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The role of switching over the visual structures in solving mathematical problems – eye-tracking study[*]

Abstract. The article concerns the research on students' perception of visual structures in van Hiele's sense (1986). The empirical studies described in the paper make use of a combined methodology – eye-tracking and a written study questionnaire. There was an analysis of the results of 14 pupils in the paper the 1st grade of a middle school and 19 pupils from a secondary school. The subjects were shown slide on a computer screen with task and were requested to solve them. Further eye-tracking parameters suggest that the ability of switching over from one structure to another, as described by van Hiele (1986), between the geometric and arithmetic structures of the presented objects guarantees success in solving the task. Constructing various structures is not only fundamental for solving problems, but also for the sake of the development of many mathematical concepts and their properties in relation to mathematics, as well as in the context of developing other key competences.

1. Introduction

The research presented in the article belong to the current cognitive research connected with searching and explaining the functioning of mind. Particularly important will be the characteristics of solving a certain mathematical task that requires structurisation of an object shown in the figure. The characteristics of the cognitive processes of the tested subjects was based on the eye-tracking method which enables to follow the eyeball movement while solving a certain mathematical task, and then to draw conclusions regarding particular aspects of the brain activity typical for mathematical thinking.

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2. Structurisation of visual objects

The educators of mathematics, Krygowska (1979) and Freudenthal (1987) have emphasised repeatedly that discovering structures helps us understand better the organisation of our knowledge and plays a crucial role in creating mathematical concepts.

Considering the concepts and mathematical facts it can be said that they include the structures whose comprehending means deliberate abstracting them from the diversity of other structures by taking into account the chosen relations among the elements of this concept and by neglecting unnecessary information.

Van Hiele in his thesis (1986) expresses very wide view over the structures accompanying humans. However, he does not make an attempt to define them, he tries to characterize them by citing examples and drawing attention to their certain qualities.

He begins his considerations with visual structures whose properties are visible most clearly.

> The first thing to be observed in these structures is that they can be extended. It is very easy to extend a structure of squares: If you give a young child a set of square tiles, he will in most cases be able to extend the pattern. So if you want to start geometry with a well-known structure and the child is no younger than 6 years, you can always begin with the structure of square-grid paper (van Hiele, 1986, p. 13)

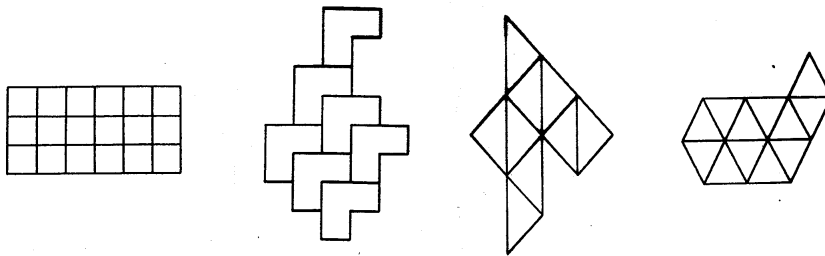Examples of such visual structures include the following (Van Hiele, 1986) (Fig. 1):

Fig. 1. Visual structures – examples

Van Hiele points to the fact that:

> The construction of visual structures in our mind very often does not need a language. You can ask a child to continue a given structure (as in the preceding pictures) and he or she will succeed without having learned a language that accompanies the structure (van Hiele, 1986, p. 16–17)

The article will present a part of a bigger research devoted to the perception of visual structures by pupils on different levels of mathematical experience. The aim of the research was an introductory characteristics of the abilities of switching over the structures that will enable finding the correct solution to the research task. The research described here had been carried out with the eye-tracking method.

3. Switching over from one structure to another in van Hiele's sense

Van Hiele (1986) points to the necessity of changing perception of a structure of a certain object in a situation when the structure does not function and cannot help to find a solution. The author explains the idea of switching from one visual structure to another while solving new problems with the following mathematical tasks, shown in in examples 1 and 2:

Task 1. *A square is given.* ☐ *Draw a square with the area two times bigger.* The person, for whom the situation is new, often starts drawing a square with the sides twice as big as the sides of the given square (Fig. 2 – upper left). However, it can be easily seen that in such case the area of the new square is four times bigger than that of the starting square. Having noticed that, the person starts looking for another solution: *Perhaps the sides of the wanted square should be one and a half times longer than the starting side?*

The idea is not bad but in the Figure 2 (upper right) we can see that the area of the newly created square is 2.25 times bigger than the area of the starting square. Now the person can split "the square structure" and spot another, new "triangular structure": they can notice that a square can be divided with a diagonal into "triangular" halves. In this completely new structure, four triangles can form a new square and reveal the solution to the problem. (Fig. 2 – down).
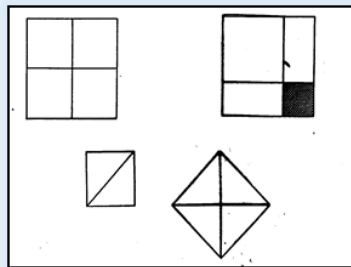

Fig. 2. Searching for solution to Task 1

Oakley (2015) describes the solution of a similar problem, pointing at two states of mind (two modes): the state of concentration and the state of distraction which allow to find a new approach that brings the expected result. The first task is connected with the arrangement of a square out of two congruent right-angled triangles. The second task involves forming a square out of four such triangles. In the second task, while focused, we tend to implement the previous strategy. As a result we can build two squares which does not lead us to the solution. It is only after the switching into the distraction mode and searching for another approach that allows us to build a square out of four triangles, such as in van Hiele's example.

Task 2. *Draw 4 lines with a solid line (without taking the pen off it) to pass through all 9 dots in the drawing (Fig. 3a)*
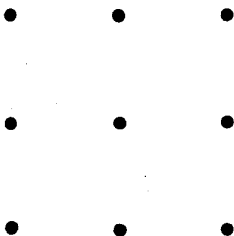
Fig. 3. a) Task 2

The solution to this task often does not come at once.

> The difficulty consists in the nine points forming a square and the taboo
> against leaving that square. If we can ignore this taboo, we find the solution
> (van Hiele, 1986, p. 13).

Most frequently in such an arrangement of dots the person solving the problem in
the first instance notices the square structure. Unfortunately the square structure
blocks out the finding of the correct solution. The ability to ignore and abandon
the square structure can be helpful in solving the problem. If one manages to
switch to a different structure beyond the square, such a solution, as in the figure
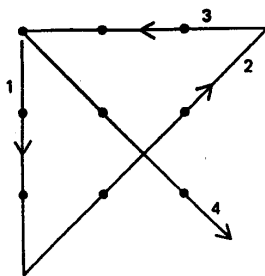3b, can be found:

Fig. 3. b) Solution to Task 2

Switching from one structure to another, a more appropriate one, is not an easy
task. The first structure is chosen because of its encouraging character, because
of the knowledge that this structure is proper for solving many problems. The
necessity to switch or change can be comprehended only after having seen that
the chosen structure does not lead to the required solution.

This mental process, described by van Hiele, leading to the solution of the
task that involves switching over the structures marked in the object, is connected
with the idea known in psychology as "the insight" (or the introspection). Insight

is defined as a sudden change in problem perception. This classic definition contains two elements: experience of suddenness of a solution and radical change in problem perception (Nęcka, 2012, p. 104). Max Wertheimer (1945) described the idea of the insight using the example of a girl calculating the area of a rhomboid (parallelogram). The girl was shown a rhomboid and asked to calculate its area. The method of adding up single squares previously used while calculating the area of a rectangle, failed. After a few unsuccessful attempts the girl asked for a pair of scissors. Then she cut off a triangle from the right side of the rhomboid and placed it on the left side thus forming a rectangle. The problem of the rhomboid before the solution was obviously "the wrong figure" and required the change of its structure. The solution was found suddenly although it came after a short break and it indicates understanding of the problem by the girl. Another example of an issue that required the insight described by psychologists is "the problem of ten coins" (Isaac, Just, 1995). The task involves "reversing the triangle" i.e. converting model A into model B provided that only three coins are moved (fig. based on Nęcka, 2012, p. 107). Contemporary research are heading for discovering cognitive mechanisms due to which insight, understood as the ability to perceive various structures in visual objects and switching between them, would be possible. Such theoretical considerations formed the basis for applying exploratory methodology connected with the observation of an eyeball while solving a certain task.

4. Eye-tracking research methodology

Eye-tracking is being successfully used in many fields of study for analysis of visual behaviours concerning solving cognitive tasks. "Eye movements reflect both visual properties of the world and human thought processes, balancing between its perception and recognition. Eye movements can become for the scientists a source of data about the dynamics of psychological processes which lead to giving an answer, performing an action or making decisions" (Pilipczuk, 2014).

Eye-tracking studies are a collection of research techniques which enable following eyeball movement – that is, changes of their location over a defined period of time (Nowakowska-Buryła, Joński, 2012). Present knowledge which concerns functioning of the optic system proves unquestionably that eye movement reflects brain action. It can be stated that an eye is a "single-image" system which means that it can recognise one image at a time and then moves to another one. Such intensive eye movements are saccades i.e. fast, rapid, saccadic eyeball movements connected with *shifting sight to other points of visual surroundings* (Demidow, 1989, p. 72). During its roaming an eye stops for brief moments at certain elements of an image. That is so called fixation which is used to describe a relative rest of an eye connected with looking at a certain direction. However, it does not mean that we do not see this part of an image at which our sight did not stop. Brain reproduces and completes these parts using millions of images that had passed in front of our eyes and were stored in our memory. The larger is the store of our resources of sight impressions, the more intense are new impressions encountered by an eye and the greater are our vision possibilities.

In view of the above it seems that following the eyeball movements while solving mathematical tasks can bring answers to numerous interesting research questions.

A summary of up-to-date eye-tracking research from the years 2000-2012 was presented in the paper of the Taiwanese researchers (Lai et al., 2013). The authors point at the wide possibilities of application of the eye tracker in research concerning analysis of the text reading process, visual perception and methods of problem solving or reasoning processes. Eye-tracking research enables deeper understanding of conceptual development and many cognitive and behavioral aspects in humans.

Surveys using the mentioned method allow to raise new research questions also in the area of educational studies (Stolińska et al., 2014; Błasiak et al., 2015; Andrzejewska et al., 2016).

A fresh look onto cognitive processes can be obtained owing to the eye tracking method also in the field of mathematical didactics research. Schwank (2001, 2003) has determined two types of mathematical thinking: functional thinking and predicative thinking. The author, analysing the movement of the eyeballs of the tested subjects, has noticed and differentiated typical elements of eyeball movement for each of these types of thinking – tested persons compare the different elements using functional thinking; while using predicative thinking they compare similar elements.

In previous papers (Rożek, 2014; Rożek et al., 2015) there has been described the process of solving mathematical test tasks of single choice in which major part of crucial task conditions was included in the figure.

5. Methodology and methods

The aim of the empirical research was the characteristics of the eye-tracker parameters leading to the solution of the mathematical task connected with the perception of the visual structures. In the context of the established research goal the following research questions have been formulated:

- Which ways of structurising are used by the subjects while solving a certain task? – the analysis of the maps and sight paths of the tested pupils
- Which eye-tracking parameters are characteristic to the text reading and which ones to the figure analysis?
- Which elements of the process of solving the research task are crucial for discovering the correct answer?

The research method was eye-tracking. Recording the eyeball movement was performed with the use of a fixed eye tracker manufactured by Senso Motoric Instruments, model iView X Hi-Speed 1250. Analysing the obtained data was possible with the Be Gaze software. The participants were 14 middle school (MSch) and 19 secondary school (SSch) pupils. The participants were shown a variety of problems

on a computer screen and were requested to solve them. Slide 1 (Fig. 4) contained textual instruction, a drawing and five number answers, out of which only one was correct.

Slide 1: *How many triangles can be seen in this figure?*
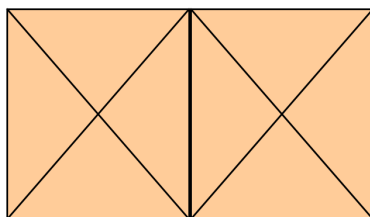
A. 8   B. 10   C. 12   D. 16   E. 18

Fig. 4. Task "about triangles"

6.   Results and discussion of the results

The correct solution to the task "about triangles" is the answer: "it can be seen maximum 18 triangles", i.e. the answer E. Counting the triangles might have proceeded as follows: to begin with counting the 8 "separable" triangles, then, as a result of switching over to another perception, there could be seen a new structure of 8 triangles formed by the halves of the squares marked out by their diagonals. Finally, the next switch to yet another structure allowed to notice 2 triangles more, marked out by the longer side of the rectangle and perpendicular to each other diagonals of the 2 squares. The correct answer to this task was given by 27% of middle school pupils (Fig. 5) and 58% of secondary school pupils (Fig. 6). The incorrect answers were given by 73% of middle school pupils: 4 pupils pointed to answer A, 2 pupils – to answer C and 5 pupils – to answer D. The secondary school pupils, choosing the incorrect answer pointed only to the answer D which made 42% of all given answers.
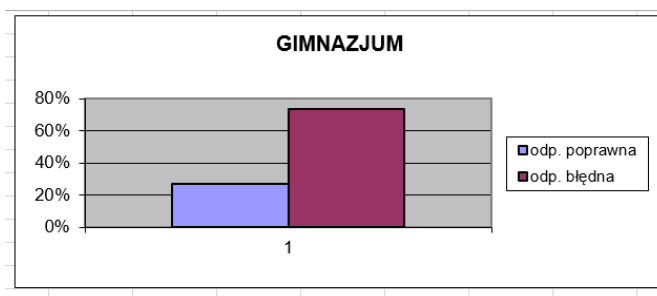

Fig. 5. Results of the solutions in Middle School (MSch)
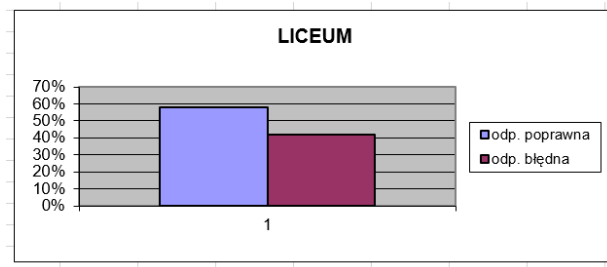(correct answer – incorrect answer)

Fig. 6. Results of the solutions in Secondary School (SSch)
(correct answer – incorrect answer)

Heat maps (Fig. 7) – show graphically a thermal arrangement of attention directed towards an inspected element distinguishing elements both noticed and ignored during the sight scan. Thus, such a map displays, using colours, the total attention focus of a subject directed onto a particular element. Then the brighter the area on the map, the more attention was given to it by a subject. On a colour map (Heat Map), the redder (warmer) the area, the greater the attention focus on this element by a subject.
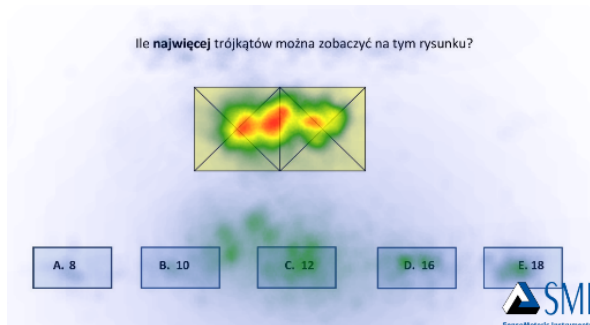

Fig. 7. Heat maps all

The Gaze plot (Fig. 8, Fig. 9) – shows which elements of the picture displayed on the screen were being watched by the tested person. **Saccades** are seen as continuous lines while **fixations** as circles of different diameters. The bigger the diameter, the longer the fixation on a given element.
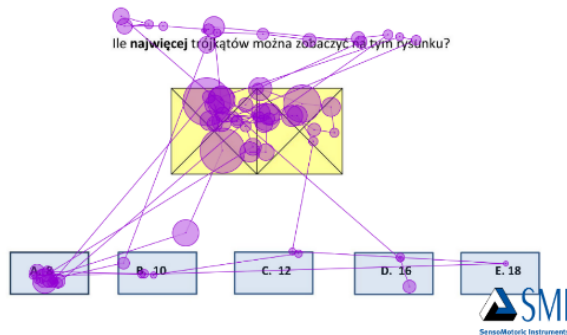

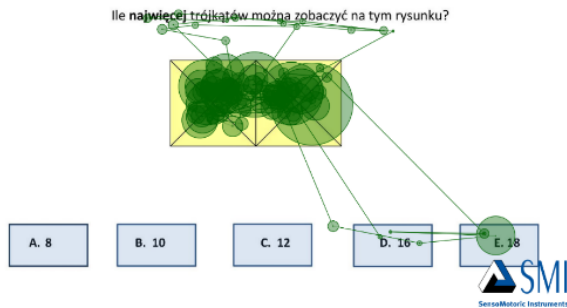**Fig. 8.** Pupil P23: incorrect answer A

Fig. 9. Pupil P10: correct answer E

There have been analysed various eye tracking parameters recorded with the use of the eye tracking method.

Defining the Areas of Interest (AOI) allowed to generate the results as the Key Performance Indicators (KPI) (Fig.10):

AOI 001 – **Text** (light pink colour)
AOI 002 – **Drawing** (green colour)
AOI 003 – **Answer boxes** (dark pink colour)
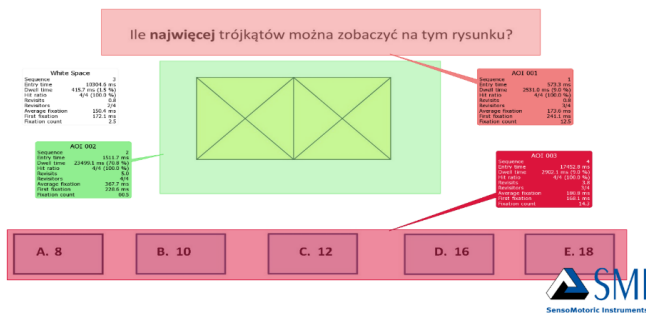
Fig. 10. KPI for the correct answer E (Middle school)

For every defined area of interest statistical results are presented in the tables for the following groups:

- Middle school pupils giving:
  - correct answer E (marked as MSch correct)
  - incorrect answers, i.e. answers A, C, D (marked as MSch incorrect)
- Secondary School pupils giving:
  - correct answer E (marked as SSch correct)
  - incorrect answer D (marked as SSch incorrect)

Tables 1–3 based on generated statistical data in KPI (Key Performance Indicators) present various data obtained by designated groups:

Table 1. Statement of data based on KPI for the area AOI 001 – Text

	MSch correct	MSch incorrect	SSch correct	SSch incorrect
Entry time [ms]	573,3	576,8	614,6	729,7
Dwell time [ms]	2568	2582,2	2928,6	1323,5
Dwell time [%]	9%	10,50%	8,50%	5,20%
Average fixation [ms]	172,6	197,9	199,6	139,2
First fixation [ms]	241,1	147,4	134,8	117,8
Fixation count	12,8	10,7	12,9	6,3

Table 1 concerning the text area AOI 001 proves that almost all tested groups (first three designated groups) obtained comparable parameters in dwell time, average fixation and fixation count. However, it is surprising to see much lower parameters for the fourth group – secondary school pupils pointing the answer D.

Table 2. Statement of data based on KPI for the area AOI 002 – Drawing

	MSch correct	MSch incorrect	SSch correct	SSch incorrect
Entry time [ms]	1511,7	1746,6	992,2	4232,1
Dwell time [ms]	23499,1	19005,3	20309,1	17183,3
Dwell time [%]	70,80%	56,90%	66,30%	64,30%
Average fixation [ms]	367,7	368,4	399	429
First fixation [ms]	228,6	150,7	157,5	183,3
Fixation count	60,5	44,8	49,4	37,5

An interesting conclusion coming out of the statements in table 2 is the fact that all the groups analysing the figure spent in percentage terms comparable amount of all time of the solution (between 60% to 70%). However, the differences in the registered dwell time in the area AOI 002 clearly indicate some significant differences. This time was much longer for the group of correct answers than in the group of the incorrect answers (both for the middle school and secondary school pupils). It is possible that longer time of processing data included in the figure was favourable to passing from one structure to another and enhanced chances to find out the correct solution.

Table 3. Statement of data based on KPI for the area AOI 003 – *Answer boxes*

	MSch correct	MSch incorrect	SSch correct	SSch incorrect
Entry time [ms]	17452,8	9310,7	8846,6	11142,4
Dwell time [ms]	2902,1	3398,8	2828,8	6078,9
Dwell time [%]	9,00%	13,80%	9,50%	14,30%
Average fixation [ms]	180,8	194	172,5	187,9
First fixation [ms]	168,1	153,3	138,8	152,8
Fixation count	14,3	15,4	13,8	20,5

The results shown in the table 3, concerning AOI 003 – Answers, are a little surprising. Percentage dwell time in this area for the group of the incorrect answers (circa 14% – for both middle school and secondary school pupils) turned out to be higher than for the group of the correct answers (circa 9% – for both middle school and secondary school pupils) – then the dependence is opposite to the area AOI 002 with the figure. These reversed dependences hold true also for the area AOI 003 regarding dwell time in ms. (milliseconds). Analysing the research results the researchers were looking for eye tracking parameters typical for the two groups: the group of pupils giving wrong answers and the group of pupils solving the

task correctly. Among the subjects there have been chosen several pairs of pupils complying with the requirements:

- one of the pupils gave the wrong answer A or C, the other chose the correct answer E;
- both pupils in the pair spent similar amount of time solving the task;

For such selected pairs a Sequence Chart has been generated. Along a horizontal axis was shown the place where the pupil's sight stayed within a given time unit [ms] on a corresponding area of interest – Areas of Interest (AOI 001: Text – light pink colour, AOI 002: Drawing – green colour; AOI 003: Answers – dark pink colour and white colour for the sight staying on a place beyond the defined areas.) In every of the three figures the upper bar was generated for the pupil giving the wrong answer while the lower bar relates to the correct answer.
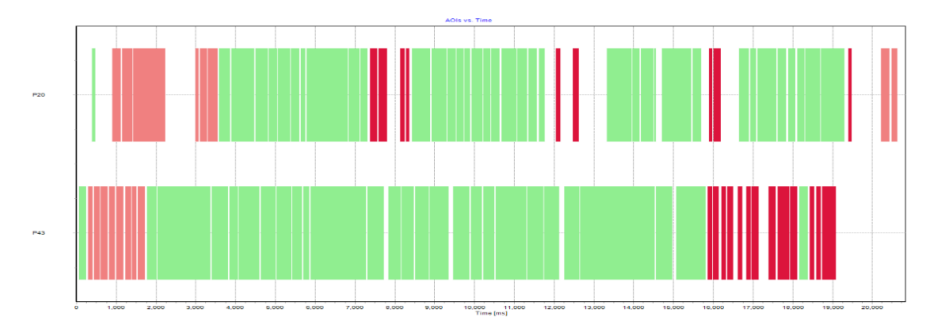

Fig. 11. Sequence Chart for two pupils with the answer A (upper bar) and the answer E (lower bar) within approximately 20 seconds

Fig. 12. Sequence Chart for two pupils with the answer A (upper bar) and the answer E (lower bar) within approximately 25 seconds
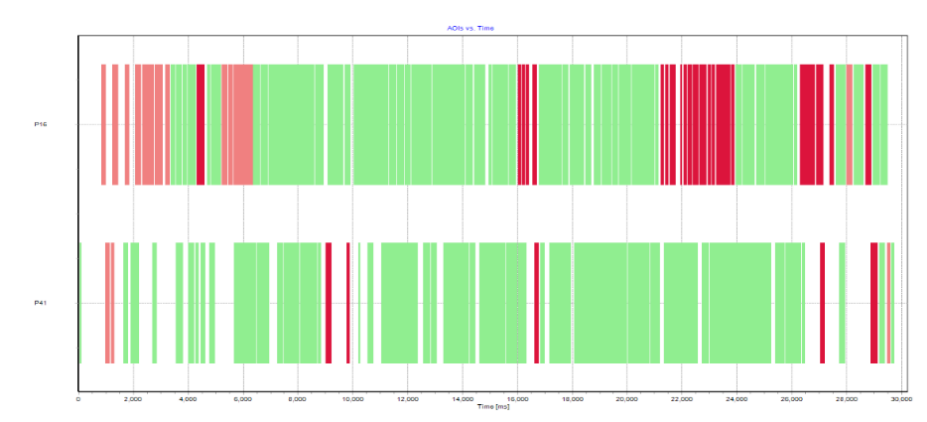
Fig. 13. Sequence Chart for two pupils with the answer A (upper bar) and the answer E (lower bar) within approximately 30 seconds

Matching comparable time of solving the task by particular pairs allowed to notice essential differences connected with transferring sight by the subjects from both groups. It can be concluded from the graphic record of transferring sight from one AOI to another AOI for such selected pairs that the dynamic of the stay on certain areas of interest for the group of pupils with the wrong answers is significantly different from the group of pupils with the correct answer.

It can be seen that the pupils giving the wrong answers A or C more often shift their sight from the drawing (green colour) to other areas (answer boxes or text boxes) while the individuals from the group pointing to the correct answer E less frequently leave the box with the drawing and remain there for longer time units despite some visible relocations to other areas. Longer stays of sight on the drawing might mean deeper data processing which perhaps allowed the subjects to notice other visual structures within the shown object.

7.   Conclusions

The research results indicate certain eye-tracking parameters which characterise the way of searching for the solution by the groups of people giving different kinds of answers. The success in finding the correct answer was connected with the ability to search and switch over to new structures present in the same object. As the research results show, all the subjects spend the majority of task solving time on the visual analysis of the figure. However, it turned out that the percentage time of the figure analysis was comparable both in the group of the correct answers and in the group of the incorrect answers. It was very surprising, though that the group of the incorrect answers spends more percentage time analysing the *Answer Boxes* than the group with the correct answers. It might testify lower ability to concentrate on the figure, lower concentration on searching for new structures. On the contrary, they underline searching for help in ready-made, given answers. Such conclusion is also corroborated by the test results presented in the Sequence Chart. On graphic presentations of the sight relocation among the marked areas

AOI by few persons it can be seen that the subjects choosing the incorrect answers make more "jumps" among the marked areas *Text*, *Drawing*, *Answer boxes*, and particularly to the area *Answer boxes.*

It can be seen in the research results that the ability to switch over among the visual structures is not obvious for the pupils, even for those at the age concerned. On the other hand, there are many examples in school geometry in which finding the solution requires various structuring of a certain object and the ability of switching over these structures. The ability of dual view on the same task condition, the ability of various structuring of the same object, the ability to detect two different structures in the same object and the possibility to switch over these structures depending on the aim of the given task – these are the elements indispensable in learning not only mathematics.

> The existence of thinking structures is of enormous importance, and the need to progress now and then and to change to other thinking structures may bring mental distress. We must realize that in many cases not such simple structures as those above are involved; structures that have been studied for years are extended, refined, taken up as a part of a more inclusive structure, brought to a higher level, and so forth (van Hiele, 1986, p. 133).

The ability of switching over from one structure to another is fundamental not only for the purpose of solving mathematical problems and developing mathematical concepts, but also with respect to human thinking and acting.

It might be worth paying more attention to shaping these skills during school education and what is more, perhaps it would be valuable to develop didactic concept that would allow to master these abilities during many years of empirical shaping.

Deeper data processing enables a new perception of the issue to appear. Longer dwelling time on the area beyond the figure can give a chance for so called "incubation break" (Nęcka, 2012) which guarantees finding another, new triangle structure in the object.

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