(E2) (\forall a, b \in M)(\exists c \in M)(a > b \Rightarrow a = b + c).$$

$$(E3) (\forall a, b, c \in M)(a > b \Rightarrow a + c > b + c).$$

$$(E4) (\forall a \in M)(\forall n \in \mathbb{N})(\exists b \in M)(nb = a).$$

$$(E5) (\forall a, b, c \in M)(\exists d \in M)(a : b :: c : d).$$

Here, the term  $na$  stands for  $\underbrace{a + a + \dots + a}_{n\text{-times}}$ ; it is, in Euclid's words, a *multiple of the magnitude* represented by  $a$ . The addition of magnitudes (of the same kind) as well as the relation *greater than* are primitive concepts. Through textual analysis one can show that  $+$  is a commutative and associative operation, and  $<$  is a transitive relation that obeys the law of trichotomy.

Axiom E1 interprets definition V.4, the so-called Archimedean axiom. It is applied once in Book V: in the proof of proposition V.8. It can be shown that the use of E1 in this proof is essential.<sup>7</sup>

E4 is implicitly applied in proposition V.5. From the modern perspective, this axiom is not essential, as it can be derived from the other four axioms. Nevertheless, it represents, in a concise way, the ancient Greek dogma concerning geometric objects encapsulated in the following slogan: *divisible into parts that are infinitely divisible*. Here, the term  $nb = a$  interprets the phrase [a is] *divisible into parts*; a recursive procedure of applying E4 to  $b$ , then to its parts, and so on, interprets the phrase *infinitely divisible*.

Axiom E5 represents the so-called fourth proportional. In Book V, it is applied in proposition V.5. It is also a building block of the exhaustion method, as developed in Book XII. Axioms E2 and E3 can be identified all throughout Book V, specifically in its most intricate propositions, V.8 and V.18.

We interpret Euclid's definition of proportion, V. def. 5, by the following formula

$$a : b :: c : d \Leftrightarrow_{df} (\forall m, n \in \mathbb{N})[(na >_1 mb \Rightarrow nc >_2 md) \wedge \\ \wedge (na = mb \rightarrow nc = md) \wedge (na <_1 mb \Rightarrow nc <_2 md)];$$

the assumption regarding magnitudes  $a, b$ , on the one hand, and  $c, d$ , on the other hand, being of the same kind is formalized by  $a, b \in \mathfrak{M}_1 = (M_1, +, <_1)$ , and  $c, d \in \mathfrak{M}_2 = (M_2, +, <_2)$ .

The conjunction that occurs in the definiens above is sometimes shortened into the following term

$$na \overset{\geq}{\underset{\leq}{\approx}} mb \Rightarrow nc \overset{\geq}{\underset{\leq}{\approx}} md.$$

In fact, it aptly represents the word pattern “when greater than greater [...] when equal [than ...] equal [...] when less [than ...] less” applied throughout Book V on regular basis.<sup>8</sup>

Then, by definition V.7, a relation *greater-lesser* between pairs of magnitudes is introduced as follows<sup>9</sup>

<sup>7</sup>See (Błaszczyk, Mrówka 2013, pp. 180–181).

<sup>8</sup>This notation was introduced by H. Hankel in his 1874's *Zur Geschichte der Mathematik in Altertum und Mittelalter*. Then it was employed by Heiberg in his translation of Book V into Latin.

<sup>9</sup>Here and in Definition 4 above, we apply the same symbol  $\succ$ . From the mathematical perspective, these two relations have nothing in common.

$$a : b \succ c : d \Leftrightarrow_{df} (\exists m, n \in \mathbb{N})[(na >_1 mb) \wedge (nc \leq_2 md)].$$

Now, we present propositions 7 to 25 of Book V. Although they are stylized on algebra, the only purpose of this modern attire is to reveal the similarities between proportions and the arithmetic of fractions. Equality as it occurs below, stands for equal figures, as explained in (Błaszczuk, 2018).

$$\text{V.7} \quad a = b \rightarrow a : c :: b : c, \quad a = b \Rightarrow c : a :: c : b.$$

$$\text{V.8} \quad a > c \Rightarrow a : d \succ c : d, \quad a > c \Rightarrow d : c \succ d : a.$$

$$\text{V.9} \quad a : c :: b : c \Rightarrow a = b.$$

$$\text{V.10} \quad a : c \succ b : c \Rightarrow a > b, \quad c : b \succ c : a \Rightarrow b < a.$$

$$\text{V.11} \quad a : b :: c : d, \quad c : d :: e : f \Rightarrow a : b :: e : f$$

$$\text{V.12} \quad a : b :: c : d, \quad a : b :: e : f \Rightarrow a : b :: (a + c + f) : (b + d + f).$$

$$\text{V.13} \quad a : b :: c : d, \quad c : d \succ e : f \Rightarrow a : b \succ e : f.$$

$$\text{V.14} \quad a : b :: c : d, \quad a > c \Rightarrow b > d.$$

$$\text{V.15} \quad a : b :: na : nb.$$

$$\text{V.16} \quad a : b :: c : d \Rightarrow a : c :: b : d.$$

$$\text{V.17} \quad (a + b) : b :: (c + d) : d \Rightarrow a : b :: c : d.$$

$$\text{V.18} \quad a : b :: c : d \Rightarrow (a + b) : b :: (c + d) : d.$$

$$\text{V.19} \quad (a + b) : (c + d) :: a : c \Rightarrow b : d :: (a + b) : (c + d).$$

$$\text{V.20} \quad a : b :: d : e, \quad b : c :: e : f, \quad a \geq c \Rightarrow d \geq f.$$

$$\text{V.21} \quad a : b :: e : f, \quad b : c :: d : e, \quad a \geq c \Rightarrow d \geq f.$$

$$\text{V.22} \quad a : b :: d : e, \quad b : c :: e : f \Rightarrow a : c :: d : f.$$

$$\text{V.23} \quad (a : b :: e : f, \quad b : c :: d : e) \Rightarrow a : c :: d : f.$$

$$\text{V.24} \quad a : c :: d : f, \quad b : c :: e : f \Rightarrow (a + b) : c :: (d + e) : f.$$

$$\text{V.25} \quad (a : c :: e : f, \quad a > c > f, \quad a > e > f) \Rightarrow a + f > c + e.$$

When  $a : b :: c : d$  is replaced with  $\frac{a}{b} = \frac{c}{d}$ , and  $a : b \succ c : d$  with  $\frac{a}{b} > \frac{c}{d}$ , the above propositions will turn into simple, but random, rules of the arithmetic of fractions, given  $a, b, c, \dots$  are positive. Since proportions and fractions rely on different foundations, it is not surprising that Euclid needs axiom E1 to prove V.8, while the law  $a > c \Rightarrow \frac{a}{d} > \frac{c}{d}$ , given  $d > 0$ , holds in any ordered field, regardless of whether it is Archimedean or non-Archimedean.

We apply fractions in the area method, insofar as they represent the operations studied in Book V. For example, the following paraphrase of proposition V.9

$$\frac{a}{c} = \frac{b}{c} \Rightarrow a = b$$

is legitimate. However, this one

$$\frac{a}{b} = \frac{c}{d} \Rightarrow a = \frac{b \cdot c}{d},$$

is not. To put it briefly, any transformation of fractions must be justified by the above propositions V.7 to V.25.

Note there is one obvious difference between objects  $S_{ABC}$  and Euclid's triangles: signed areas of triangles form an additive ordered group  $(\mathbb{F}, +, 0, <)$ , while magnitudes of the same kind, specifically, triangles, form an additive ordered semi-group  $(M, +, <)$ . To put it simply: within Euclid's framework, there are neither *zero-* nor *negative-triangles*.

The idea that  $S_{ABC}$  can take negative values is represented in axioms A2, A3, and A6. In fact, A6 describes the basic relation for the area method when provides foundation for automated proofs in synthetic geometry. In fact, it was designed for this specific role. But in what follows, we will not explore these three axioms at all. To revise Euclid's proportion and reconstruct his propositions and proof technique, we need to adopt a tiny portion of the area method.

Contrarily, Hilbert's arithmetic of segments, as developed in (Hilbert, 1902, ch. 3), mimics Euclid's language, but applies technique foreign to Greek mathematics. For instance, his theorem 23 simulates Euclid's VI.2. The theorem reads (Hilbert, 1902, p. 52):<sup>10</sup>

*If two parallel lines cut from sides of the arbitrary angle the segments  $a, b$  and  $a', b'$  respectively, then we have always the proportion*

$$a : b = a' : b'.$$

The term  $a : b = a' : b'$ , in Hilbert's words, "expresses nothing else than the validity of the equation"  $ab' = ba'$ . Here,  $ab$  stands for a specifically introduced product of segments.<sup>11</sup> For the first time in the history of mathematics, the idea of a product of two line segments represented by another line was introduced in Descartes 1637's *La Géométrie*. Hilbert's definition of product builds on a different idea, nevertheless, the product of two lines is another line, and this concept of product is foreign to the ancient Greeks mathematics. Clearly, Hilbert's definition of proportion cannot be applied to other figures than segments. Therefore within his system we can not reconstruct Euclid's proofs, specifically the proof of proposition VI.2, since it involves a proportion of triangles.

Robin Hartshorne, specifically in chapters 4–5 of his (Hartshorne, 2000), seeks to develop Hilbert's project further to reconstruct Book VI of the *Elements*. Since he applies Hilbert's proportions, he can not reinterpret the starting point of this book, that is proposition VI.1. Indeed, he writes: "Having developed the theory of proportion abstractly in Book V, Euclid proceeds to apply his theory to geometry in Book VI, and develops what we recognize as the familiar theory of similar triangles. The key result here, which forms the basis of the subsequent development, is (VI.2), which says that a line parallel to the base of a triangle, if it cuts the sides, cuts them proportionately, and conversely. Euclid's proof is a tour de force, using the theory of area previously developed in Book I to establish this result" (Hartshorne, 2000, p. 167). In fact, most of the results of Book VI can be

<sup>10</sup>The converse of this theorem also holds in Hilbert's system.

<sup>11</sup>For details on how Hilbert defines the product of line-segments, consult (Hilbert, 1902, ch. 3) or (Baldwin, Mueller, 2019, § 3.).

derived from proposition VI.2. However, propositions VI.31, which crowns the ancient Greek theory of similar figures, via VI.20 relies on VI.1. While proposition VI.1 explicitly refers to proportions of triangles and parallelograms, it can not be interpreted by Hilbert's proportion.

Why then Hartshorne seeks to developed alternative to Euclid's proportion theory. Here are his reasons: "There are two reasons for us to seek an alternative development of the theory of similar triangles: One is to free ourselves from dependence on Archimedes' axiom, and the other is to avoid Euclid's use of the theory of area, which we have not yet treated satisfactorily" (Hartshorne, 2000, p. 167).

Euclid's theory of equal figures is developed with no reference to Archimedean axiom.<sup>12</sup> Yet, Archimedean axiom is essential in his theory of proportion, since we can show that without this axiom proposition V.8, the backbone of this theory, can not be proved.<sup>13</sup> However, in our interpretation, Euclid's propositions V.7–V.25 are modeled by fractions in an ordered field, be it Archimedean or non-Archimedean.

### 3.2. The Starting Point of Book VI

In Book VI, Euclid refers explicitly to definition V.5 only once, namely in the proof of proposition VI.1. It is about parallelograms and triangles "between the same parallels". In regard to triangles, it reads

"Let ABC and ACD be triangles, [...] of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD".

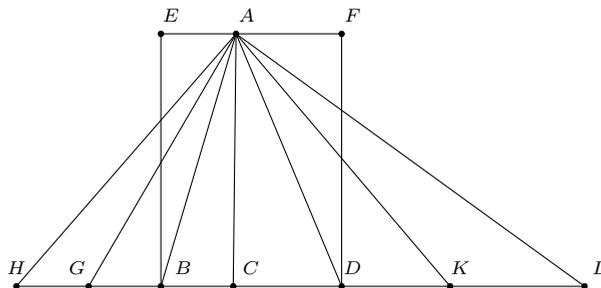


Fig. 4. Elements, VI.1.

We formalize it as follows

$$\triangle ABC : \triangle ADC :: BC : DC. \quad (1)$$

Euclid's proof is cumbersome, to say the least; it applies non-defined concepts of the addition of triangles encoded in the notion of *multiple* and requires to compare triangles in terms of *greater-lesser*. The accompanying diagram is to represent relations  $\triangle AHC = 3\triangle ABC$ , and  $\triangle ALC = 3\triangle ADC$ . Somehow, we are to decide that  $\triangle AHC < \triangle ALC$  whenever  $HC < LC$ . We can get rid of deliberations of

<sup>12</sup>See (Błaszczyk, 2018).

<sup>13</sup>See (Błaszczyk, Mrówka, 2013, pp. 180–181).

this kind by accepting (1) as an axiom. That is how, instead of proposition (1), we adopt axiom A10

$$\frac{S_{ABC}}{S_{ADC}} = \frac{\overline{BC}}{\overline{DC}}. \quad (2)$$

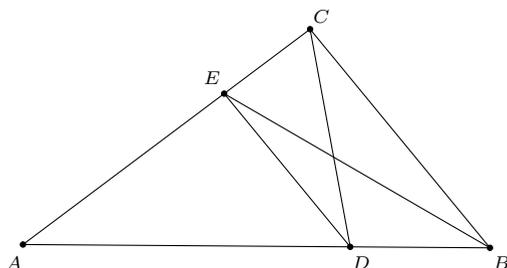
In the next section, we show how to derive other propositions of Book VI from axioms A1 to A12.

### 3.3. Sample Propositions. VI.2

We start with Euclid's proposition VI.2, the so-called Thales theorem. The English translation reads:

#### THEOREM 3

*If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the sides of the triangle proportionally. And if the sides of a triangle are cut proportionally then the straight-line joining the cutting will be parallel to the remaining side of the triangle.*



**Fig. 5.** Thales Theorem.

First, we present Euclid's proof in a schematized form. To this end, we apply standard symbols such  $\parallel$  or  $\perp$ , meaning parallel or perpendicular lines respectively. Second, we use specific symbols, such as  $\xrightarrow{I.38}$ . Here, the arrow represents the connective "for" rather than the logical implication, and the subscript I.38 means that there is a reference to proposition I.38; in the *Elements*, references are rendered by some characteristic phrases. Thus, the first two sentences of Euclid's proof, namely: "Thus, triangle BDE is equal to triangle CDE. For they are on the same base DE and between the same parallels DE and BC", we represent by the following scheme

$$DE \parallel BC \xrightarrow{I.38} \triangle(BDE) = \triangle(CDE).$$

In fact, to simplify the presentation, we skip the assumption that triangles BDE and CDE are "on the same base DE".

Euclid's second argument is covered by three sentences, namely: "And ADE is some other triangle. And equal have the same ratio to the same. Thus, as



[3] By A2, we can permute the names of vertexes, then by A10, the following equalities obtain

$$\frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}}, \quad \frac{S_{CDE}}{S_{DAE}} = \frac{\overline{CE}}{\overline{EA}}.$$

[4] By transitivity of equality in an ordered field,

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

Similarly, we can turn this proof into the following scheme

$$\begin{array}{lcl} DE \parallel BC & \xrightarrow{Df2} & S_{DEB} = S_{DEC} \\ S_{DEB} = S_{DEC} & \xrightarrow{fr} & \frac{S_{DEB}}{S_{DAE}} = \frac{S_{DEC}}{S_{DAE}} \\ & \xrightarrow{A2, A10} & \frac{S_{DEB}}{S_{DAE}} = \frac{S_{EBD}}{S_{EDA}} = \frac{\overline{BD}}{\overline{DA}} \\ & \xrightarrow{A3, A10} & \frac{S_{DEC}}{S_{DAE}} = \frac{S_{DCE}}{S_{DEA}} = \frac{\overline{CE}}{\overline{EA}} \\ & \xrightarrow{fr} & \frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}. \end{array}$$

In this scheme, the abbreviation *fr* stands for *by the arithmetic of fractions*, Df 2 – represents Definition 2, and A10 – axiom A10, as presented in sections § 1.1–1.2 above.

For the proof of the second part, suppose  $\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}$ .

[1] By A10:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}}, \quad \frac{S_{CED}}{S_{AED}} = \frac{\overline{CE}}{\overline{EA}}.$$

[2] By transitivity of equality:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{S_{CED}}{S_{AED}}.$$

[3] By axiom A2,  $S_{DAE} = S_{AED}$ . By applying the arithmetic of an ordered field to the above equality, we have

$$S_{BDE} = S_{CED}.$$

[4] By definition of parallel lines,  $DE \parallel BC$ . □

The schemes of Euclid's proof and the area method proof are almost identical. The only difference is that, within the area method, we must respect the order of end-points of line segments and the vertexes of triangles.

**3.4. Sample Propositions. VI.19**

We proceed to Euclid's proposition VI.19. The English translation reads: "Similar triangles are to each other as the double ratio of corresponding sides".

Since "ratio of corresponding sides" means the similarity scale, "double ratio of corresponding sides" means, in modern terms, the square of the similarity scale. Let triangles  $ABC$  and  $EDC$ , as represented in Fig.6, be similar. Thus, the similarity scale is (informally)  $\frac{AB}{DE}$  or  $\frac{BC}{EF}$ . Within Euclid's framework, it could be rendered as follows  $AB : DE :: BC : EF$ . The square of the similarity scale, say  $(BC : EF)$  is not a legitimate object, therefore Euclid constructs point  $G$  such that the proportion obtains

$$BC : EF :: EF : BG,$$

or, informally, again

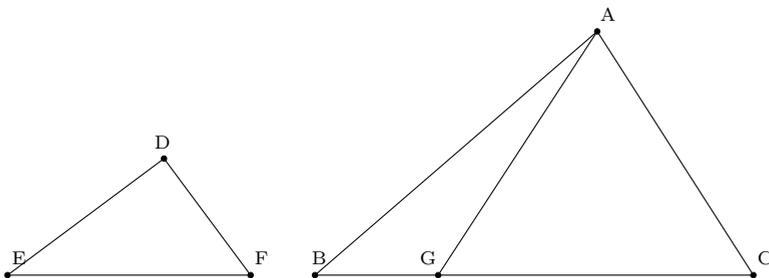
$$BC : BG = (BC : EF)(EF : BG) = (BC : EF)(BC : EF).$$

Hence, in the scheme below, the term  $(BC : EF)^2$  stands for "double ratio of corresponding sides". Thus, we paraphrase proposition VI.19 as follows

**THEOREM 4**

*Similar triangles are to each other as the square of the similarity scale:*

$$\triangle(ABC) : \triangle(DEF) :: BC : BG.$$



**Fig. 6.** Theorem 5.

Here is the scheme of Euclid's proof.

$$\begin{array}{lcl}
 AB : BC :: DE : EF & \xrightarrow{\text{V.16}} & AB : DE :: BC : EF \\
 BC : EF :: EF : BG & \xrightarrow{\text{V.11}} & AB : DE :: EF : BG \\
 AB : DE :: EF : BG & \xrightarrow{\text{VI.15}} & \triangle(ABG) = \triangle(DEF) \\
 BC : EF :: EF : BG & \xrightarrow{\text{df.V.9}} & BC : BG = (CB : EF)^2 \\
 CB : BG :: \triangle(ABC) : \triangle(ABG) & \rightarrow & \triangle(ABC) : \triangle(ABG) = (BC : EF)^2 \\
 \triangle(ABG) = \triangle(DEF) & \rightarrow & \triangle(ABC) : \triangle(DEF) = (BC : EF)^2.
 \end{array}$$

Now, a scheme of a proof within the area method.

$$\begin{array}{ccc}
 \frac{\overline{AB}}{\overline{BC}} = \frac{\overline{DE}}{\overline{EF}} & \xrightarrow{fr} & \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} \\
 \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{EF}}{\overline{BG}} & \xrightarrow{fr} & \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{EF}}{\overline{BG}} \\
 \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{EF}}{\overline{BG}} & \xrightarrow{VI.15} & \frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}} \\
 \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{EF}}{\overline{BG}} & \xrightarrow{fr} & \frac{\overline{BC}}{\overline{BG}} = \frac{\overline{EF}^2}{\overline{BG}^2} \\
 & \xrightarrow{A2, A10} & \frac{S_{ACB}}{S_{AGB}} = \frac{S_{ABC}}{S_{ABG}} = \frac{\overline{BC}}{\overline{BG}} \\
 \frac{S_{ACB}}{S_{AGB}} = \frac{\overline{BC}}{\overline{BG}} & \xrightarrow{fr} & \frac{S_{ACB}}{S_{AGB}} = \frac{\overline{EF}^2}{\overline{BG}^2} \\
 S_{ABG} = S_{DEF} & \xrightarrow{fr} & \frac{S_{ACB}}{S_{DFE}} = \left( \frac{\overline{EF}}{\overline{BG}} \right)^2.
 \end{array}$$

□

One step is missing in this scheme: since we skipped the proof of proposition VI.15, we ask the reader to take it for granted.

#### 4. Co-side Theorem

The so-called Co-side theorem is fundamental within the area method. It states:

**THEOREM 5**

*For four distinct points  $A, B, P, Q$ , let  $M$  be the intersection of the lines  $AB$  and  $PQ$  such that  $Q \neq M$ . Then the following equality obtains*

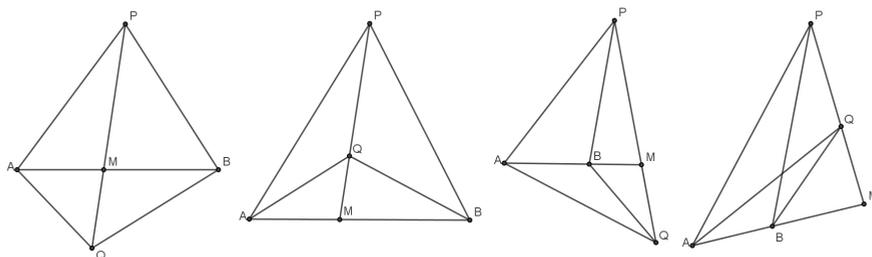
$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Below we present two proofs. The first comes from (Chou, Gao, Zhang, 1994, p. 8–17). It builds on an arithmetic trick to represent the fraction  $\frac{S_{PAB}}{S_{QAB}}$  as the product of three other fractions, namely  $\frac{S_{PAB}}{S_{PAM}}$ ,  $\frac{S_{PAM}}{S_{QAM}}$ ,  $\frac{S_{QAM}}{S_{QAB}}$ . Then, due to axiom A10, these ratios of triangles are reduced to ratios of line segments, namely<sup>14</sup>  $\frac{S_{PAB}}{S_{PAM}} = \frac{\overline{AB}}{\overline{AM}}$ ,  $\frac{S_{PAM}}{S_{QAM}} = \frac{\overline{PM}}{\overline{QM}}$ , and  $\frac{S_{QAM}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}$ .

Proof 1.

$$\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PAB}}{S_{PAM}} \cdot \frac{S_{PAM}}{S_{QAM}} \cdot \frac{S_{QAM}}{S_{QAB}} = \frac{\overline{AB}}{\overline{AM}} \cdot \frac{\overline{PM}}{\overline{QM}} \cdot \frac{\overline{AM}}{\overline{AB}} = \frac{\overline{PM}}{\overline{QM}}.$$

<sup>14</sup>In fact, this trick can also be performed within the framework of Book V.



**Fig. 7.** Co-side theorem.

□

The next proof considers four cases, represented in Fig. 7, depending on whether or not  $M$  lies between  $P, Q$  and between  $A, B$ . We base it on proposition VI.1 of the *Elements* as well as propositions from Book V which enable to transform ratios, specifically these two modifications

V.12'

$$a : b :: c : d \Rightarrow a : b :: (a + c) : (b + d)$$

V.19'

$$(a + b) : (c + d) :: a : c \Rightarrow b : d :: a : c,$$

and V.11, that is transitivity of proportion.

Regarding proportions, we observe that within the framework of Book V, there is no universal rule that covers all these four cases; each case refers to a specific proposition of Book V, namely V.12 or V.19.

Although we apply fractional notation, the proof is carried out within Euclid's framework. In this way, we aim to demonstrate that the insight behind the Co-side theorem can be tackled by the original theory of proportion.

In figures Fig. 8–11 below, we use symbols  $S_1, \dots, S_4$  to represent triangles. They are used to ease the reading of formulas applied in the proof.

Proof 2.

Case 1. Point  $M$  lies between  $P, Q$  and between  $A, B$ .

By proposition VI.1, we have the following equalities (proportions)

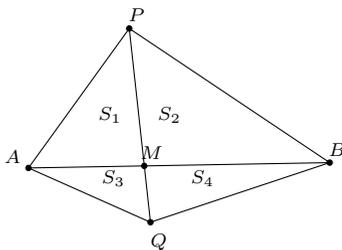
$$\frac{S_1}{S_3} = \frac{\overline{PM}}{\overline{QM}}, \quad \frac{S_2}{S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

Applying transitivity of proportion and V.12', we get

$$\frac{S_1}{S_3} = \frac{S_2}{S_4} = \frac{S_1 + S_2}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

Since

$$S_1 + S_2 = S_{PAB}, \quad S_3 + S_4 = S_{QAB},$$

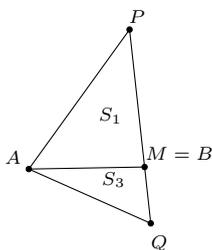


**Fig. 8.** Co-side theorem, case 1.

finally, the required equality obtains, namely

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Note, when  $B = M$ , the Co-side theorem is the same as Euclid's VI.1, namely

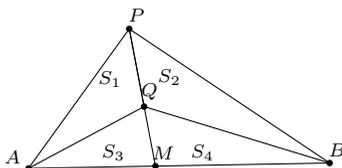


**Fig. 9.** Co-side turns into VI.1.

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PB}}{\overline{BQ}} = \frac{\overline{PM}}{\overline{MQ}}.$$

Case 2.

Point  $M$  lies between  $A, B$ , but not between  $P, Q$ ; see Fig. 10.



**Fig. 10.** Co-side, case 2.

By VI.1

$$\frac{S_1 + S_3}{S_3} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_2 + S_4}{S_4}.$$

Applying V.12' to the above equalities, we get

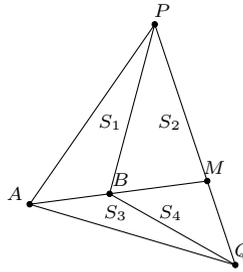
$$\frac{S_1 + S_2 + S_3 + S_4}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

Hence,

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Case 3.

Point  $M$  lies between  $P, Q$  and not between  $A, B$ ; see Fig. 11.



**Fig. 11.** Co-side, case 3.

By VI.1

$$\frac{S_1 + S_2}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_2}{S_4}.$$

Applying VI.19' to these equalities, we get

$$\frac{S_1 + S_2 - S_2}{S_3 + S_4 - S_4} = \frac{S_1}{S_3} = \frac{\overline{PM}}{\overline{QM}}.$$

Hence,

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Case 4.

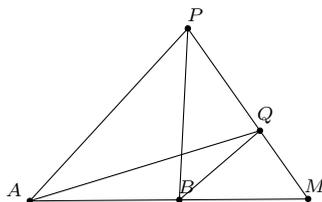
Point  $M$  does not lie between  $P, Q$  and between  $A, B$ ; see Fig. 12.

By VI.1

$$\frac{S_{PAB} + S_{PBM}}{S_{QAB} + S_{QBM}} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_{PBM}}{S_{QBM}}.$$

By VI.19', finally, we have

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$



**Fig. 12.** Co-side, case 4.

□

To sum up, in Euclid's VI.1, the co-side requirement is to guarantee that the triangles are of the same height ("between the same parallels"), while their bases can differ. In the Co-side theorem, triangles have a common base, while their heights can differ. Since in modern mathematics, the choice of the base of a triangle is conventional, we have co-side rather than co-base theorem.

Euclid's VI.1 enables to reduce the geometric pattern represented in the diagram Fig. 4 to the proportion of lines given by formula (2). The Co-side theorem provides us with four more geometric patterns, which can be reduced to the proportions of lines. We will show how to exploit these new patterns in school geometry lessons in a forthcoming paper.

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