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Leonardo Pisano called Fibonacci, between advanced mathematics, history of mathematics and mathematics education: three examples drawn from *Liber Quadratorum**

Abstract. In this research, three problems faced by Leonardo Pisano, called Fibonacci, are dealt with. The main purpose is to show that history of mathematics can offer interesting material for mathematics education. The approach to the use of history of mathematics in mathematics education cannot be merely historical, but adapted to the needs of the explanations in a classroom. In the course of this paper, the meaning of such assertion will be clarified. A further purpose is a trying to explore the relations history of mathematics-mathematics education-advanced mathematics. Last, but not the least, a further aim is to offer a specific series of interesting material to the teacher in order to develop stimulating lessons. The material here expounded is conceived for pupils frequenting the third and the fourth years of the high school, but, with minor modifications, it can be adapted to the fifth year of the high school and to the initial years of the scientific faculties at the university.

1. Introduction

Federigo Enriques (1871-1946) claimed on several occasions there is not a clear distinction between advanced mathematical research, history of mathematics and mathematics education.[1] Given the extreme specialization of advanced mathematics (which existed already in Enriques' epoch) and since history of mathematics and mathematics education have also become specialized fields of research, Enriques' assertion ought seem paradoxical or wrong, at all. In contrast to this, the

*Od matematyki wyższej, przez historię do dydaktyki. Trzy przykłady zaczerpnięte z *Liber Quadratorum* Leonarda z Pizy zwanego Fibonaccim

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[1] Enriques, 1921, 2003, p.12-14.

ideas expounded by Enriques hold a profound meaning and express a part of the truth, though, probably, not the whole truth.

I will refer to a specific example, drawn from Fibonacci's (1170-1245?) *Liber Quadratorum* (1225), in which a conspicuous series of ideas to catch the relations between advanced mathematics, history of mathematics and mathematics education exists. Since the main approach of this paper is mathematics-education centred and, after all, a possible concrete use of Fibonacci's technique in an educational context will be shown, the references to history of mathematics will be limited to what is strictly necessary to prove the plausibility of Enriques' thesis. Thence, this research offers a double reading-key:

1) A support to Enriques' conception, according to which – at least in some circumstances – mathematical research, mathematics education and history of mathematics can be tied by a thorough link;

2) The possible use of some techniques and methods used along history of mathematics in an education context. In particular the content of this paper can be presented to the pupils when they deal with the principle of mathematical induction, this means, in general, when they frequent the third-fourth year at the high school.

I have no claim to provide a general theory of the relations history of mathematics-mathematics education since, as I have explained in other works[2], I am convinced that such a general theory does not exist, but profound connections do.

2. References to the figure of Fibonacci and to his works

The figure of Leonardo Pisano, called Fibonacci, is so well known that here I will only refer to some of his main features, perhaps useful for the reader not involved in history of Medieval mathematics[3]: Leonardo was the son of Guglielmo Bonacci, who was *publicus scriba* (namely a delegate) of Pisa maritime republic abroad. In 1185 Guglielmo was transferred to Bugia (nowadays in Algeria) as a responsible of Pisa's commercial business in that land. The young Leonardo followed his father in Bugia. In the *Incipit* of his famous *Liber abaci* (1202, 1228)[4] Fibonacci himself writes to have lived in Bugia and to have travelled to Egypt, Syria,

[2] Bussotti, 2014; Bussotti, 2015.

[3] I mention here some significant works on Fibonacci without any claim to be exhaustive (the literature on Fibonacci is huge): Boncompagni, 1852; Boncompagni, 1854; Bussotti, 2008a, p. 234-249; Bussotti, 2008b, p. 43-61; Favaro, 1874; Franci, 2002; Genocchi, 1855a; Genocchi, 1855b, Genocchi, 1855c, Genocchi, 1857; Grimm, 1973; Horadam, 1991; Lüneburg, 1991; Picutti, 1979; Picutti, 1983; Pisano, Bussotti, 2015; Rashed, 1994; Rashed, 2003; Woepcke, 1854-55; Woepcke, 1860-61.

[4] Fibonacci, 1228, 1857, p. 1. The works by Fibonacci have been published by Baldassare Boncompagni. See Fibonacci, 1228, 1857 (Liber Abaci); Fibonacci, 1862a, p. 227-252 (Flos); Fibonacci, 1862b, p. 253-283 (Liber Quadratorum); Fibonacci, 1862c, p. 1-224 (Practica Geometriae). The Liber Abaci has been translated into English by L.E. Sigler (Sigler, 2002); The Liber Quadratorum has been translated into French by P. Ver Eecke (Fibonacci, 1952) and into English by L. E. Sigler (Fibonacci, 1988). The Practica Geometriae has been edited and translated into English with great care by B. Hughes (Hughes, 2008).

Sicily and Byzantium to improve his knowledge of mathematics. Thence Leonardo reached a – presumably – good knowledge of Arabic mathematics. Fibonacci came in contact with the Hindu-Arabic digits and decimal positional system, imported into the Arabic countries from India, as well as with Arabic algebra and Diophantine tradition, which was alive and flourishing in the Arabic world. Fibonacci wrote a series of important works, in which his ingenious originality, the knowledge of Euclid's *Elements* and the acquisitions deriving from the Arabic mathematics converge to create a quite stimulating picture:

A) The *Liber abaci* is surely the most famous of Fibonacci's works. It is a huge book divided into fifteen chapters, where Fibonacci introduces the calculations with the Hindu-Arabic digits, namely the four arithmetical operations, the calculations with fractions, the way to extract square and cubic roots and a numerous series of problems – in general concerning – transactions, exchange of moneys, business societies among partners and so on. Usually, but not always, these problems can be solved by system of equations of first degree. Other problems concern numbers. In the 12^{th} chapter of the *Liber abaci*, the well known rabbit-problem is dealt with, where the Fibonacci-series is introduced. The *Liber* was a monumental work and impressed Fibonacci's contemporaries[5]. Although the introduction of the Hindu Arabic digits is documented in the Western world before Fibonacci[6], there is no doubt that the previous works on this subject were not comparable with Fibonacci's as to completeness, competency and clearness. The *Liber* can be considered an advanced handbook, with several ideas and problems also belonging to the then high mathematics.

B) Another significant work is the *Flos*, written around 1224-1225, in which Leonardo, answering to a problem proposed to him by the court mathematician of the Emperor Friedrich the second of Swabia, Giovanni da Palermo, showed that the equation

$$x^3 + 2x^2 + 10x = 20$$

does not have any Euclidean irrational as a solution. Here Fibonacci develops a series of arguments, which, nowadays, could be transformed into interesting didactical consideration on the limits of the use of some instruments to solve certain mathematical problems. In the specific case, the use of rule and compass, which could be connected to the three classical Greek geometrical problems: duplication of the cube, squaring of the circle, trisection of the angle and to modern algebraic approach to these questions.

Around 1220-1221 Leonardo also wrote a *Pratica Geometriae.* Though this book has some interesting aspects, it is, in my perspective, less interesting than the other works. Two further contributions by Leonardo: *the libro di*

[5] For the references to this fascinating and difficult historiographical problem, I refer to Pisano, Bussotti, 2015. In that paper we also offer precise indications on the state of art regarding this subject.

[6] See previous note.

merchaanti o detto di minor guisa and the *Libro sopra il* 10° *di Euclide* are lost.

C) However, the most surprising and profound work by Fibonacci is the *Liber Quadratorum*. The origin of this contribution is due to a further problem posed by Giovanni da Palermo to Leonardo: to solve in whole or rational numbers the system of two equations:

$$\begin{cases} x^2 + 5 = y^2 \\ x^2 - 5 = z^2 \end{cases}$$

As a matter of fact, Leonardo proposes an entire and coherent theory - developed in the twenty propositions of the *Liber* - concerning a sequence of interrelated problems with the squares. The most interesting aspect of the *Liber* is probably that methodological: Leonardo used an approach, which is in part influenced by Euclid's geometrical tradition, in part by Diophantus' and Arabic tradition, but which is, in great part, original. As we will see, it is based on the idea to use the sums of consecutive integer numbers to solve the problems connected to the squares. Since my aim is to exploit Leonardo's method in an education perspective I refer to the literature for both the mathematical and genetic-historical problems connected to the *Liber Quadratorum*[7]. I will focus on three propositions. Each section will include the explanation of Leonardo's approach and – in a separate box – the didactical considerations. The section 5 will also present a box regarding the relations history of mathematics – mathematics education – advanced mathematics.

It is necessary to point out that, as to the figure of Leonardo, there are several questions: a) what exactly were his sources of inspirations; b) why did he succeed in introducing the Hindu-Arabic numerals in Western Europe; c) what is his heritage. In particular: how much did he directly influence the Western mathematics and mathematics education – specifically the abacus schools and the abacus treatises – in the late Middle Age. On these fascinating historical problems, I have to refer to the literature[8], because, as already pointed out, my aim in this paper is not purely historical.

3. Squares obtained as sum of the successive odd numbers

The first proposition of the *Liber Quadratrum* is a rather dense introduction to the topics dealt with in the course of the book. In the second proposition Leonardo writes:

I want to prove why the ordered series of the squares arises from the ordered collection of the odd numbers starting from 1 to the infinity.[9]

[7] As to the Liber Quadratorum see specifically: Bussotti, 2008a, p. 234-249; Bussotti, 2008b, p. 43-61; Genocchi, 1855b; Genocchi, 1855c; Picutti, 1979; Rashed, 1994; Rashed, 2003; Woepcke, 1854-55. Useful references in Fibonacci, 1952 and Fibonacci, 1988.

[8] The literature on this subject is huge. Detailed references exist in Pisano, Bussotti, 2013 and Pisano, Bussotti, 2015.

[9] Fibonacci, 1225, 1854, p. 255. Original Latin text: "Volo demonstrare quare ex ordinata imparium collectione, ad uno incipiendo ad infinitum egrediatur ordinata series quadratorum".

Fibonacci does not claim the priority of this discovery. But – as far as we know – his way of reasoning is original. He argues like this

$$A \quad B \quad G^T \quad D^K \quad E^L \quad Z^M \quad I^N$$

|___ |___|___|___|___|___|

Ryc. 1.

The segments AB, BG, GD, ... represent the successive numbers, so that $AB = 1$; $BG = 2$; $GD = 3$, and so on. The exponents T, K, L, ... have a different meaning as they indicate sums: each one indicates the sum of the two previous numbers. So that T is the sum of AB and BG, thence $T = 3$; K is the sum of BG and GD, so that $K = 5$, and so on. Therefore, the exponents indicate the series of the odd numbers. Each odd number has a double representation in this figure by Fibonacci. For example, 3 is represented by GD and by T; 5 is represented by EZ and by K, and so on. Fibonacci reasons as follows:

1. Given any number ZI, the relation
$$ZI^2 = EZ^2 + EZ + ZI \tag{1}$$
holds, being $EZ + ZI = N$. In modern terms this simply means that $(n+1)^2 = n^2 + n + (n+1)$. Relying upon the formula for the square of a binomial proved by Euclid in Elements, II, 4, Fibonacci had proved identity (1) in the first proposition of the *Liber Quadratorum*.
2. Applying the same formula as (1), one obtains $EZ^2 = DE^2 + DE + EZ$, where $DE + EZ = M$. This formula can be applied till reaching $BG^2 = AB^2 + AB + BG = 1 + T = 1 + 3$.
3. Therefore $GD^2 = BG^2 + BG + GD = 1 + T + K = 1 + 3 + 5$, because $BG^2 = 1 + T$ and $BG + GD = K$. It is clear that this procedure can be iterated obtaining each square as the sum of the numbers $1+T+K+L+M+N+\ldots$, namely the successive odd numbers. This proves the proposition.

Didactical considerations.

A) *The methodology.* From an educational point of view the logical and methodological considerations are important for the pupils to catch that mathematics is not only a series of results obtained in a mechanical way once and forever. It is also a set of different methods and way of reasoning, by means of which the men reached the results, often with a conspicuous work and labour. The proof of this proposition by Fibonacci is easy, but it is paradigmatic of a procedure he also applied to more complicated problems. The logic of this method has to be clearly explained to the pupils. For the explanation, let us symbolize like this:

Proposition q="every square is the sum of an initial segment of odd numbers" (this is the proposition to prove).

Proposition p="$(n+1)^2 = n^2 + n + (n+1)$".

By the application of proposition p to any number (we indicate it by $n+1$, Leonardo by ZI), it possible to *descend* to 2^2. But for 2^2, proposition q holds, too, because $2^2 = 1 + 3$. Then the *ascending* phase begins and, by the replacements showed in the item 3., the proposition q is proved. Thence, Leonardo's methodology has a *descending* and an *ascending* phase.

Let us now propose to the students a modern demonstration by mathematical induction:

Basis of the induction: $2^2 = 1 + 3$

Inductive step: let us suppose that $n^2 = 1 + 3 + \cdots + (2n - 1)$, then

$$(n + 1)^2 = 1 + 3 + \cdots + (2n - 1) + (2n + 1)$$

which proves the proposition.

The similarities between the two proofs are evident and the teacher has to point out: i) in the modern inductive proof, there is the basis $2^2 = 1 + 3$, identity, which, as seen, is fundamental in Fibonacci's argument, too; ii) the inductive step in Fibonacci's demonstration can be directly obtained by applying the proposition p and supposing that proposition q is valid until a number n. Therefore the two procedures are perfectly equivalent from a logical point of view. Then the question: why did Fibonacci add the *descending* phase, which is useless in a merely mathematical perspective? The answer to this question is very instructive because it points out the human aspect of mathematics. The main consideration concerns exactly the inductive step: in Leonardo's reasoning the function of our inductive step is replaced, in fact, by the series of descending identities

$$\begin{cases} ZI^2 = EZ^2 + EZ + ZI \\ EZ^2 = DE^2 + DE + EZ \\ \ldots \\ BG^2 = AB^2 + AB + BG \end{cases}$$

The ascending phase consists in a mere replacement of values, according to what already seen, starting from the identity $BG^2 = AB^2 + AB + BG = 1 + T$, which is the basis of the induction.

Once explained the logic of Fibonacci's method, let us consider the reason why he presented such a demonstration: in the epoch in which Leonardo wrote, a set of standard methods – as mathematical induction – applicable to number theory had not yet been created. It should be clarified to the pupils that the formalization of the inductive basis and inductive step needs a certain level of symbolization and formalization:

Symbolization, because – in the case of our proof by mathematical induction, we have written the last addend of the sum, which constitutes the square of n, as $2n+1$. This implies the capability to use a letter as n to indicate a number *in general* and the idea to write a number in function of a modulus, which is 2 in the case of $2n + 1$. These skills were acquired only in the 17^{th} century with mathematicians as Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662), although – as we will see in section 5.1.- Fibonacci seems, in some circumstances, to have arrived close to the concept of modulus. It is not a case that mathematical induction in modern terms is present in the *Triangle Arithmetique* by Pascal. For, even if before Pascal some methods similar to mathematical induction were conceived and some scholars date back this method to a period preceding Pascal, I am convinced that the founder of modern mathematical induction was Pascal[10]. Fermat developed the method of infinite descent, which has similarities – but also profound differences – with mathematical induction[11].

Formalization, which consists in operating on the numerical forms according to the rules and formulas of our algebra. But, in Fibonacci's period, despite the conspicuous progresses made by the Arabic mathematicians, the formal calculus – we could say the literal calculus - almost did not exist. The numbers were, in general, associated to segments of straight lines or anyway, at least in the Western world, the geometrical representation played still a pivotal role. Fibonacci made considerable progresses in this sense because, for example, the idea to represent the number in two different ways within a sole proof is rather audacious: in the analysed case, i.e., 5 is represented by K, but also by $BG+GD$, according to the necessity. Furthermore, the geometrical representation is a sort of psychological-didactical support rather than a necessity intrinsic to the proof. This means that Fibonacci probably had something similar to the concept of *any number*, but his expressive means allowed him to obtain a certain level of formalization, by which it was difficult to express and conceive the necessary formal steps typical of mathematical induction. Notwithstanding, he conceived a method, which resembles mathematical induction and for which, his level of symbolization and formalization was enough. This is a very interesting lesson for the pupils because they can understand, by means of an easy example, the profound sense in which mathematics is a historical and human activity tied to the scientific context of the period: Fibonacci's method is redundant only in our eyes. In fact, it is extremely innovative.

B) *Relations with infinity.* The following considerations could be developed jointly with the teacher of philosophical and historical disciplines: the descending phase of Leonardo's proof could also be interpreted as a reluctance to resort to infinity, also in its potential form, or at least as the intention to limit as much as possible the resort to infinity. Indeed, this phase – which, in Leonardo's way of reasoning, not in ours, is quite important – operates between two finite limits: the initial number ZI, which is arbitrary, but finite and given in the specific context, and the number $BG = 2$. This scheme is completely finitary in the sense that it involves a finite segment of integer numbers in descent, from ZI to 2. One of the reasons why Fibonacci tried to tie the ascending phase, which involves the whole potentially infinite series of the integers, to the descending one, is also probably the attempt to create a logical dependence of a potentially infinite ascent from a finite descent. For, it is clear that Fibonacci had an almost dynamical vision of his procedure; a sort of movement, which, starting from an initial point (ZI) reached a minimum (2) after which the ascending phase began. Although the modern mathematical induction is not seen by us as a dynamical procedure, it is a scheme in ascent, in which the potential infinity is directly involved and there is no attempt to create a dependence of its on a finitary scheme as Fibonacci's descent. According to my interpretation, this is a proof of how the mental and psychological presuppositions can also play a role in the way in which mathematics is created and developed. The discussion on the concept of infinity in the middle ages as well as on the relations between science-mathematics and mentality of a certain epoch, are typically interdisciplinary fields, which – at least on some occasions – could be shared by the teachers of mathematics and of history or philosophy.

Now I will propose an example of a more complex proof carried out by Leonardo's method to show that its range of application is wider one could think of.

[10] As to Pascal and mathematical induction see Palladino, Bussotti, 2002; Bussotti, 2004.

[11] A complete work on the infinite descent, where Fermat plays a fundamental role, is Bussotti, 2006.

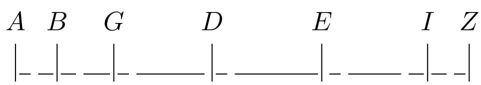
4. Leonardo's proof of the identity $(2n-1)(2n+1)[(2n-1)+(2n+1)] = 12[1^2+3^2+5^2+...+(2n-3)^2+(2n-1)^2]$

The 7^{th} and 8^{th} proposition of the *Liber Quadratorum* proves two interesting propositions by means of a technique similar to the one expounded in the previous section. In the 7^{th} proposition Leonardo proves that

$$n(n+1)[n+(n+1)] = 6(1^2+3^2+5^2+\cdots+n^2),$$

while in the 8^{th} one, he demonstrates the identity referred to in the title of this section. I will focus on proposition 8.

Fibonacci considers the series of the odd numbers. He represents them in a line so that $AB = 1$; $BG = 3$; $GD = 5$; $DE = 7$; $EZ = 9, \ldots$



Ryc. 2.

Since, $DZ = DE + EZ$, Leonardo claims he has to prove that

$$DE \cdot EZ \cdot DZ = 12(AB^2 + BG^2 + GD^2 + DE^2), \tag{2}$$

which, posing $DE = 2n-1$ and $EZ = 2n+1$, is exactly our proposition.

Leonardo considers $DE = EI$, so that $IZ = 2$. Since the successive odd numbers are indicated in the figure, it is $GD = DE - 2$ and $GE = 2DE - 2$. Hence $GD \cdot DE = DE^2 - 2DE$. Therefore

$$CD \cdot DE \cdot GE = (DE^2 - 2DE)(2DE - 2) = 2DE^3 - 6DE^2 + 4DE \tag{3}$$

Now Fibonacci adds $12DE^2$ to the value in (3), thus obtaining

$$GD \cdot DE \cdot GE + 12DE^2 = 2DE^3 + 6DE^2 + 4DE \tag{4}$$

But $DE = EI$, thence $EZ = DE + IZ = DE + 2$ and $DZ = 2DE + 2$. This means that $DE \cdot EZ = DE^2 + 2DE$. Therefore,

$$DE \cdot EZ \cdot DZ = 2DE^3 + 6DE^2 + 4DE \tag{5}$$

That is, from (4) and (5)

$$GD \cdot DE \cdot GE + 12DE^2 = DE \cdot EZ \cdot DZ \tag{6}$$

By the same reasoning, Fibonacci constructs a descent. For, by analogous steps as those used to prove identity (6), it is possible to show that

$$BG \cdot GD \cdot BD + 12GD^2 = GD \cdot DE \cdot GE \tag{7}$$

Thus, from (6) and (7)

$$BG \cdot GD \cdot BD + 12(GD^2 + DE^2) = DE \cdot EZ \cdot DZ \tag{8}$$

By the same reasoning, it will be shown that

$$BG \cdot GD \cdot BD = AB \cdot BG \cdot AG + 12BG^2 \tag{9}$$

So that, from (8) and (9), one gets

$$AB \cdot BG \cdot AG + 12(BG^2 + GD^2 + DE^2) = DE \cdot EZ \cdot DZ \tag{10}$$

However, Fibonacci continues, since $AB = 1$, $BG = 3$ and $AG = 4$, the product $AB \cdot BG \cdot AG = 12$, or $12AB$, if one wishes to indicate the unit, as Fibonacci, by AB. Thence

$$12(AB^2 + BG^2 + GD^2 + DE^2) = DE \cdot EZ \cdot DZ \tag{11}$$

Being AB, BG, GD,... the successive odd numbers, this proves the theorem. Obviously the reasoning works whatever the numbers DE, EZ and DZ are. Thence it is true in general.

Fibonacci adds some interesting corollaries on which I do not focus. I only add that by propositions 7 and 8, it is possible to deduce the famous formula by Archimedes on the sum of squares[12]. For, from proposition 6, it follows immediately

$$1^2 + 3^2 + 5^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{12}$$

Didactical considerations. The same considerations as those presented in the previous section can be developed. It is important to point out that, in this case, the particular method of Leonardo, consisting of a descent followed by an ascent, is applied to prove a far more difficult theorem, this means it is not only of historical interest, but it is a performative method, too. The teacher should point out that, in this case, even more than in the theorem presented in the previous section, the decisive inductive step is represented by the formula, which allows Fibonacci to reach the identity (10). His inductive step is in descent, whereas for us, it is in ascent. The basis of the induction, which permits the prove of the theorem - namely the identity $AB \cdot BG \cdot AG = 12$, represents, hence, the last step in Fibonacci's methods, while, in the modern mathematical induction, it is the first one.

5. The equation $\frac{x^2-y^2}{y^2-z^2} = \frac{a}{b}$ (a>b)

In this equation - which constitutes the 18^{th} proposition of the *Liber Quadratorum* - every letter indicates an integer. I have translated the language of Fibonacci into the modern symbolism because, in what follows, I will not focus on the problems of symbolism and, hence, the transcription in modern symbols makes the explanation easier.

From a mathematical standpoint, this equation represents a far more difficult problem than the identities analysed in the sections 3. and 4. A complete theory,

[12] This result is obtained by Archimedes in the 10^{th} proposition of the *Spirals*.

inside which it was solved, was discovered by Lagrange (1737-1813)[13]. Nonetheless, the considerations developed by Fibonacci are elementary – though rather refined – and they are particularly suitable to catch: a) his technique of the "successive addends"; b) the relations history of mathematics-mathematics education-advanced mathematics.

Before entering into Fibonacci's reasoning, it is necessary to remind the reader two easy statements he proved in the 17^{th} proposition of the *Liber*:

1) Given two consecutive odd numbers $(2n+1)$ and $(2n+3)$, it holds

$$(2n+3)^2 = (2n+1)^2 + 8(n+1) \tag{13}$$

2) Given two consecutive even numbers $2n$ and $(2n+2)$, it holds

$$(2n+2)^2 = (2n)^2 + 4(2n+1) \tag{14}$$

5.1. The equation $\frac{x^2-y^2}{y^2-z^2} = \frac{a}{b}$ the three elementary cases

Fibonacci distinguishes his solution in cases.

The *first* analysed case takes places when $a = b + 1$ (in what follows it is supposed $a > b$, but the reasoning is the same in the opposite hypothesis). The solution is easy and can be obtained for every a. Fibonacci offers an example, which, however, is *paradigmatic* of a general demonstration. He considers $a = 11$; $b = 10$. Referring to proposition (13), it is

$$21^2 = 19^2 + 10 \cdot 8; \quad 23^2 = 21^2 + 11 \cdot 8$$

Thence

$$\frac{23^2 - 21^2}{21^2 - 19^2} = \frac{21^2 + 11 \cdot 8 - 21^2}{19^2 + 10 \cdot 8 - 19^2} = \frac{11}{10}.$$

Therefore $x = 23$, $y = 21$, $z = 19$ are the solutions.

To get a general proof, it is enough to pose $a = n + 1$; $b = n$. The solutions are: $x = 2n + 3$; $y = 2n + 1$; $z = 2n - 1$, because, according to proposition (13), it is

$$\frac{(2n+3)^2 - (2n+1)^2}{(2n+1)^2 - (2n-1)^2} = \frac{(2n+1)^2 + 8(n+1) - (2n+1)^2}{(2n-1)^2 + 8n - (2n-1)^2} = \frac{n+1}{n} = \frac{a}{b}$$

The *second* analysed case concerns the condition $a = b + 2$.

Once again, Fibonacci provides a paradigmatic example. He considers $a = 11$; $b = 13$. He adds the two numbers, obtaining 24, divides 24 by 4 and multiplies the result by 2, thus obtaining 12 (in practice he divides the sum by 2). The number 12 is the "central" square, that is y. The other two numbers are $x = 14$, $z = 10$. For, it is

[13] Lagrange's fundamental work on this subject is Lagrange, 1769. For detailed explanations of Lagrange's method, see Bussotti, 2006, p. 299-341.

$$\frac{14^2-12^2}{12^2-10^2}=\frac{12^2+4\cdot 13-12^2}{10^2+4\cdot 11-10^2}=\frac{13}{11}.$$

The general demonstration of Fibonacci's method relies upon the identity (14), because, posed $a=2n+1$; $b=2n-1$, obviously their sum divided by 2 is $2n$. One has

$$\frac{(2n+2)^2-(2n)^2}{(2n)^2-(2n-2)^2}=\frac{(2n)^2+4(2n+1)-(2n)^2}{(2n)^2+4(2n-1)-(2n-2)^2}=\frac{2n+1}{2n-1}=\frac{a}{b}.$$

What follows is very interest and shows the profoundness of Fibonacci's though: Leonardo poses this problem, let us suppose that $a=21$, $b=19$. Now he requires three specific squares who solve our equation, with the further condition their bases are numbers of the form $3n$. He offers the following canon: sum the two numbers, $19+21=40$, divide (as in the previous case) by 4, obtaining 10, multiply 10 by 3, obtaining 30. This is the central square, the other two being 33 and 27. Indeed, it is

$$\frac{33^2-30^2}{30^2-27^2}=\frac{21}{19}$$

How did he reach this amazing result? Probably Fibonacci had studied, for every modulus, the differences of two consecutive squares (this is the circumstance in which Leonardo seems to have caught important aspects of the concept of modulus). Thence mod. $3n$, we have

$$(3n+3)^2-(3n)^2=9(2n+1) \tag{15}$$

Therefore, let us pose $a=2n+1$; $b=2n-1$, so that $a+b=4n$, number which, divided by 4, produces n. Let us now considers $x=3n+3$; $y=3n$; $z=3n-3$. It will be:

$$\frac{(3n+3)^2-(3n)^2}{(3n)^2-(3n-3)^2}=\frac{(3n)^2+9(2n+1)-(3n)^2}{(3n)^2+9(2n-1)-(3n-3)^2}=\frac{2n+1}{2n-1}=\frac{a}{b}.$$

As Fibonacci points out, this procedure can be applied to every modulus, $4n$, $5n$, $6n$, and so on. For example mod. $5n$, the difference between two consecutive numbers $5n+5$ and $5n$ is, obviously, $25(2n+1)$[14]. It will be, hence, necessary to divide $a+b$ by 4 and multiply the result by 5, thus obtaining the central square, whose root is $5n$, the other two roots will be $5n+5$ and $5n-5$. For example, if $a=21$, $b=19$, the three roots are $x=55$, $y=50$, $z=45$, thus obtaining

$$\frac{55^2-50^2}{50^2-45^2}=\frac{21}{19}.$$

The *third* and last elementary case takes place when both a and b are squares. Let us pose $a=x_1^2$; $b=y_1^2$. The three solutions are then given by $x=x_1^2$; $y=x_1y_1$;

[14] In general, such a difference, for any modulus kn is $k^2(2n+1)$, which is an extension of the identities (13) and (14).

$z = y_1^2$. This is the least interesting case, whose proof derives immediately by applying the properties of the proportions:

$$x_1^4 : x_1^2 y_1^2 = x_1^2 y_1^2 : y_1^4$$
$$(x_1^4 - x_1^2 y_1^2) : x_1^4 = (x_1^2 y_1^2 - y_1^4) : x_1^2 y_1^2$$
$$(x_1^4 - x_1^2 y_1^2) : (x_1^2 y_1^2 - y_1^4) = x_1^4 : x_1^2 y_1^2 = x_1^2 y_1^2 = a : b.$$

Fibonacci offers the example of $a = 25$, $b = 16$. In this case, it will be $x = 25$, $y = 20$, $z = 16$.

Didactical considerations. What dealt with in this section induces several observations.

A) *Relation between proofs* and *numerical examples.* In order to prove his statements Fibonacci uses specific numerical values and not general symbols, hence one could thought that his proofs are not valid, but are limited to those specific cases analysed by him. In fact, this is not true because his examples are *paradigmatic*. Through this expression, I indicate an example in which the general numerical forms can be replaced to the specific numbers because all the numbers have a precise form, which can be identified (being this form $2n + 1$ or $3n$, or $5n$, or whatever it is), thus obtaining a rigorous proof. All the demonstrations examined in the present section have this feature. Therefore, Leonardo's arguments are valid in general, though apparently based on examples. I think, it is a common experience, shared by the teachers of mathematics, to observe that – especially while dealing with theorems concerning integers – that several – not all - pupils do no exactly catch the fundamental difference between a demonstration and a proposition verified for all the examined examples, but not proved. After having explained Fibonacci's argumentations, in order to clarify this difference, a teacher could consider the famous theorem by Fermat that every prime of the form $4n + 1$ is the sum of two squares. He/she could begin to decompose these primes beginning with $5 = 1^2 + 2^2$; $13 = 2^2 + 3^2$; $17 = 1^2 + 4^2$; $29 = 2^2 + 5^2$..... It could be comprehensible that the pupils asked if, in this case, a replacement of the single numbers by a general form is possible, so to get a demonstration of the statement. The teacher should then explain this is not possible: the examples of Fermat's theorem are not *paradigmatic*: it is not possible – at least as far as we know until now – to deduce from them a general proof of this theorem, which, in fact, is independent from the way in which such primes are decomposed into two squares[15]. Therefore, a clear explanation of Fibonacci's technique jointly with the replacement of the general forms to the specific numbers used by Leonardo: i) will make it clear his reasoning works in general; ii) will be a support to clarify the difference between a paradigmatic and a not-paradigmatic example; iii) will be a support to distinguish between a non-paradigmatic example and a rigorous proof. All these aspects are fundamental for a pupil to catch the true logical foundations of mathematics as well as the nature of this discipline. History of mathematics is, in this case, a precious help for mathematics education.

B) *Solutions of undetermined equations.* Fibonacci provides one or more solutions to his equations. From a didactical point of view an interesting question arises: did he supply all the possible solutions? The answer is negative: let us consider once again the case $a = 13$; $b = 11$. Let us divide their sum by 4 and do not multiply by 2. We get the number 6, which offers the solution $x = 7$, $y = 6$, $z = 5$, which does not exist in the *Liber Quadratorum*. Given $\frac{a}{b} = \frac{2n+1}{2n-1}$, the least y who solves the equation is $y = n$, with $x = n + 1$; $z = n - 1$ because $\frac{(n+1)^2-n^2}{n^2-(n-1)^2} = \frac{2n+1}{2n-1} = \frac{a}{b}$. This number is simply obtained dividing $a + b$ by 4, thence completely remaining within Fibonacci's methodology. These considerations have to induce the teacher to clearly specify that, given an undetermined equation and, in general a mathematical problem, there are four different questions: i) to proof if the equation has a solution; ii) to show a solution of the equation; iii) to show possible other solutions of the equation by an algorithm; iiii) to show that the found solutions are all the solutions. These problems *ought* be connected: for example, it is clear that if one finds a solution to an equation, the question i) is automatically solved, but they are not *necessarily* connected. In the case of our equation, i. e., with $a = 2n + 1$; $b = 2n - 1$, Fibonacci's technique offers an infinite number of solutions, but we cannot be sure they are all the solutions. Others could exist, which are not included into Fibonacci's algorithm. The difficulties involved in the answers to the questions i)-iiii) are, in general, quite different because by means of ingenious devices, which, in some circumstances, arrive at resembling a theory, it is possible to offer a solution or also an infinity of solutions. However, as I will explain more clearly in the didactical considerations added to the next section, the answer to the problem iiii) for this equation is connected to the general solution of all the undetermined equations of second degree with two unknowns, problems which goes far beyond the possibilities of a mathematician living in the $12^{th} - 13^{th}$ century and far beyond the programme of mathematics in a high school. This is the connection of which I was speaking with regard to history of mathematics-mathematics education-advanced mathematics: a problem, as that of Fibonacci, starts from history, allows us a series of didactical considerations useful at high school level and is connected with an aspect of advanced mathematics, whose complete solution was offered only in the late 18^{th} century. To follow this itinerary – though it can comprehend only a part of the mathematical details – is quite instructive for the high schools' pupils.

C) *To play with numbers.* Apart from more general considerations, the algorithm by Fibonacci is useful for the pupils to learn to deal with numbers and numerical forms. For example: given, in our equation $a = 2n+1$, $b = 2n-1$, the teacher, after having explained Fibonacci's reasoning, could ask the pupils to find the numbers of various forms: $6n$, $7n$, $8n$,... which solve the equation. This is a useful exercise because the ability to manipulate the numbers – which is so important in mathematics – is progressively disappearing. Furthermore, to play with numbers inside an algorithm might also be enjoyable for the pupils.

5.2. The equation $\frac{x^2-y^2}{y^2-z^2} = \frac{a}{b}$ the general solution

To deal with the general solutions, Fibonacci offers a series of arguments, which can be summarized in the following reasoning: if it is possible to decompose $a + b$ or a multiple of this sum $k(a + b)$ into a series of $m + n$ consecutive addends, such that the first m represent kb and the other n the same multiple ka of a, then the problem is solved. For, let us consider the series of successive addends

$$h, h + 1, h + 2, \ldots, h + m, h + m + 1, h + m + 2, \ldots, h + m + n \qquad (16)$$

[15] For detailed explanations on the $4n + 1$ primes-theorem, see Bussotti, 2006.

with

$$h + (h + 1) + (h + 2) + \ldots (h + m) = kb$$

$$h + m + 1) + (h + m + 2) + \ldots + (h + m + n) = ka$$

Then the three roots are $z = 2h - 1$, $y = 2(h + m) + 1$; $x = 2(h + m + n) + 1$ because:

$$\begin{aligned}
&(2h + 1)^2 = (2h - 1)^2 + 8h \\
&[2(h + m + 1) + 1]^2 = [2(h + m) + 1]^2 + 8(h + m + 1) \\
&(2h + 3)^2 = (2h + 1)^2 + 8(h + 1) \\
&[2(h + m + 2) + 1]^2 = [2(h + m + 1) + 1]^2 + 8(h + m + 2) \\
&\ldots\ldots\ldots\ldots\ldots \\
&[2(h + m) + 1]^2 = [2(h + m) - 1]^2 + 8(h + m) \\
&[2(h + m + n) + 1]^2 = [2(h + m + n - 1) + 1]^2 + 8(h + m + n) \\
&[2(h + m) + 1]^2 = (2h - 1)^2 + 8(h + h + 1 + \cdots + h + m) \\
&[2(h + m + n) + 1]^2 = [2(h + m) + 1]^2 + 8(h + m + 1 + \cdots + h + m + n)
\end{aligned}$$

Thus obtaining

$$\frac{x^2 - y^2}{y^2 - z^2} = \frac{[2(h + m + n) + 1]^2 - [2(h + m) + 1]^2}{[2(h + m) + 1] - (2h - 1)^2} =$$

$$\frac{[2(h + m) + 1]^2 + 8(h + m + 1 + \ldots + h + m + n) - [2(h + m) + 1]^2}{(2h - 1)^2 + 8(h + h + 1 + \ldots + h + m) - (2h - 1)^2} = \frac{a}{b}$$

The question is: is it always possible to find a decomposition as (16)? Fibonacci does not offer a general answer, but a method by trials and errors, which let us induce to think he was aware that such decomposition is always possible. However, this is a purely historical problem[16]. What is very interesting from a didactical standpoint is that the solutions to our equation are easy if the difference $a - b$ is an odd number or a number of the form 2^n or $2^n(2m + 1)$, $n > 1$. In these cases, a direct numerical application of the technique of the *consecutive addends* offers a solution. By the way, the numerical examples of Fibonacci belong to these two cases. While, if $a - b = 2(2m + 1)$, a different demonstration is necessary. Thence, I will divide the argument into two cases; 1) $a - b = (2n + 1)$ or $a - b = 2^n$ or $a - b = 2^n(2m + 1)$, $n > 1$; 2) $a - b = 2(2m + 1)$.

First case: $a - b = (2n + 1)$ or $a - b = 2^n(2m + 1)$. Let first consider $a - b = (2n + 1)$. I offer the following canon, which is not exactly present – in this form – in Fibonacci's work, but strictly tied to his general approach:

a) Let us consider the difference $a - b = d$;

b) Multiply a by d and b by d;

c) Let us consider the sum $a + b = s$ and the number $ds = d(a + b) = da + db$;

[16] See Genocchi, 1855b, p. 348-353.

d) Let us now decompose da and db in the sum of d consecutive addends, of which a and b are the central addends. This is always possible because d is an odd number. This construction ensures that in the decomposition of da and db in consecutive addends, the places between b and a are exactly $d = a - b$, hence we have a decomposition of $d(a+b)$ in a series of consecutive addends of which the first d represent da and the last d represent db. Therefore the equation is solved.

An example will completely clarify the situation: let the equation $\frac{x^2-y^2}{y^2-z^2} = \frac{14}{5}$ be given. In this case $a - b = 9$, Thence we have to decompose $9 \cdot 5$ and $9 \cdot 14$ into nine consecutive addends, of which 5 and 14 are the central addends, thus obtaining the required decomposition of $ds = 9(5 + 14)$, which is

$$1+2+3+4+5+6+7+8+9| \ +10+11+12+13+14+15+16+17+18$$

The first nine addends are the decomposition of $9 \cdot 5$, the last nine of $9 \cdot 14$. Therefore, using the same symbols as those adopted for the general method, it is $h = 1$; $m = 8$, $n = 9$. The three roots are hence $z = 2h-1 = 1$; $y = 2(h+m)+1 = 19$; $x = 2(h+m+n)+1 = 37$. Indeed, it is

$$\frac{37^2 - 19^2}{19^2 - 1^2} = \frac{14}{5}$$

Second case: $d = a - b = 2^l$ ($l > 1$, because the cases $n = 0, 1$ have already been dealt with). If $d = 4$, the problem is quite easy because $a + b$ can, then, be decomposed into four consecutive addends, of which the first two are equal to b and the second tow are equal to a. If $n > 2$, it is always possible – as it is easy to prove – to decompose $2^{l-2}a$ and $2^{l-2}b$ in a sum of 2^{l-1} consecutive addends, in which the numbers $\frac{a-1}{2}$, $\frac{a+1}{2}$; $\frac{b-1}{2}$, $\frac{b+1}{2}$ are the central addends of the respective sums.

Example: $\frac{x^2-y^2}{y^2-z^2} = \frac{27}{11}$. It is $27 - 11 = 16 = 2^4$, thence $l = 4$. It is hence necessary to decompose $4 \cdot 11$ and $4 \cdot 27$ into 8 consecutive addends, in which the central pairs are respectively

$$\left(\frac{b-1}{2}, \frac{b+1}{2}\right) = (5, 6)$$

and

$$\left(\frac{a-1}{2}, \frac{a+1}{2}\right) = (13, 14).$$

The decomposition is thence:

$$2+3+4+|5+6|+7+8+9|\,|+10+11+12+|13+14|+15+16+17+18$$

Where one vertical bar indicates the central pairs of every decomposition and two vertical bars the step from the decomposition of $4 \cdot 11$ to that of $4 \cdot 27$. Thus: $h = 2$; $m = 7$, $n = 8$. The three roots are hence $z = 2h-1 = 3$; $y = 2(h+m)+1 = 19$; $x = 2(h+m+n)+1 = 35$. We have:

$$\frac{35^2 - 19^2}{19^2 - 3^2} = \frac{27}{11}$$

If $d = a - b = 2^l(2q+1)$ $(l > 1)$, the reasoning is analogous. The difference is that $2^{l-1}(2q+1)a$ and $2^{l-2}(2q+1)b$ will de decomposed in the sum of $2^{l-1}(2q+1)$ consecutive addends. I provide an example: let us suppose that $\frac{a}{b} = \frac{17}{5}$. It is $a-b = 12 = 3\cdot 2^2$, $2q+1 = 3$, $2^{l-2} = 1$. This means that $2^{l-2}(2q+1)a = 3\cdot 17 = 51$ and $2^{l-2}(2q+1)b = 3\cdot 5 = 15$ will be decomposed into 6 consecutive addends, whose central pairs are $8, 9$ and $2, 3$. The decomposition is hence

$$0 + 1 + |2 + 3| + 4 + 5|\ | + 6 + 7 + |8 + 9| + 10 + 11$$

So that it is $z = 1$, $y = 11$, $x = 23$.

The particularity of this method is that we have always $n = m + 1$ because the equimultiples of a and b, ka and kb, are decomposed into the same number of addends, which is not necessarily true in the general conditions of solution.

This method cannot be applied if the difference $a - b$ is of the form $2(2n+1)$. In this case, there is no particular device connected to a numerical canon: it is necessary to offer a general solution. Fibonacci – as told above – offers a method by trials and errors, which works because of the following considerations due to Angelo Genocchi[17]: Let us suppose (see (16)) that

$$h, h+1, h+2, \ldots, h+m, h+m+1, h+m+2, \ldots, h+m+n$$

can be constructed so that kb is the sum of the first m terms and ka of the last n. It is hence

$$\begin{cases} hm + \frac{m(m+1)}{2} = kb \\ (h+m)n + \frac{n(n+1)}{2} = ka \end{cases}$$

Therefore

$$\begin{cases} 2k = \frac{mn(m+n)}{am-bn} \\ 2h + 1 = \frac{2bmn+bn^2-am^2}{am-bn} \end{cases}$$

Since a and b are coprime, the equation $am - bn = 1$ has solutions, for example $(m, n) = (b+1, a)$; $(m, n) = (b, a-1)$ and so on. This means that the decomposition (16) is possible and that hence our equation has always solutions. In particular

$$\begin{cases} z = 2h + 1 = 2bmn + bn^2 - am^2 \\ y = 2(h+m) + 1 = bn^2 + am^2 \\ x = 2(h+m+n) + 1 = 2amn - bn^2 + am^2 \end{cases}$$

Didactical considerations.

Relations history of mathematics-mathematics education-advanced mathematics.

1. The general solution of Fibonacci's equation is an ideal example to detect the relations between history of mathematics, mathematics education and advanced

[17] Genocchi, 1855a, p. 186-189; Genocchi, 1855b, p. 348-353.

mathematics: let us start from the case in which the difference $a-b$ is an odd number or a number, which can be divided by 4 or a bigger power of 2. I have offered a general explanation, but, in this case, the method of the *paradigmatic examples* still works. The explanation why the examples are paradigmatic is a little more complicated than in the cases in which $a-b=1$ or 2, but the teacher can offer it without resorting to any general formula. It is enough to show the canon and to give an explanation in words, which is quite intuitive. If the difference $a-b$ is of the form $2(2n+1)$ – and only in this case – my canon cannot be applied. It is hence necessary to show, as Genocchi did, that the series (16) can be anyway obtained by a general and formal reasoning. This is instructive from an educative standpoint because the pupils have a tangible example how much different can be the difficulties behind an apparently similar problems. It is significant the teacher explains why my canon works when the difference $a-b$ is, for example, of the form $2^2(2n+1)$, but not when it is $2(2n+1)$. Nuances, but nuances are the essence of mathematics. And history plays an important role in this context, because there is a historiographical counterpart to the previous didactical question: Fibonacci, probably, knew the canon, but is it possible to interpret, as Genocchi seems to do, his method by trials and errors as an evidence of the fact the he knew the form of the general solution? This is a historiographical question, to which it is not possible to answer in this paper. Notwithstanding, I cared about pointing out the parallelism between some historiographical and educative questions. The system by which Genocchi proves that a general solution exists is tied to an appropriate symbolism, but the reasoning is, in itself, rather easy. The same consideration holds if we look for other solutions to Fibonacci's equations (as Fibonacci himself highlights), however the request to find rigorously all the solutions drives to a field, which is completely different, because no reasoning structured as Genocchi's is of help. It is possible to show that this problem can be reduced to the integral solutions of the equation $At^2+B=u^2$ (each letter indicates an integer), whose general theory is due to Lagrange (see note 13) and drives to a sector of the theory of numbers which, for a long period, belonged to advanced mathematics. Obviously, it is not possible to deal with Lagrange's solution in a high school, but it is possible the students catch how a single problem can open the doors on interesting educative perspectives, profound historical questions and subjects concerning high mathematics.

2. To complete the canon when the difference $a-b$ is an odd number or a number of the form $2^n(2m+1)$, $n>1$, it is necessary to add a consideration, which can be interestingly exploit in an educative perspective: let us consider the equation $\frac{x^2-y^2}{y^2-z^2}=\frac{22}{5}$. Since $22-5=17$, which is an odd number, it is possible to decompose $17\cdot 5$ and $17\cdot 22$ into the sum of seventeen consecutive addends, of which 5 and 17 are the central addends. Nevertheless, here the canon works perfectly only if we admit negative numbers. The decomposition is this:

$$-3-2-1-0+1+2+3+4+5+6+7+8+9+10+11+12+13+|14+15+ \\ 16+17+18+19+\cdots+30$$

Namely:

$$4+5+6+7+8+9+10+11+12+13+|14+15+16+17+18+19+\cdots+30$$

So that the roots are $z=7$; $y=27$, $x=61$.

The question is from a historical point of view: could Fibonacci resort to a canon, which, though elementary, works if one admits negative numbers? The answer should be negative because it is problematic to admit Fibonacci was able to operate with negative numbers, albeit Genocchi hypothesizes, in reference to another problem dealt with by Fibonacci[18], he was, or, at least, he had the idea of negative numbers. Whatever the answer is, it is interesting to point out a particularity of the relation history of mathematics-mathematics education: history of mathematics is a fundamental support for mathematics education, even though the solutions to a certain problem, presented by the teacher in the classroom, might not exactly coincide with those likely offered in the historical period in which the problem was posed. It is important that: 1) history provides stimulating material for mathematics education; 2) the reference to history induces to address the subjects by means of a more critical and, at the same time, intuitive approach, which is different from those connoting most of handbooks; 3) the creativity of the pupils can be stimulated, so that they learn to play with numbers (and this paper deals with such a problem) and with the geometrical figures, the two ingredients, which, for centuries and centuries, were at the basis of both advanced mathematics and mathematical education and which nowadays risk to go lost for a more abstract approach based on the structures and their laws.

6. Conclusion

I hope to have clarified – although by means of a sole example – why Enriques' opinion on the lack of a very board-line between history of mathematics, mathematics education and advanced mathematics is plausible, even though not necessarily true in any aspect. To support this thesis I have considered an example which could be used at the third-fourth year in the high schools. Others could be conceived for the university level.

Rather than to summarize the content of this paper, I prefer to suggest seven lessons to develop in a classroom in order to present the material here proposed.

- Lesson 1 (after the explanation of the principle of mathematical induction): Fibonacci's method to prove that each square is the sum of a series of consecutive addends. Comparison between this method and the mathematical induction. The importance of a good mathematical symbolism. The concept of generality (a letter which indicate any number).
- Lesson 2: further examples of Fibonacci's method and comparison with modern demonstrations by mathematical induction.
- Lesson 3 and 4: interdisciplinary lessons with the teachers of history and philosophy (in the nations and schools where this discipline is taught). The problem of infinity. How the relation of the man with mathematical infinity has changed from Fibonacci's time to nowadays. Relation with the infinity in the history of philosophy. These subjects will not be – this is obvious – completely analysed in two lessons. But two lessons are sufficient to offer stimulating and interesting ideas to the pupils.

[18] Genocchi, 1855b, p. 172-175.

- Lesson 5: the equation $\frac{x^2-y^2}{y^2-z^2} = \frac{a}{b}$, when $a - b = 1$ or 2. Relation between *paradigmatic examples* and rigorous proofs. Relation between paradigmatic examples and empirical examples.
- Lesson 6 and 7: $\frac{x^2-y^2}{y^2-z^2} = \frac{a}{b}$ in the other cases. The technique of the successive addends. Its range of applicability. Extension of this technique when it is necessary to introduce negative numbers. Cases in which this technique cannot be applied. The general algorithm. *Very important*: the teacher should point out the limits inside which a method can be used. Each method has its ray of applicability. Point out that apparently similar problems need different procedures of solution.

It is easy to think of exercises connected to the methods dealt with here. For example: to solve the equations

$$\frac{x^2-y^2}{y^2-z^2} = \frac{43}{11};\ \frac{x^2-y^2}{y^2-z^2} = \frac{41}{11};\ \frac{x^2-y^2}{y^2-z^2} = \frac{17}{5}.$$

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