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Eugeniusz Barcz

On the golden number and Fibonacci type sequences*

Dedicated to my grandchildren: Mathilda, Maximilian

Abstract. The paper presents, among others, the golden number φ as the limit of the quotient of neighboring terms of the Fibonacci and Fibonacci type sequence by means of a fixed point of a mapping $f(x) = 1+\frac{1}{x}$ of a certain interval with the help of Edelstein's theorem. To demonstrate the equality $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$, where f_n is n-th Fibonacci number also the formula from Corollary 1 has been applied. It was obtained using some relationships between Fibonacci and Lucas numbers, which were previously justified.

1. Introduction

Leonardo Fibonacci invented its sequence around 1200, dealing with the issue of rabbit population growth. Mathematicians began to discover more and more interesting properties of Fibonacci numbers being terms of its sequence. Édouard Lucas, creator of the Towers of Hanoi puzzle, conducted detailed research on these numbers in the second half of the 19th century. Lucas popularized the name of the Fibonacci numbers and he is also the creator of the sequence called his name. This sequence is an example of Fibonacci type sequence. It is worth adding that, using the property of Fibonacci numbers, he proved that the number of Mersenne's $2^{127} - 1$ is the prime number.

In this article we will present some basic properties of Fibonacci and Lucas numbers and the golden number. We will prove in a simple way compact formulas expressing Fibonacci numbers and the numbers of Lucas sequence using the golden number. We will also present facts about the convergence of the quotient of consecutive terms (n-th to $(n-1)$-th) of the Fibonacci type sequences to the

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golden number using a fixed point of a mapping $f(x) = 1 + \frac{1}{x}$ which is a contractive mapping of certain interval. An indirect effect of the fixed point of f can be seen in the case of the above convergence for the Fibonacci type sequence. In addition, we will discuss various methods of proving convergence to the golden number.

2. Examples of Fibonacci type sequences and the convergence of quotients of terms of the sequences mentioned

Definition 1
A golden section of the segment of length d is called a division into smaller sections of lengths x and $d - x$, in which

$$\frac{d}{x} = \frac{x}{d-x}.$$

By solving the quadratic equation $x^2 + dx - d^2 = 0$ resulting from the above equation we get $x = d\frac{\sqrt{5}-1}{2}$ and a golden proportion $\frac{d}{x}$, which is expressed in the golden number

$$\varphi = \frac{\sqrt{5}+1}{2} \approx 1,6180339887...$$

Definition 2
For a given rectangle with side lengths in the ratio $1 : x$, we will call the golden proportion of the only ratio $1 : \varphi$ at which the original rectangle can be divided into a square and a new rectangle which has the same ratio of sides $1 : \varphi$.

Definition 3
The golden rectangle is called a rectangle in which the ratio of the lenght of its sides is $1 : \varphi$.

Directly from this definition we get a quadratic equation

$$x^2 - x - 1 = 0,$$

and thus for $x > 1$ we have $x = \varphi$.

Let $\hat{\varphi} = \frac{-1}{\varphi} = \frac{1-\sqrt{5}}{2}$,then

$$x^2 - x - 1 = (x - \varphi)(x - \hat{\varphi}) = 0.$$

Therefore $\varphi^2 = \varphi + 1$ and $\hat{\varphi}^2 = \hat{\varphi} + 1$. Consequently,

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2},\ \hat{\varphi}^n = \hat{\varphi}^{n-1} + \hat{\varphi}^{n-2},\ n \geq 2. \tag{1}$$

Definition 4
Fibonacci sequence is a sequence defined recursively as follows:

$$f_1 = f_2 = 1, f_{n+1} = f_{n-1} + f_n, n \geq 2$$

(sometimes formally accepted $f_0 = 0$ and then the recursive formula is valid for $n \geq 1$).

Definition 5
Fibonacci numbers are called consecutive terms of the sequence (f_n).

Let us consider the following sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, ...$$

By recurrence
$$f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ n \geq 2. \tag{2}$$

We will now sketch the proof of the following formula:

$$f_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \hat{\varphi}^n\right),\ n \geq 0. \tag{3}$$

For $n = 0$ we get $f_0 = 0$. For $n = 1$ we get correctly $f_1 = \frac{1}{\sqrt{5}} (\varphi - \hat{\varphi}) = 1$. For the higher powers we use (1) and by induction we confirm validity of (3).

Remark 1
We will show how to get the formula (3) *and confirm its truth. Using the following formula:*
$$f_k = a \left(\frac{1+\sqrt{5}}{2}\right)^k + b \left(\frac{1-\sqrt{5}}{2}\right)^k$$

for $k = 0$ *and* $k = 1$ *we have:*

$$a + b = 0,\ a \left(\frac{1+\sqrt{5}}{2}\right) + b \left(\frac{1-\sqrt{5}}{2}\right) = 1,$$

thus
$$a = \frac{1}{\sqrt{5}}, b = \frac{-1}{\sqrt{5}},\ f_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).$$

Therefore we get $f_k = \frac{1}{\sqrt{5}} \left(\varphi^k - \hat{\varphi}^k\right)$.
We will now show that the obtained result is always true. For $k = 0, 1$ *we have* $f_0 = 0, f_1 = 1$. *Assuming that equality* $f_m = \frac{1}{\sqrt{5}} (\varphi^m - \hat{\varphi}^m)$ *is true for every* $m \leq k$ $(m \geq 0)$ *we show that* $f_{k+1} = \frac{1}{\sqrt{5}} \left(\varphi^{k+1} - \hat{\varphi}^{k+1}\right)$. *Based on formula (1) we have*
$$f_k + f_{k-1} = \frac{1}{\sqrt{5}} \left(\varphi^k - \hat{\varphi}^k\right) + \frac{1}{\sqrt{5}} \left(\varphi^{k-1} - \hat{\varphi}^{k-1}\right) =$$

$$= \frac{1}{\sqrt{5}} \left(\left(\varphi^k + \varphi^{k-1}\right) - \left(\hat{\varphi}^k + \hat{\varphi}^{k-1}\right)\right) = \frac{1}{\sqrt{5}} \left(\varphi^{k+1} - \hat{\varphi}^{k+1}\right) = f_{k+1}.$$

In this way we proved the truth of formula (3).

Remark 2
Since $|\frac{1-\sqrt{5}}{2}| < 1$, *we see that* $\lim_{k\to\infty} \hat{\varphi}^k = 0$ *and* f_k *is the nearest integer to* $\frac{1}{\sqrt{5}}\varphi^k$. *From Lawrynowicz i inni, 2018 we have* $\frac{\sqrt{5}-1}{2} = \sqrt{1 - \sqrt{1 - \sqrt{1 - \ldots}}}$

which is equivalent $\hat{\varphi} = -\sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}$. *The equation* $\hat{\varphi}^2 = 1 + \hat{\varphi}$ *produces a root* $\sqrt{1-\sqrt{1-\sqrt{1-\ldots}}} = \lim_{n\to\infty} A_n$ *(by definition), where sequence* (A_n) *is defined by* $A_1 = \sqrt{1-\frac{1}{\varphi}}, A_{n+1} = \sqrt{1-A_n}$ $(n \geqslant 1)$. *Because the sequence* (A_n) *is monotonic and bounded, so* $\lim_{n\to\infty} A_n = A$, *moreover* $A_{n+1}^2 = 1 - A_n$. *Therefore we get* $A^2 = 1 - A$, *and hence* $A = \frac{\sqrt{5}-1}{2}$. *To demonstrate the equality* $\frac{\sqrt{5}-1}{2} = \sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}$ (without proof in Lawrynowicz i inni, 2018), *we raise both sides of equality* $A = \sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}$ *to square. We get* $A^2 = 1-\sqrt{1-\sqrt{1-\ldots}} = 1 - A$, *so we have equality* $A^2 + A - 1 = 0$, *hence* $A = \frac{\sqrt{5}-1}{2}$. *Also from* $\varphi^2 = 1 + \varphi$ *we can get* $\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$ *in a similar way; here* $A_1 = \sqrt{1}, A_{n+1} = \sqrt{1+A_n}$.
We have more:

$$\hat{\varphi} + \hat{\varphi}^2 + \hat{\varphi}^3 + \ldots = -\hat{\varphi}^2,$$
$$-\hat{\varphi} + \hat{\varphi}^2 - \hat{\varphi}^3 + \ldots = \varphi.$$

Definition 6
A sequence (F_n) *of the form* $F_{n+1} = F_n + F_{n-1}, n \geq 2$, *where* F_1 *and* F_2 *are given positive integers we call a Fibonacci type sequence.*

For example, this sequence is the so-called Lucas sequence (l_n):

$$1, 3, 4, 7, 11, 18, 29, \ldots$$

These numbers can be described by a formula

$$l_1 = 1,\ l_2 = 3,\ l_{n+1} = l_n + l_{n-1}, n \geq 2.$$

By using induction, we notice that

$$l_{n+1} = f_n + f_{n+2} \tag{4}$$

Now using the formula (4) we get

$$f_{n+1} = \frac{f_n + l_n}{2} \tag{5a}$$

and hence the generalization using recursive formulas for (f_n) and (l_n) in induction relative to m

$$f_{n+m} = \frac{1}{2}(f_m l_n + l_m f_n) \tag{5b}$$

Using (4) we get from here

$$f_{n+m} = f_m f_{n+1} + f_{m-1} f_n. \tag{5c}$$

Based on formula (5c) we get by putting $m = n$ equality $f_{2n} = f_n f_{n+1} + f_{n-1} f_n = (f_{n+1} + f_{n-1}) f_n = l_n f_n$. Finally we have

$$f_{2n} = l_n f_n \ (n \geq 1). \tag{5d}$$

Remark 3
*Based on the formula (*5c) assuming $m = n+1$ you can get equality $f_{2n+1} = f_n^2 + f_{n+1}^2$ $(n \geq 1)$, while using (5d) we get $f_{2n} = f_n(2f_{n-1} + f_n)$ $(n \geq 1)$. Let's add that both obtained formulas are useful when finding a specific term in a Fibonacci sequence using a calculator or a computer.

By creating a sequence of proportions (x_n), where $x_n = \frac{f_{n+1}}{f_n}$, we get

$$x_n = 1 + \frac{1}{x_{n-1}} \tag{6}$$

and consequently assuming that this sequence is convergent

$$g = \lim_{n\to\infty} x_n = 1 + \frac{1}{\lim_{n\to\infty} x_{n-1}} = 1 + \frac{1}{g}.$$

From identity $f_{n+1} = \varphi f_n + \hat{\varphi}^n$, which will be presented in Corollary 1, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \left(\varphi + \frac{\hat{\varphi}^n}{f_n}\right) = \varphi$$

because $\lim_{n\to\infty} \frac{\hat{\varphi}^n}{f_n} = 0$, so this sequence is convergent. Moreover, $g = \varphi$.

An other proof of convergence of the sequence (x_n) will be shown further in the proof of Theorem 2(a).

The following Edelstein's theorem will be useful

Theorem 1
(Goebel, 2005, Edelstein, 1962) *Let X be a compact metric space and let $f : X \to X$ be a contractive mapping, that is $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ in X. Then f has a unique fixed point. Further, for any $x \in X$, the iterative sequence $(f^n(x))$ converges to the fixed point.*

Let $C = \langle 1, 2\varphi - 1\rangle$. The map f specified on C by a formula $f(x) = 1 + \frac{1}{x}$ meets the conditions:

1° $f(C) \subset C$, bacause f is decreasing and $f(2\varphi - 1) = 1 + \frac{1}{2\varphi-1} = \frac{2\varphi}{2\varphi-1} > 1$ and $f(1) = 2 < 2\varphi - 1$;

2° for each $x, x' \in C |f(x) - f(x')| = \left|\frac{1}{x} - \frac{1}{x'}\right| = \frac{|x-x'|}{xx'} < |x - x'|$, because $x > x' \geqslant 1$ either $x' > x \geqslant 1$.

Therefore using Theorem 1 we can get the following

Lemma 1
Let $C = \langle 1, 2\varphi - 1\rangle$. The mapping $f : C \to C$, $f(x) = 1 + \frac{1}{x}$, has a unique fixed point u, and $\lim_{n\to\infty} x_n = u = \varphi$.

Dowód. It is easy to see that the sequence (x_n), $x_n = \frac{f_{n+1}}{f_n} = 1+\frac{1}{x_{n-1}}$ $(n \geqslant 1)$, starting from $x_0 = \frac{f_2}{f_1} \in C$ can be described by the formula $x_n = f^n(x_0)$. Because $f : C \to C$ is a concractive mapping (from the condition 2°), so by Edelstein's theorem (Theorem 1) f has a unique fixed point $u \in C$. Because $u = 1 + \frac{1}{u}$, so $u = \varphi$. Further the iterative sequence $(f^n(x_0))$ converges to u. We finally have $\lim_{n\to\infty} x_n = u = \varphi$.

Remark 4

The result: $\hat{x} = \varphi = \lim_{n\to\infty} x_n$ *can also be obtained by using the Darboux property and knowing that* f, $f(x) = 1 + \frac{1}{x}$, *is a continuous mapping of the interval* $C =< 1, 2\varphi - 1 >$ *into itself.*
Indeed, let $g(x) = x - f(x)$. *Then*

$$g(1) = 1 - f(1) = 1 - 2 < 0,$$

$$g(2\varphi - 1) = 2\varphi - 1 - f(2\varphi - 1) = 2\varphi - 1 - 1 - \frac{1}{2\varphi - 1} =$$

$$= 2\varphi - 2 - \frac{1}{2\varphi - 1} > 2(\varphi - 1) - \frac{1}{2(\varphi - 1)} > 0.$$

So there is a point $\hat{x} \in C$, *that* $g(\hat{x}) = 0$. *It means that* $\hat{x} - \frac{1}{\hat{x}} - 1 = 0$, *therefore*

$$\hat{x} = \varphi,$$

because f *is a decreasing map.*
The rest of the proof is obvious because the iteration method for the equation $g(x) = 0$ *is always convergent, independently of the choice of point* $x_o \in C$, *and the half dividing method immediately gives the root of this equation.*

Consider the sequence (b_n), where $b_n = 2f_{n+1}$; this is sequence:

$$2, 4, 6, 10, \ldots$$

The corresponding sequence of proportions (p_n) has a form

$$p_n = \frac{b_{n+1}}{b_n} = \frac{2f_{n+2}}{2f_{n+1}} = \frac{f_{n+2}}{f_{n+1}} = x_{n+1}. \tag{7}$$

For Lucas sequence (l_n) we have a sequence of proportions (q_n) defined by equality $q_n = \frac{l_{n+1}}{l_n}$. Because $b_n = f_n + l_n$ (from (5a)), so $l_n = b_n - f_n$. Therefore

$$q_n = \frac{b_{n+1} - f_{n+1}}{b_n - f_n}. \tag{8}$$

Theorem 2

The following equality occurs:

(a) $\lim_{n\to\infty} x_n = \varphi$,

(b) $\lim_{n\to\infty} p_n = \varphi$,

(c) $\lim_{n\to\infty} q_n = \varphi$.

Dowód. (a) Other proof of this point (based on Corollary 1) has been provided earlier. We can also get it by showing the monotonicity and boundedness of the subsequences (x_{2n}) and (x_{2n+1}) of the sequence (x_n), which will cause the convergence of the sequence (x_n) to the limit equal to the limit of both subsequences, and as a consequence we will get the limit of the sequence (x_n) equal to φ (see Foryś, 2014). Instead, we will use Lemma 1. Under this Lemma we get $\lim_{n\to\infty} x_n = \varphi$.

(b) results from (a).

(c) We already know that the sequence of quotients $\frac{l_{n+1}}{l_n}$, i.e. the sequence (q_n) is specified by $q_n = \frac{b_{n+1}-f_{n+1}}{b_n-f_n} = \frac{\frac{b_{n+1}-f_{n+1}}{f_n}}{\frac{b_n-f_n}{f_n}}$.
We have the following equality in the case of the numerator $\frac{b_{n+1}-f_{n+1}}{f_n} = \frac{2f_{n+2}-f_{n+1}}{f_n} = \frac{\frac{2f_{n+2}}{f_{n+1}}}{\frac{f_n}{f_{n+1}}} - \frac{f_{n+1}}{f_n}$, so $\frac{b_{n+1}-f_{n+1}}{f_n} \to 2\varphi^2 - \varphi$ at $n \to \infty$.

In the case of the denominator we have $\frac{b_n-f_n}{f_n} = \frac{b_n}{f_n} - 1 = \frac{2f_{n+1}}{f_n} - 1 \to 2\varphi - 1$ at $n \to \infty$.
Therefore $\lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{l_{n+1}}{l_n} = \frac{2\varphi^2-\varphi}{2\varphi-1} = \varphi$.

Theorem 3
$l_n = \varphi^n + \hat{\varphi}^n, n \geq 1$, where l_n is an n-th Lucas number given by $l_n = f_{n-1} + f_{n+1}$.

Dowód. (Version 1) Based on (5d) and (3) we get

$$l_n = \frac{f_{2n}}{f_n} = f_{2n}\frac{\sqrt{5}}{\varphi^n - \hat{\varphi}^n} = f_{2n}\frac{\varphi^n + \hat{\varphi}^n}{\varphi^{2n} - \hat{\varphi}^{2n}}\sqrt{5} = (\varphi^n + \hat{\varphi}^n)\frac{f_{2n}}{f_{2n}} = \varphi^n + \hat{\varphi}^n.$$

(Version 2) We can prove this theorem using the following formula $l_n = \alpha\varphi^n + \beta\hat{\varphi}^n$ and based on the recursive definition of the sequence (l_n): $l_0 = 2, l_1 = 1, l_n = l_{n-1} + l_{n-2}$ for $n > 1$. We have a system of equations

$$\begin{cases} \alpha + \beta = 2 \\ \alpha\varphi + \beta\hat{\varphi} = 1 \end{cases}.$$

Thus $\beta = 2 - \alpha$ and

$$\alpha\varphi + (2-\alpha)\hat{\varphi} = \alpha(\varphi - \hat{\varphi}) + 2\hat{\varphi} = 1,$$

from this

$$\alpha = \frac{1-2\hat{\varphi}}{\sqrt{5}} = 1, \beta = 1.$$

Therefore

$$l_n = \varphi^n + \hat{\varphi}^n.$$

We will now show that obtained result is always true.
For $n = 0, 1$ we obviously have $l_0 = 2, l_1 = \varphi + \hat{\varphi} = 1$.
Assuming that $l_{n-1} = \varphi^{n-1} + \hat{\varphi}^{n-1}, l_{n-2} = \varphi^{n-2} + \hat{\varphi}^{n-2}$ $(n > 1)$ we get by (1) $l_{n-1} + l_{n-2} = \left(\varphi^{n-1} + \varphi^{n-2}\right) + \left(\hat{\varphi}^{n-1} + \hat{\varphi}^{n-2}\right) = \varphi^n + \hat{\varphi}^n = l_n$.

Lemma 2

$$\varphi^n = \varphi f_n + f_{n-1}, n \geq 1.$$

Dowód. For $n = 1$, we get correctly $\varphi = f_1\varphi + f_0$.
Assuming the equality $\varphi^{n-1} = \varphi f_{n-1} + f_{n-2}$ $(n \geq 2)$ and multiplying both sides of this equality by φ we get

$$\varphi^n = \varphi^2 f_{n-1} + \varphi f_{n-2} = (\varphi + 1) f_{n-1} + \varphi f_{n-2} = \varphi f_n + f_{n-1}.$$

Corollary 1

$$f_{n+1} = \varphi f_n + \hat{\varphi}^n, n \geq 0.$$

Dowód. From Lemma 2 we have $\varphi f_n = \varphi^n - f_{n-1}$ for $n \geq 1$. Based on Theorem 3 we have

$$\varphi^n - f_{n-1} = f_{n+1} - \hat{\varphi}^n,$$

so

$$\varphi f_n = f_{n+1} - \hat{\varphi}^n.$$

For $n = 0$ we get

$$f_1 = 1 = \varphi f_0 + \hat{\varphi}^0.$$

Finally

$$f_{n+1} = \varphi f_n + \hat{\varphi}^n.$$

Remark 5
For three sequences $(f_n), (b_n)$ and (l_n), it can be shown that the quotients of neighboring terms tend to number φ finding a unique fixed point of the function $f(x) = 1 + \frac{1}{x}$ specified on the interval $< 1, 2\varphi - 1 >$ and applying Lemma 1 to the appropriate quotients. For example, for Lucas' sequence we have

$$\frac{l_{n+1}}{l_n} = \frac{l_n + l_{n-1}}{l_n} = 1 + \frac{l_{n-1}}{l_n}, \textit{ hence } q_n = 1 + \frac{1}{q_{n-1}}.$$

We get $q_n = f(q_{n-1})$, where $f(q) = 1 + \frac{1}{q}$.

Remark 6
For three sequences $(f_n), (b_n)$ and (l_n), which are Fibonacci type sequences, we can show the convergence of quotients $(x_n), (p_n)$ and (q_n) using a compact form of n-th terms of this type sequences.

For example for the sequence (b_n) , $b_n = \frac{2}{\sqrt{5}}(\varphi^{n+1} - \hat{\varphi}^{n+1})$ $(n \geq 1)$, we have $\frac{b_{n+1}}{b_n} = \frac{\frac{2}{\sqrt{5}}(\varphi^{n+2} - \hat{\varphi}^{n+2})}{\frac{2}{\sqrt{5}}(\varphi^{n+1} - \hat{\varphi}^{n+1})} = \frac{\varphi - \hat{\varphi}(\frac{\hat{\varphi}}{\varphi})^{n+1}}{1 - (\frac{\hat{\varphi}}{\varphi})^{n+1}} \to \varphi$ (as $n \to \infty$), bacause $|\frac{\hat{\varphi}}{\varphi}| = \frac{1}{\varphi^2} < 1$. Therefore $\lim_{n\to\infty} p_n = \varphi$.

Now we will present the relationship between the number φ and dependence (6) using the concept of a continued fraction for the golden ratio φ. We know that $\varphi = 1 + \frac{1}{\varphi}$. Therefore in the continued fraction $[a_0; a_1, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$ all the quotients $a_0, a_1, a_2, \dots$ are equal to 1. Thus the number φ represents the continued fraction $[1; 1, 1, \dots]$. Based on the continued fraction corresponding to φ we can also give the form of the continued fraction for $\frac{1}{\varphi}$:

$$\frac{1}{\varphi} = [0; 1, 1, \dots].$$

Taking into account that moving one space does not change the structure of this fraction, we get a recursive relationship

$$x_n = 1 + \tfrac{1}{x_{n-1}} \text{ with } x_0 = 1.$$

Therefore $\lim_{n\to\infty} x_n = \varphi$.

Now we will present examples of Fibonacci type sequence in conjunction with the golden number.

Let us consider the sequence (t_n), where $t_n = f_{n-1} + f_{n+3}$; this is sequence $3, 6, 9, \dots$ It is easy to see that $f_{n-2} + f_{n+2} = 3f_n$ $(n \geq 2)$. Indeed $f_{n-2} + f_{n+2} = f_n - f_{n-1} + f_{n+1} + f_n = 3f_n$.
Now for $r_n = \frac{t_{n+1}}{t_n}$ we have $r_n = \frac{3f_{n+2}}{3f_{n+1}}$ and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{f_{n+2}}{f_{n+1}} = \varphi$.

The sequences $(f_n), (b_n)$ and (t_n) are special cases of the sequence (g_n), where $g_n = kf_{n+1}$ (k is a fixed positive integer).

For $s_n = \frac{g_{n+1}}{g_n}$ we have $s_n = \frac{f_{n+2}}{f_{n+1}} \to \varphi$ at $n \to \infty$.

Of course, (g_n) is Fibonacci type sequence and $\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \varphi$.

The above results can be generalized in the form of

Theorem 4

$$\lim_{n\to\infty} X_n = \varphi,$$

where

$$X_n = \frac{F_{n+1}}{F_n}, n \geq 1.$$

Dowód. Let's assume that $F_1 = A$ and $F_2 = B$, where A and B are positive integers. Based on the definition on the Fibonacci type sequence we have: $F_3 = A + B,\ F_4 = A + 2B,\ F_5 = 2A + 3B, \dots$ Given the definition on the Fibonacci sequence we have from here:

$$F_3 = Af_1 + Bf_2,\ F_4 = Af_2 + Bf_3,\ F_5 = Af_3 + Bf_4, \dots$$

Using the recursive formula $F_{n+1} = F_n + F_{n-1}$ $(n \geq 2), F_1 = A, F_2 = B$, is easy to show that

$$F_{n+1} = Af_{n-1} + Bf_n.$$

Therefore

$$X_n = \frac{Af_{n-1} + Bf_n}{Af_{n-2} + Bf_{n-1}} = \frac{A + Bx_{n-1}}{\frac{A}{x_{n-2}} + B} \to \frac{A + B\varphi}{\frac{A}{\varphi} + B} \ (n \to \infty)$$

by Theorem 2 (a). Because $\varphi = \frac{\sqrt{5}+1}{2}$, so

$$\frac{A+B\varphi}{\frac{A}{\varphi}+B} = \frac{A+B\frac{\sqrt{5}+1}{2}}{\frac{2A}{\sqrt{5}+1}+B} = \frac{2A+B\sqrt{5}+B}{2} \cdot \frac{\sqrt{5}+1}{2A+B\sqrt{5}+B} = \frac{\sqrt{5}+1}{2}.$$

Finally

$$\lim_{n\to\infty} X_n = \varphi.$$

Remark 7
Convergence to the number φ of the iterative sequence (X_n) given by the formula $X_n = f^n(X_0)$, $X_0 = \frac{F_3}{F_2} > 1$, also we get based on a slightly modified Lemma 1, when instead of C we take $K = \langle 1, M \rangle, M \geqslant 2$, and the mapping f is still defined by the same formula as in Lemma 1, while $f(K) \subset K$ (because $f(M) > 1$ and $f(1) = 2 \leqslant M$). The proof of this version of the Lemma 1 is still based on Edelstein's theorem.

Remark 8
(*LRE*,n.d.) *A generalized version of Fibonacci numbers has recurrence*

$$G_n = G_{n-1} + G_{n-2}$$

with $G_1 = a$ and $G_2 = b$, where $a, b > 0$ has solution by $G_n = \frac{1}{2}[(3a-b)f_n + (b-a)l_n]$.

Of course, every Fibonacci type sequence is included in the generalized version of Fibonacci numbers.

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University of Warmia and Mazury
Faculty of Mathematics and Computer Science
Chair of Complex Analysis
Słoneczna 54 Street, 10-710 Olsztyn
Poland
e-mail: ebarcz@matman.uwm.edu.pl