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Iterations of homographic functions and recurrence equations involving a homographic function*

Abstract. The formulas for the m-th iterate ($m \in \mathbb{N}$) of an arbitrary homographic function H are determined and the necessary and sufficient conditions for a solution of the equation $y_{m+1} = H(y_m)$, $m \in \mathbb{N}$ to be an infinite n-periodic sequence are given. Based on the results from this paper one can easily determine some particular solutions of the Babbage functional equation.

1. Preliminaries

The recurrence equations involving a homographic function where studied in (Graham, Knuth, Patashnik, 2002). The authors stated that the only known examples of such equations possessing periodic solutions are

$$y_{m+1} = 2\mathrm{i} \sin \pi r + \frac{1}{y_m}, \qquad m \in \mathbb{N},$$

where r is a rational number such that $0 \leq r < \frac{1}{2}$.

Various approaches to the sequences given by the recurrence equation

$$y_{m+1} = H(y_m), \qquad m \in \mathbb{N}, \tag{1}$$

where H is a homographic function may be found in (Koźniewska, 1972; Levy, Lessman, 1966; Uss, 1966; Wachniccy, 1966).

In this paper we prove formulas determining all solutions of (1). We also give the necessary and sufficient conditions for a solution of (1) to be periodic.

We also determine some particular solutions of the Babbage functional equation

$$\varphi^m(x) = x, \qquad x \in X, \tag{2}$$

where m is an arbitrary fixed integer. Recall that ψ^n for $n \in \mathbb{N}$ denotes the n-th iterate of a function $\psi \colon X \to X$, i.e. $\psi^0 = \mathrm{Id}_X$ and $\psi^n = \psi \circ \psi^{n-1}$ for integer $n \geq 1$.

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*Iteracje funkcji homograficznej i równanie rekurencyjne zadane funkcją homograficzną

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Some results concerning (2) may be found in (Kuczma, 1968). In particular the following

Theorem 1 (Kuczma, 1968, p. 291)
If φ is a meromorphic solution of equation (2), then

$$\varphi(x) = \frac{a'x+b'}{c'x+d'}$$

for some $a', b', c', d' \in \mathbb{C}$.

Theorem 2 (Kuczma, 1968, p. 291)
If $L(x) = \alpha x + \beta$, where $\alpha \neq 0$, then φ satisfies (2)if and only if $L^{-1} \circ \varphi \circ L$ does so.

Theorem 3 (Kuczma, 1968, p. 291)
Let $K_{-1} = 0$, $K_0 = 1$, $K_m = \gamma K_{m-1} + \delta K_{m-2}$ for $m \in \mathbb{N}_+$ and let $S(x) = \gamma + \frac{\delta}{x}$, where $\gamma, \delta \in \mathbb{C}$ and $\delta \neq 0$, then

$$S^m(x) = \frac{K_m x + \delta K_{m-1}}{K_{m-1}x + \delta K_{m-2}} \quad \textit{for } m \in \mathbb{N}_+. \tag{3}$$

Let

$$H(x) := \frac{ax+b}{cx+d}, \quad \text{where } a,b,c,d \in \mathbb{C},\ c \neq 0,\ ad - bc \neq 0 \tag{4}$$

In the sequel we assume that the domain of H is the set D defined as follows

$$D := \bigcap_{m \in \mathbb{N}_+} D_{H^m},$$

where D_{H^m} denotes the domain of H^m and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. We also set $H^0 := \mathrm{Id}_D$.

Let $H\colon D \to \mathbb{C}$ be a function given by (4) and let

$$y_0 = x_0 \quad \text{and} \quad y_{m+1} = H(y_m) \quad \text{for an } x_0 \in D \text{ and } m \in \mathbb{N}.$$

Notice that

$$y_m = H^m(x_0) \qquad m \in \mathbb{N} \tag{5}$$

and $x_0, H(x_0), H^2(x_0), \ldots, H^{m-1}(x_0) \in D$.

Based on the theory of recurrence linear equations of order 2 with constant coefficients (Koźniewska, 1972, p. 59) we get

Lemma 1
Let $K_{-1} = 0$, $K_0 = 1$, $K_m = \gamma K_{m-1} + \delta K_{m-2}$ for $m \in \mathbb{N}_+$, $\delta \neq 0$ and let $\Delta = \gamma^2 + 4\delta$. Then for $m \in \mathbb{N} \cup \{-1\}$,

1° $K_m = (m+1)\left(\frac{\gamma}{2}\right)^m$, *if* $\Delta = 0$,

2° $K_m = \frac{1}{\sqrt{\Delta}}\left(\left(\frac{\gamma+\sqrt{\Delta}}{2}\right)^{m+1} - \left(\frac{\gamma-\sqrt{\Delta}}{2}\right)^{m+1}\right)$, *if* $\Delta \neq 0$, *where* $\sqrt{\Delta}$ *denotes one of the complex square roots of* Δ*.*

A consequence of Lemma 1 is

Lemma 2
If $K_{-1} = 0$*,* $K_0 = 1$*,* $K_m = \gamma K_{m-1} + \delta K_{m-2}$ *for* $m \in \mathbb{N}_+$*,* $\delta \neq 0$*,* $\gamma, \delta \in \mathbb{R}$ *and* $\Delta = \gamma^2 + 4\delta$*, then for* $m \in \mathbb{N} \cup \{-1\}$*,*

1° $K_m = (m+1)\left(\frac{\gamma}{2}\right)^m$*, if* $\Delta = 0$*,*

2° $K_m = \frac{1}{\sqrt{\Delta}}\left(\left(\frac{\gamma+\sqrt{\Delta}}{2}\right)^{m+1} - \left(\frac{\gamma-\sqrt{\Delta}}{2}\right)^{m+1}\right)$*, if* $\Delta > 0$*,*

3° $K_m = (\sqrt{-\delta})^m \cos\frac{m\pi}{2}$*, if* $\gamma = 0$ *and* $\Delta < 0$*,*

4° $K_m = (\sqrt{-\Delta})^m(\cos m\psi + \cot\psi \sin m\psi)$*, if* $\gamma \neq 0$ *and* $\Delta < 0$*,*

where ψ *is the principal value of an argument of the complex number* $\frac{\gamma}{2} + \mathrm{i}\frac{\sqrt{-\Delta}}{2}$*.*

2. Periodic solutions of the recurrence equation

Definition 1
An infinite sequence $(y_m)_{m\in\mathbb{N}}$ *is called periodic with period* n *(or* n*-periodic), where* $n \in \mathbb{N}$*,* $n \geq 1$*, if* $y_{m+n} = y_m$ *for every* $m \in \mathbb{N}$*.*

Consider equation (1) with the initial condition $y_0 = x_0$, where $H\colon D \to \mathbb{C}$ is a function defined by (4) and $x_0 \in D$. By (5) we get

Lemma 3
Let $H\colon D \to \mathbb{C}$ *be a function defined by* (4) *and let* $n \geq 2$ *be a fixed integer. Every solution of* (1) *is periodic with period* n *if and only if*

$$H^n = \mathrm{Id}_D. \tag{6}$$

Proof. Assume that for some integer $n \geq 2$ equation (6) holds, then by (5) for every $m \in \mathbb{N}$ we have

$$y_{m+n} = H^{m+n}(x_0) = H^m(H^n(x_0)) = H^m(x_0) = y_m,$$

where $y_0 = x_0 \in D$. For the converse suppose that every solution of (1) is n-periodic. Let $x_0 \in D$, so $H^m(x_0) \in D$ for every $m \in \mathbb{N}$. Put $y_m := H^m(x_0)$, $m \in \mathbb{N}$. The sequence $(y_m)_{m\in\mathbb{N}}$ satisfies (1), so it is n-periodic. Thus

$$H^n(x_0) = y_n = y_0 = H^0(x_0) = x_0.$$

Hence (6) holds.

Observe that Lemma 3 holds true if H is an arbitrary function with a proper domain satisfying (2).

Theorem 4
Let $S\colon D' \to \mathbb{C}$ be a function defined as $S(x) = \gamma - \frac{\delta}{x}$, where $\gamma, \delta \in \mathbb{C}$, $\delta \neq 0$ and $D' := \bigcap_{m\in\mathbb{N}_+} D_{S^m}$, where D_{S^m} denotes the domain of S^m. Every sequence $(y_m)_{m\in\mathbb{N}}$ satisfying the following recurrence relation

$$y_{m+1} = S(y_m), \quad m \in \mathbb{N} \tag{7}$$

is 2-periodic if and only if $\gamma = 0$.

Proof. In view of Lemma 3 it follows that every sequence $(y_m)_{m\in\mathbb{N}}$ satisfying (7) is 2-periodic if and only if

$$S^2(x) = \gamma + \frac{\delta x}{\gamma x + \delta} = \mathrm{Id}_{D'}(x), \quad x \in D'.$$

Which is equivalent to the fact that $\gamma = 0$.

Now we prove the following results.

Theorem 5
Let S be as in Theorem 4, $\Delta = \gamma^2 + 4\delta$ and let $n \in \mathbb{N}$ be such that $n \geq 3$.

(i) *If every sequence $(y_m)_{m\in\mathbb{N}}$ satisfying (7) is n-periodic, then $\Delta \neq 0$ and $\delta = \frac{-\gamma^2}{4\cos^2\frac{k\pi}{n}}$ for some $k \in \{1, 2, 3, \ldots, n-1\}$.*

(ii) *If $k \in \{1, 2, 3, \ldots, n-1\}$ and $\gamma^2 + 4\delta \neq 0$ and $4\delta\cos^2\frac{k\pi}{n} = -\gamma^2$, then every sequence $(y_m)_{m\in\mathbb{N}}$ satisfying (7) is n-periodic.*

Proof. To show (i) observe that by Lemma 3 we get

$$S^n(x) = x, \quad x \in D'. \tag{8}$$

By Theorem 3 and Lemma 1, (8) is equivalent to the following conditions

$$\begin{aligned}
\frac{K_n x + \delta K_{n-1}}{K_{n-1}x + \delta K_{n-2}} - x\frac{K_{n-1}x + \delta K_{n-2}}{K_{n-1}x + \delta K_{n-2}} &= 0, \quad x \in D',\\
\frac{\gamma K_{n-1}x + \delta K_{n-2}x + \delta K_{n-1} - K_{n-1}x^2 - \delta K_{n-2}x}{K_{n-1}x + \delta K_{n-2}} &= 0, \quad x \in D',\\
\frac{K_{n-1}(-x^2 + \gamma x + \delta)}{K_{n-1}x + \delta K_{n-2}} &= 0, \quad x \in D',\\
K_{n-1} &= 0,
\end{aligned}$$

$$\Delta \neq 0 \;\text{ and }\; \left(\frac{\gamma + \sqrt{\Delta}}{2}\right)^n = \left(\frac{\gamma - \sqrt{\Delta}}{2}\right)^n,$$

$$\Delta \neq 0 \;\text{ and }\; (\gamma + \sqrt{\Delta})^n = (\gamma - \sqrt{\Delta})^n,$$

$$\Delta \neq 0 \text{ and } \exists\, k \in \{1, \ldots, n-1\} : \;\; \gamma + \sqrt{\Delta} = (\gamma - \sqrt{\Delta})\left(\cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n}\right). \tag{9}$$

Now notice that

$$\gamma+\sqrt{\Delta}=(\gamma-\sqrt{\Delta})\Big(\cos\frac{2k\pi}{n}+\mathrm{i}\sin\frac{2k\pi}{n}\Big)$$

is equivalent to the following conditions

$$\sqrt{\Delta}\Big(1+\cos\frac{2k\pi}{n}+\mathrm{i}\sin\frac{2k\pi}{n}\Big)=\gamma\Big(\cos\frac{2k\pi}{n}+\mathrm{i}\sin\frac{2k\pi}{n}-1\Big),$$

$$2\sqrt{\Delta}\cos\frac{k\pi}{n}\Big(\cos\frac{k\pi}{n}+\mathrm{i}\sin\frac{k\pi}{n}\Big)=2\gamma\sin\frac{k\pi}{n}\Big(\mathrm{i}\cos\frac{k\pi}{n}-\sin\frac{k\pi}{n}\Big),$$

$$2\sqrt{\Delta}\cos\frac{k\pi}{n}\Big(\cos\frac{k\pi}{n}+\mathrm{i}\sin\frac{k\pi}{n}\Big)=2\gamma\mathrm{i}\sin\frac{k\pi}{n}\Big(\cos\frac{k\pi}{n}+\mathrm{i}\sin\frac{k\pi}{n}\Big). \tag{10}$$

Thus condition (9) is equivalent to

$$\Delta\neq 0 \text{ and } \exists\; k\in\{1,\dots,n-1\}\ \sqrt{\Delta}\cos\frac{k\pi}{n}=\gamma\mathrm{i}\sin\frac{k\pi}{n},$$

$$\Delta\neq 0 \text{ and } \exists\; k\in\{1,\dots,n-1\}\ \sqrt{\Delta}=\gamma\mathrm{i}\tan\frac{k\pi}{n},$$

$$\Delta\neq 0 \text{ and } \exists\; k\in\{1,\dots,n-1\}\ \Delta=-\gamma^2\tan^2\frac{k\pi}{n},$$

$$\Delta\neq 0 \text{ and } \exists\; k\in\{1,\dots,n-1\}\ \delta=\frac{-\gamma^2}{4\cos^2\frac{k\pi}{n}},$$

which completes the proof of (i).

For the implication (ii) consider two cases:

a. n is an even number and $k=\frac{n}{2}$,

b. n is an even number and $k\neq\frac{n}{2}$ or n is odd and $k\in\{1,2,3,\dots,n-1\}$.

In the case a, we get $\gamma=0$ and according to Theorem 4 every sequence satisfying (7) is 2-periodic and hence n-periodic.

For the case b, notice that for every $k\in\{1,2,3,\dots,n-1\}$, $\cos\frac{k\pi}{n}\neq 0$ we have $\Delta\neq 0$ and $\delta=\frac{-\gamma^2}{4\cos^2\frac{k\pi}{n}}$ which yields $\Delta=-\gamma^2\tan^2\frac{k\pi}{n}$. Denote by $\sqrt{\Delta}$ the number $\gamma\mathrm{i}\tan\frac{k\pi}{n}$, thus $\sqrt{\Delta}\cos\frac{k\pi}{n}=\gamma\mathrm{i}\sin\frac{k\pi}{n}$ which is equivalent to (10). Now reversing the reasoning from the case (i) – from condition (10) to (8) (without condition (9)) – finishes the proof.

The results obtained above will be now applied to examine the sequences defined by (1).

Theorem 6

If $H\colon D\to\mathbb{C}$ is a function given by (4) *and* $(K_{-1},K_0,K_1,\dots)$ *is a sequence defined in Theorem 3 for which* $\gamma=a+d$ *and* $\delta=bc-ad$, *then*

$$H^m(x)=\frac{1}{c}\frac{cK_mx+dK_m+\delta K_{m-1}}{cK_{m-1}x+dK_{m-1}+\delta K_{m-2}}-\frac{d}{c}\quad \textit{for } m\in\mathbb{N}_+,\ L(x)\in D. \tag{11}$$

Proof. Let $\gamma = a+d$, $\delta = bc-ad$, $L(x) = \frac{x}{c} - \frac{d}{c}$ and $S(x) = (L^{-1}\circ H \circ L)(x)$, $L(x) \in D$. We have

$$L^{-1}(x) = cx + d,$$
$$(H\circ L)(x) = \frac{ax+bc-ad}{cx},$$
$$S(x) = (L^{-1}\circ H\circ L)(x) = a+d+\frac{bc-ad}{x} = \gamma + \frac{\delta}{x},$$

for $L(x) \in D$, where $\delta \neq 0$. By Theorem 3 we obtain

$$S^m(x) = \frac{K_m x + \delta K_{m-1}}{K_{m-1}x + \delta K_{m-2}} \quad \text{for } m \in \mathbb{N}_+,\ L(x) \in D.$$

Now observe that

$$S^m = L^{-1}\circ H^m \circ L \quad \text{for } m \in \mathbb{N}_+,$$

thus

$$H^m = L \circ S^m \circ L^{-1} \quad \text{for } m \in \mathbb{N}_+,$$

which gives (11).

Lemma 3 and Theorem 2 yield

Theorem 7
Let $H\colon D \to \mathbb{C}$ be a function defined by (4), $L(x) = \frac{x}{c} - \frac{d}{c}$, $S(x) = (L^{-1}\circ H\circ L)(x)$, $L(x) \in D$ and let $n \geq 2$ be a fixed integer. Then every solution of (1) is n-periodic if and only if

$$S^n = \mathrm{Id}_D.$$

3. Examples

From the proof of Theorem 5 it follows that condition

$$\gamma^2 + 4\delta \neq 0 \ \text{ and } \ \exists\, k \in \{1,\dots,n-1\}\ \sqrt{\gamma^2+4\delta} = -\gamma \mathrm{i} \tan\frac{k\pi}{n} \tag{12}$$

is equivalent to the fact that every sequence $(y_m)_{m\in\mathbb{N}}$ satisfying equation

$$y_{m+1} = \gamma + \frac{\delta}{y_m}$$

or equation

$$y_{m+1} = \frac{ay_m + b}{cy_m + d},$$

where $a+d = \gamma$ and $bc-ad \neq 0$, is n-periodic with $n \geq 3$.

Moreover, it is easy to find numbers γ, δ satisfying (12) and a, b, c, d - solutions of the system $a+d=\gamma$, $bc-ad \neq 0$. Namely, for $n=3$, $\gamma = 1$, $\delta = -1$ (12) is fulfilled and numbers $a = 2$, $b = -3$, $c = 1$, $d = -1$ satisfy the system $a + d = 1$, $bc - ad = -1$, thus in view of Theorem 5 and Lemma 3 the following functions

$$S(x) = 1 + \frac{-1}{x}, \qquad H(x) = \frac{2x - 3}{x - 1}$$

fulfil the Babbage equation $\varphi^3(x) = x$ (which can be directly checked).

Now let $n = 4$, for $\gamma = 2$, $\delta = -2$ (12) holds true. Let $a = 3$, $b = -5$, $c = 1$ and $d = -1$, then $a + d = 2$, $bc - ad = -2$. Similarly as above we get that the mappings

$$S(x) = 2 + \frac{-2}{x}, \qquad H(x) = \frac{3x - 5}{x - 1}$$

satisfy equation $\varphi^4(x) = x$.

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