

Annales Universitatis Paedagogicae Cracoviensis

Studia ad Didacticam Mathematicae Pertinentia 17(2025)

ISSN 2080-9751

DOI 10.24917/20809751.17.6

Piotr Błaszczuk, Anna Petiurenko

Commentary to Book III of Euclid's *Elements**

Abstract. We provide a commentary on the propositions of Book III of Euclid's *Elements*, accompanying the Polish translation included in this volume. Focusing on the circle and its relation to triangles, we analyze Proposition III.32 and show that, within absolute geometry, if some right-angled triangle admits a circumscribed circle, then all triangles in the plane are Euclidean (with sum of angles equal two right angles). In this context, we also revisit Saccheri's theory, in which the existence of a Euclidean right-angled triangle implies the Parallel Postulate, and discuss Lobachevsky's analysis of an angle of parallelism.

Contents

| | | |
|---|---|-----|
| 1 | Introduction | 106 |
| 2 | Inside, outside, and the border. III.1–8 | 106 |
| 3 | Three points determine the circle. III.9–10 | 110 |
| 4 | Touching circles. III.11–13 | 111 |
| 5 | The distance of a point from a line. III.14–15 | 112 |
| 6 | Tangent to a circle – a definition. III.16–19 | 113 |
| 7 | Angles, arcs, and chords. III.20–30 | 117 |
| | 7.1. Segments of a circle | 119 |
| | 7.2. Equal angles on equal circumferences | 121 |
| 8 | Circumscribing and inscribing an angle in a circle. III.31–34 | 124 |

*2020 Mathematics Subject Classification: Primary 01A120; Secondary 01A45, 01A60

Keywords and phrases: *Euclid, Saccheri, Lobachevsky, Parallel Postulate, Archimedean axiom*

| | | |
|--------------|---|------------|
| 9 | An alternative to the Parallel Postulate | 127 |
| 9.1. | Axiom T | 127 |
| 9.2. | From Axiom T to Proposition I.32 | 128 |
| 9.3. | From Axiom T to Proposition III.16 | 130 |
| 10 | Tangent to a circle – an <i>iff</i> criterion. III.35–37 | 130 |
| 11 | Saccheri’s theory | 132 |
| 11.1. | Clavius’s Axiom and the Lambert Quadrilateral | 134 |
| 11.2. | From a Euclidean triangle to the Parallel Postulate | 135 |
| 11.3. | Saccheri and Lobachevsky | 139 |

1. Introduction

Book III of the *Elements* addresses foundational issues that are no less significant than those in Book I. While Book I develops much of its theory through the study of triangles, Book III turns to the geometry of the circle, systematically applying properties of isosceles and right-angled triangles.

Certain results concerning circumscribed circles can, under suitable assumptions, be related to the Parallel Postulate. The famous Proposition IV.5 (that every triangle admits a circumscribing circle) is known to be equivalent to the Parallel Postulate. Other properties of circles can also be formulated as equivalent versions of the Parallel Postulate in the presence of additional assumptions, such as the Archimedean axiom. From this perspective, we examine Proposition III.32 and show that if there exists a right-angled triangle that admits a circumscribed circle, then all triangles in the plane are Euclidean (that is, with sum of angles equal π). This result establishes a bridge between local geometric constructions and global properties of the plane.¹

In this context, we revisit Saccheri’s theory of quadrilaterals, in which he attempts to derive the Parallel Postulate from the existence of a Euclidean right-angled triangle. We also discuss Lobachevsky’s analysis of parallel lines, based on the assumption that there exist triangles whose angle sum is less than π . These approaches illuminate, from different directions, the interplay between triangle geometry, circle geometry, and the theory of parallels.

2. Inside, outside, and the border. III.1–8

Euclid already provides a definition of a circle in Book I, namely Definition I.15: “A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from a single point lying inside the figure are equal to one another.”²

The first definition in Book III provides the equality criterion for circles: “Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).”

¹Throughout the paper, π denotes two right angles, in the sense of Euclid’s Fourth Postulate.

²All translations of the *Elements* follow (Fitzpatrick, 2007).

Although Euclid does not provide explicit definitions of inside and outside of a circle, they follow easily from Definition I.15, based on whether the distance from the center is less than or greater than the radius.

In Propositions III.1–8, he assumes that the center of a circle exists. Then, in III.9, he constructs the center of a circle passing through three given points.

In **III.1**, Euclid finds the center of a given circle. The construction is now standard: he selects two points on the circumference, A and B , finds the midpoint D of the segment AB , draws a line perpendicular to AB through D , and determines a point F on this line. He then shows that F is the center of the circle.

Although the construction is simple, its justification is subtle.

Euclid selects a point G inside the circle which does not lie on the line through F and D , and shows that $AG \neq BG$. He then concludes: “So, similarly, we can show that neither is any other (point) than F .”

The argument thus proceeds as follows: by definition, a circle has a center. Since no point other than F satisfies the conditions required of the center, F must be the center.

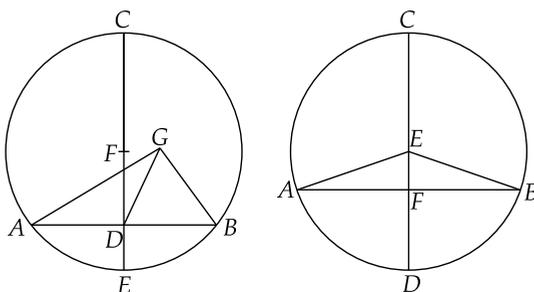


Figure 1: Finding the center of a circle: Proposition III.1 (left) and III.3 (right)

Since AB lies inside the circle (as shown in the next proposition), the point F also lies inside the circle.

Finally, let us examine how Euclid shows that $GA \neq GB$. The triangles $\triangle ADG$ and $\triangle BDG$ are congruent by Proposition I.8. Consequently,

$$\angle ADG = \angle BDG = \frac{\pi}{2}.$$

By construction, the angle $\angle ADF$ is also right. Euclid concludes: “Thus, $\angle ADF$ (is) equal to $\angle GDB$, the greater equal to the lesser. The very thing is impossible.”

In the final step, Euclid implicitly appeals to the trichotomy law for angles.

In **III.2**, Euclid considers two points A and B on a circle and proves that the straight line joining them lies entirely inside the circle (see Fig. 2). In doing so, he also establishes – albeit not explicitly – that an arc of a circle is distinct from a line segment.

Throughout the proof, Euclid considers three possibilities: the segment AB lies outside the circle, lies on the circumference, or lies inside it. The proof proceeds by *reductio ad absurdum* and seeks to dismiss the first two possibilities.

Suppose that the segment AB lies outside the circle. Euclid draws a segment from the center D to a point E on AB , which meets the circumference at F .

In triangle $\triangle ADB$, Proposition I.5 implies that $\angle A = \angle B$. Next, in triangle $\triangle ADE$, by Proposition I.16 the exterior angle $\angle BED$ is greater than $\angle A$, and hence greater than $\angle B$.

By Proposition I.19, $DB > DE$, since it subtends the greater angle. However, $DB = DF$, as both are radii of the circle. Euclid concludes: “Thus, DF (is) greater than DE , the lesser than the greater. The very thing is impossible.”

For the second possibility, Euclid writes: “So, similarly, we can show that neither (will it fall) on the circumference itself.”

Indeed, the dismissal of the case in which the arc AFB coincides with the segment AB is even simpler. In triangle $\triangle ADF$, the exterior angle at F is greater than $\angle A$, and hence greater than $\angle B$. However, in the isosceles triangle $\triangle FDB$, we have $\angle F = \angle B$. Thus, $\angle F$ is both greater than and equal to $\angle B$, which is a contradiction.

In this way, Euclid distinguishes between the arc AB and the segment AB . The key argument relies on Proposition I.16 and proceeds within absolute geometry.

Finally, let us note a similarity between the diagrams of Propositions III.2 and I.4. In III.2, Euclid considers three possible positions of the segment AB , while in I.4 he considers two possible positions of the segment EF (see Fig. 2, middle and right).

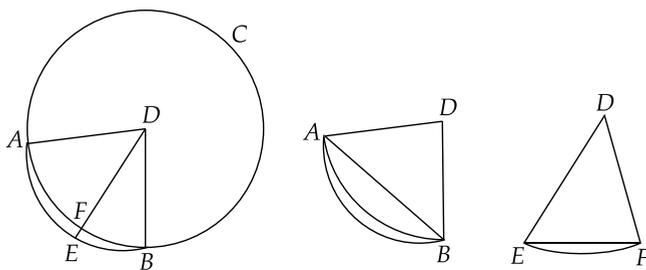


Figure 2: Chords in a circle: Proposition III.2 (left). Comparing III.2 and I.4 (middle and right)

In **III.3**, Euclid shows that if a diameter bisects a chord (other than a diameter), then it meets it at right angles. Conversely, if it meets a chord at right angles, then it also bisects it.

To prove this, he uses the fact that the triangles $\triangle EFA$ and $\triangle EFB$ are isosceles. Then, by applying the SAS and SSA criteria, he obtains the desired result (see Fig. 1).

In **III.4**, Euclid shows that intersecting chords (other than diameters) do not intersect at their midpoints.

The proof is a *reductio ad absurdum*. Suppose that the chords AB and CD bisect each other at E . Draw a diameter through E and the center F . By Proposition III.3, since the diameter passes through the midpoint E of the chord AB , it

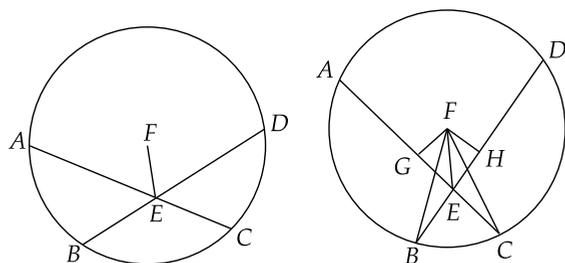


Figure 3: Propositions III.4 (left) and III.35 (right)

is perpendicular to AB ; hence, $\angle AEF$ and $\angle BEF$ are right angles.

Euclid concludes: “Thus, FEA (is) equal to FEB , the lesser to the greater. The very thing is impossible.”

We note in advance a deeper result proved later in Book III. In Proposition III.35, Euclid shows that the products $AE \cdot EC$ and $BE \cdot ED$ are equal. This result plays a central role in the geometry of the circle, revealing an invariant relation for intersecting chords.

In **III.5**, Euclid shows that intersecting circles cannot have the same center. The proof proceeds by *reductio ad absurdum* and uses the trichotomy law for line segments.

In **III.6**, he shows that internally tangent circles do not have the same center. The proof also proceeds by *reductio ad absurdum*.

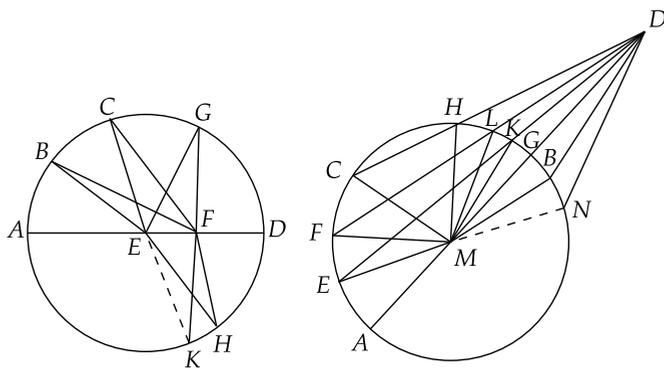


Figure 4: Propositions III.7 and III.8

In **III.7**, Euclid compares line segments such as FB , FC , and FG (see Fig. 4, left). To this end, he applies Propositions I.20–23, relying on the triangle inequality. This proposition may be viewed as a refinement of Proposition I.4: *If the angles between equal sides of two triangles differ, then their third sides also differ.*

Let us also recall that in Proposition I.7, Euclid proves that “two (given)

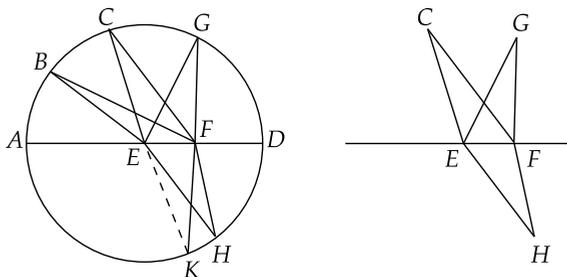


Figure 5: Comparing Proposition III.7 and its connection to I.7

straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side”. In Proposition III.7, he shows that such lines – namely, EG and FG on one side, and EH and FH on the other – can indeed be constructed, but only on opposite sides of the base EF .

In III.8, Euclid compares line segments such as DG , DK , DH , and DC (see Fig. 4, right). Throughout the proof, Proposition I.24 plays a crucial role.

3. Three points determine the circle. III.9–10

Proposition III.9 states that if points A , B , and C lie on a circle and are all equidistant from a point D , then D is the center of the circle.

Euclid assumes that A , B , and C lie on a circle. He finds the midpoint E of segment AB and the midpoint F of segment BC . He then draws ED and FD and demonstrates that these lines are perpendicular to AB and BC , respectively. By Proposition III.1, the center of the circle lies on these perpendicular bisectors; consequently, it must be their intersection point, which is D .

Note that, by Proposition III.2, the points A , B , and C are not collinear.

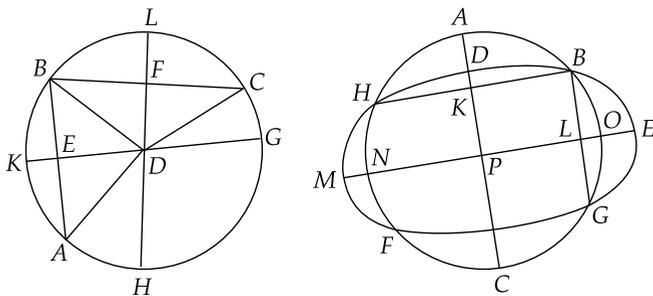


Figure 6: Propositions III.9 and III.10

Euclid applies a similar construction in Proposition IV.5, where he assumes that the perpendicular bisectors of the sides AB and AC of a triangle intersect. He then considers three possible positions of the intersection point: inside the triangle

(see Fig. 7, left), on the side BC (see Fig. 7, middle), or outside the triangle (see Fig. 7, right).

Euclid does not explicitly prove that these bisectors intersect; in fact, this assumption relies on the Parallel Postulate. The converse also holds: the existence of a circumscribing circle for an arbitrary triangle implies the Parallel Postulate.

By contrast, in Proposition III.9, the intersection point D is given in advance; thus, the proof does not rely on an unstated assumption.

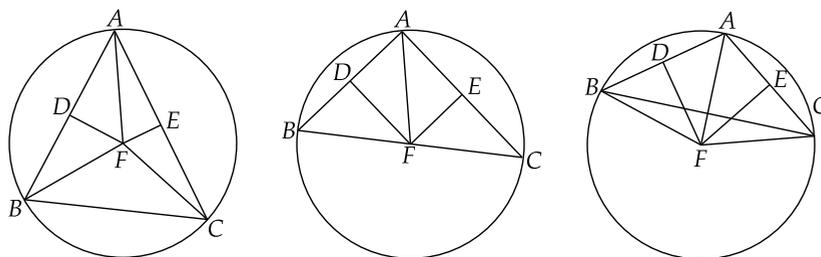


Figure 7: Proposition IV.5: circumscribing a circle around a triangle

Proposition **III.10** follows from Proposition III.9. It states that: “A circle does not cut a(nother) circle at more than two points.”

The accompanying diagram appears as if the circles share four points. Euclid shows that if two circles share three points, then they must have the same center. Therefore, by Definition III.15, the circles are equal.

4. Touching circles. III.11–13

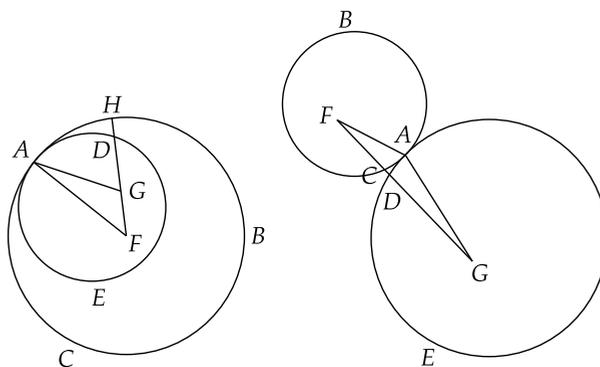


Figure 8: Propositions III.11 and III.12

In touching circles – whether they touch internally (Proposition **III.11**) or externally (Proposition **III.12**) – the line joining their centers passes through the point of contact. Furthermore, the point of contact is unique.

Both proofs proceed by *reductio ad absurdum* and rely on the triangle inequality.

These results serve as lemmas for Proposition III.13, where Euclid shows that a circle tangent to another, whether internally or externally, has exactly one point of contact.

5. The distance of a point from a line. III.14–15

In III.14, Euclid implicitly introduces the concept of the distance from a point to a line via the Pythagorean theorem – the prototype of the modern notion of distance in Euclidean spaces \mathbb{R}^n . The proposition states that in a circle, equal chords are equally distant from the center; equivalently, the center lies at equal distances from equal chords. The proof relies on the Pythagorean theorem (I.47).

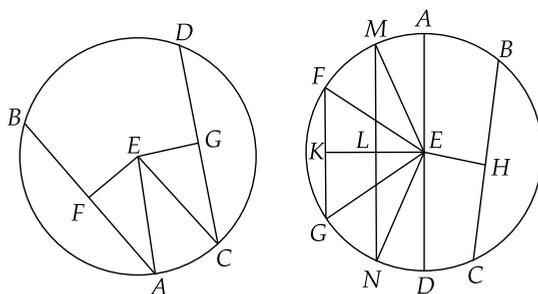


Figure 9: Propositions III.14 (left) and III.15 (right)

In the right-angled triangles $\triangle EFA$ and $\triangle EGC$, we have $FA = GC$ because the chords are equal, and $EA = EC$ because they are radii. Therefore, in modern notation:

$$FE^2 + FA^2 = EG^2 + GC^2.$$

Euclid concludes: “Thus, the remaining (square) on FE is equal to the (remaining square) on EG ,” which can be interpreted as the perpendicular distances from the center to the chords being equal.

The converse follows from the same equality, assuming $EF = EG$.

In Euclid’s framework, the Pythagorean theorem requires the *Parallel Postulate*. However, to prove Proposition III.14, the congruence criterion *Right Angle–Side–Side* (RASS) suffices.

The proof of RASS uses only the SAS criterion, as follows (see Fig. 10). Suppose that in the right-angled triangles $\triangle ACB$ and $\triangle A'C'B'$ we have $AB = A'B'$ and $AC = A'C'$, but $C'B' > CB$. Let D' be a point on $C'B'$ such that $C'D' = CB$. Then, by Proposition I.4, the triangles $\triangle ACB$ and $\triangle A'C'D'$ are congruent, and hence $AB = A'D'$. By Proposition I.16, $\angle A'D'B' > \pi/2$, which implies that in the isosceles triangle $\triangle A'D'B'$ the sum of two angles exceeds π , contradicting Proposition I.17.

In other words, the proof is carried out entirely within *absolute geometry*.

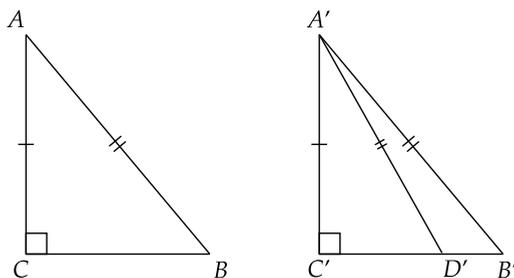


Figure 10: Proving the RASS criterion

In **III.15**, Euclid shows that a chord closer to the center of a circle is longer than one that is farther away. For the proof, he constructs a chord MN equal to BC and considers the triangles $\triangle MEN$ and $\triangle FEG$, where E and F are the feet of the perpendiculars from the center to the respective chords. Since $\angle MEN > \angle FEG$, it follows from Proposition I.24 that the base MN is greater than the base FG (see Fig. 9, right).

6. Tangent to a circle – a definition. III.16–19

Definition III.3 states: “A straight-line said to touch a circle is any (straight-line) which, meeting the circle and being produced, does not cut the circle.” In III.16, Euclid shows that a line perpendicular to a diameter at the point where it meets the circle satisfies this condition.

In III.17, he demonstrates how to construct a tangent to a circle passing through a given point. In III.18, he proves that a line satisfying Definition III.3 is perpendicular to a radius of the circle. In III.19, he shows that the perpendicular to the tangent drawn at the point of contact passes through the center of the circle.

Thus, a tangent to a circle is perpendicular to a radius, and no other line through that point is tangent to the circle.

In *La Géométrie*, Descartes understands a tangent as a line perpendicular to a diameter passing through its endpoint. Indeed, this is how we define the tangent to a circle today.

In Proposition III.16, Euclid also compares rectilinear angles with angles formed by a semicircle and a tangent. This part of Proposition III.16 was widely discussed throughout the 17th century; however, it had no lasting impact on modern mathematics.

Proposition **III.16** states that a line perpendicular to a diameter at the point where the diameter meets the circle lies outside the circle.

To prove this, Euclid considers three possibilities: the perpendicular lies inside the circle, on the circumference, or outside the circle. The condition “lies inside” means that some point of the line lies inside the circle and that the line cuts the circumference.

Suppose that AC lies inside the circle (see Fig. 11). Then, in the isosceles

triangle $\triangle CDA$, angles $\angle A$ and $\angle C$ are right angles, which contradicts I.17 (the sum of any two angles in a triangle is less than π).

The same argument applies to the second case: “So, similarly, we can show that neither (will it fall) on the circumference.” Euclid concludes: “Thus, (it will fall) outside (the circle).”

Moreover, in the statement of III.16, Euclid asserts: “And another straight line cannot be inserted into the space between the (aforementioned) straight line and the circumference. And the angle of the semicircle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).”

The proof of this claim proceeds by *reductio ad absurdum*. Suppose that FA lies in the space between the tangent AE and the circumference of the circle. Euclid drops a perpendicular from D to FA , namely the line DG . In triangle $\triangle DGA$, since $\angle DGA$ is a right angle, the side DA is greater than DG . However, it is also smaller, since $DG = GH + HD$ and $HD = DA$. “The very thing is impossible.”

The proof assumes $DG \neq DA$; in other words, according to Euclid, it is not possible for distinct lines – such as EA and FA – to have a common perpendicular like DA .

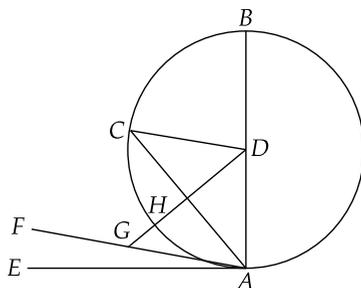


Figure 11: Proposition III.16

Finally, Euclid addresses the controversial part of Proposition III.16, namely the comparison between rectilinear angles and angles with one arm formed by a semicircle: “the semicircular angle contained by the straight line BA and the circumference CHA is greater than any acute rectilinear angle whatsoever.”³

Euclid writes: “if any rectilinear angle is greater than the (angle) contained by the straight-line BA and the circumference CHA [...], then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE – anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line BA and the circumference CHA ”.

The proof relies on the previous claim that no straight line can be inserted between EA and the arc CHA .

³Let us recall that Definitions 8 and 9 in Book I introduce the concept of an angle: Definition 8 applies to any lines, while Definition 9 refers specifically to straight lines.

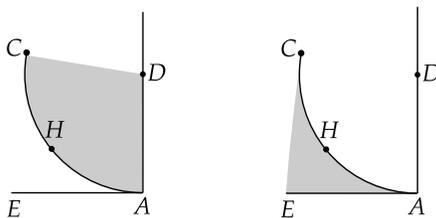


Figure 12: Angle greater than any acute angle (left). Angle less than any acute angle (right)

The second part proceeds similarly: “if any rectilinear angle is [...] less than the (angle) contained by the circumference CHA and the straight-line AE [...], then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE – anything which will make an angle contained by straight-lines less than the (angle) contained by the circumference CHA and the straight line AE ”.

Euclid thus assumes the following:

- (1) The angle formed by the arc CHA and the diameter BA is comparable with any rectilinear angle.
- (2) The angle formed by the arc CHA and the tangent EA is comparable with any rectilinear angle.
- (3) The difference between the angle formed by the arc CHA and the diameter BA is determined by a rectilinear angle.
- (4) The difference between the angle formed by the arc CHA and the tangent EA is determined by a rectilinear angle.

(Ad 1 and 2) In Euclid’s framework, the greater-than relation between rectilinear angles is not formally defined. Nevertheless, it typically reflects configurations such as $\angle EAF < \angle EAC$, where the line FA lies between the lines EA and CA (see Fig. 13, left).

In modern geometry, the relation *lies between* for lines is defined as follows: let a line l intersect BA , CA , and DA at points A_1 , A_2 , and A_3 , respectively. If the points satisfy the order $A_1 - A_2 - A_3$, then the line CA lies between the lines BA and DA .

It can be shown that the betweenness relation among the lines CA , BA , and DA does not depend on the choice of the line l .

Any straight line such as FA , other than the tangent EA , intersects the arc CHA , so we cannot provide an analogous definition for angles with one rectilinear and one curvilinear arm.

Such a relation can be illustrated by the tangent line t and another line l . In modern mathematics, the angle between a circle and a line is defined as the angle between that line and the tangent to the circle (see Fig. 13, middle).

(Ad 3 and 4) Note that there are circles – such as those with centers D , D_1 , and D_2 (see Fig. 13, right) – that share the common tangent EA . The angles formed by the tangent and the respective arcs – the so-called horn angles – may

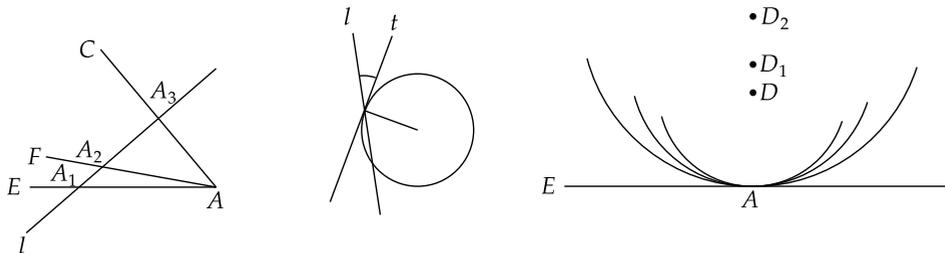


Figure 13: The relation of *betweenness* for lines (left); an angle between a line and a circle (middle); greater and lesser horn angles (right)

appear greater or smaller; however, the difference between them is not given by rectilinear angles.

From III.16 onward, a tangent is understood as a line perpendicular to a diameter at its endpoint.

In III.17, Euclid provides a construction of a tangent to a given circle from a point lying outside it. The following table explains this construction.

| | | | |
|-------------|--------------|----------------------|--------------|
| | $AE, (E, a)$ | $(A, b), AE \perp D$ | $FE, (E, a)$ |
| $A, (E, a)$ | D | F | B |

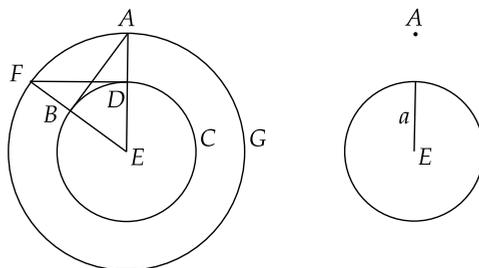


Figure 14: Proposition III.17 (left). Starting the construction (right)

Let the given circle have the radius a , that is, (E, a) is given, and let $AE = b$, with $b > a$. Triangles $\triangle ABE$ and $\triangle FDE$ share the common angle at E ; moreover, $BE = DE$ and $AE = FE$. By I.4, the triangles are equal, and since $\angle EDF = \pi/2$, it follows that $\angle ABE$ is a right angle. Therefore, BA is tangent to the circle BDC (see Fig. 14).

In III.18, Euclid shows that if a line meets a circle at a single point, then the line drawn from that point to the center is perpendicular to the line.

Let F be the center, and C the point at which the line DE meets the circle. By Proposition I.12, there exists a perpendicular to DE through F . Suppose it intersects DE at a point G , distinct from the point of contact C . Then, in the right triangle $\triangle FCG$, the side FC , being opposite the greater angle, is greater

than FG . However, FC is a radius, while FG would have to be greater—since G lies outside the circle—which leads to a contradiction (see Fig. 15, left).

In **III.19**, Euclid states the converse: if DE is tangent to a circle and C is the point of contact, then the perpendicular to DE drawn through C passes through the center of the circle (see Fig. 15, right).

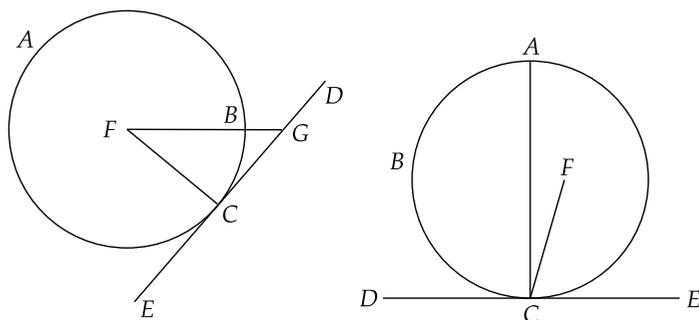


Figure 15: Propositions III.18 and III.19

7. Angles, arcs, and chords. III.20–30

Propositions III.20 and III.21 concern inscribed and central angles. Their proofs rely on earlier results—specifically, Propositions I.5, I.32, and I.13, concerning adjacent (supplementary) angles. A new concept introduced here is that of an arc; Euclid refers to it using the terms *circumference base* and *segment of a circle*.

A segment of a circle is defined in Definition III.5 as follows: “A segment of a circle is the figure contained by a straight-line and the circumference of a circle.” However, in Proposition III.21, the relevant object is technically an arc of a circle determined by a central angle greater than π .⁴

In **III.20**, Euclid shows, by a simple argument involving isosceles triangles, that

$$2\angle BAC = \angle BEC$$

(see Fig. 16, left).

The second part of the proposition, $2\angle BDC = \angle BEC$, illustrates Euclid's kind of algebra. Since

$$2\angle GDC = \angle GEC, \quad 2\angle GDB = \angle GEB,$$

and

$$2\angle GDC - 2\angle GDB = \angle GEC - \angle GEB,$$

it follows that

$$2\angle BDC = \angle BEC.$$

⁴Note that what we call an angle greater than π is not considered an angle in Euclid's system.

□

Let us now turn to the concept of an arc.

III.20 reads: “In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.”

Interestingly, the concept of a *circumference base* does not play an explicit role in this proof – it relies solely on isosceles triangles (see Fig. 16, right). We may observe that the arc BFC determines the angle $\angle BEC$. Since the circle – and therefore its radius – is given, this arc also determines the base of the triangles $\triangle BAC$, $\triangle BEC$, and $\triangle BDC$.

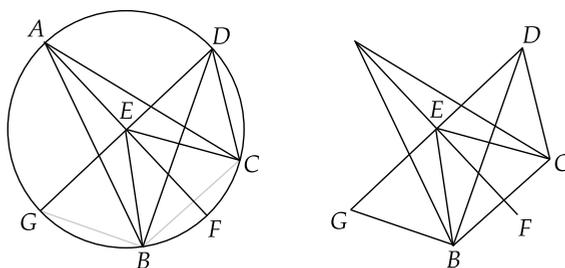


Figure 16: Propositions III.20 (left); lines relevant to its proof (right)

Proposition III.21 reads: “In a circle, angles in the same segment are equal to one another.”

Since the circle is given, the segment $BAED$ – as referred to by Euclid – determines the chord BD , the common base of the triangles $\triangle BFD$, $\triangle BAD$, and $\triangle BED$. By III.20, and essentially through arguments based on isosceles triangles, Euclid concludes that

$$\angle BAD = \angle BED$$

(see Fig. 17).

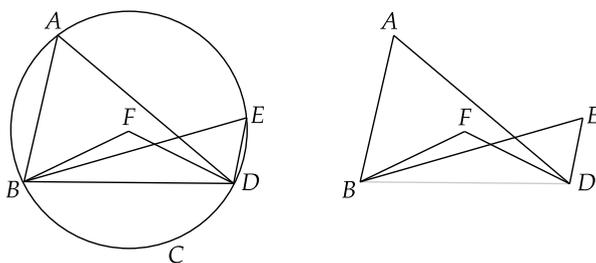


Figure 17: Proposition III.21 (left); lines relevant to its proof (right)

To summarize the role of the concepts *segment of a circle* and *circumference base* in Propositions III.20–21: the former determines a chord, the latter a central angle. In a given circle, both define the same isosceles triangle, with the radius as its sides.

Proposition **III.22** states that in a quadrilateral inscribed in a circle, opposite angles sum to π (see Fig. 18, left).

The proof draws on Propositions III.21 and I.32 (the sum of the angles in a triangle).

In Propositions I.5 and I.7, Euclid considers triangles with a common base and equal sides on the same side of the base. In Proposition I.5, the equal sides extend from different endpoints of the base, while in Proposition I.7, they extend from the same endpoints. Proposition III.21 also concerns triangles with a common base. It can be reformulated as follows: if two triangles, such as $\triangle ABD$ and $\triangle EBD$, share a common base BD , and the angles $\angle A$ and $\angle E$ subtending that base are equal, then the two triangles are inscribed in the same circle.

The standard proof proceeds by *reductio ad absurdum*. Assuming that there is a circle circumscribing triangle $\triangle ABD$, we obtain three equal angles: $\angle A = \angle E' = \angle E$. In the triangle $\triangle DE'E$, the exterior angle $\angle E'$, by I.16, is greater than the interior opposite angle $\angle E$ (see Fig. 18, right) – a contradiction.

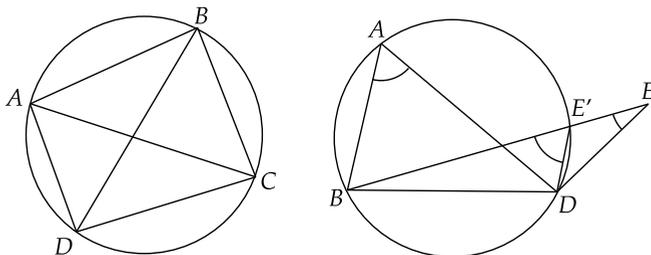


Figure 18: Proposition III.22 (left); the reverse of Proposition III.21 (right)

7.1. Segments of a circle

Definition III.8 reads: “And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.”

By Proposition III.21, all these angles are equal.

Definition III.11 reads: “Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.”

Due to this definition and Proposition III.21, similar segments of circles are segments determined by equal chords. Propositions III.22–25 explore this concept in a manner analogous to the treatment of equal triangles in Propositions I.4 and I.7.

III.23 reads: “Two similar and unequal segments of circles cannot be constructed on the same side of the same straight line.”

The proof proceeds by *reductio ad absurdum*. If such a construction were possible, then the angles of the respective segments could be arranged as shown

in Fig. 19. By I.16, the angle $\angle ACB$ is greater than the angle $\angle ADB$ – a contradiction.

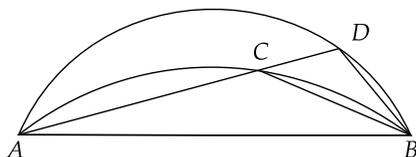


Figure 19: Proposition III.23

III.24 reads: “Similar segments of circles on equal straight lines are equal to one another.”

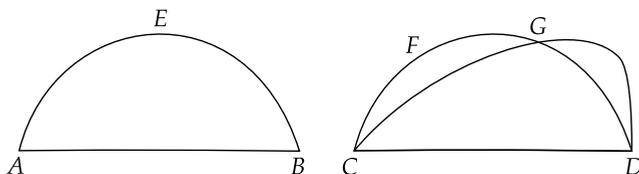


Figure 20: Proposition III.24

To prove this, Euclid applies segment AEB to segment CFD , and as part of this operation, segment AB to segment CD . He then argues: “For if the straight line AB coincides with CD , and the segment AEB does not coincide with CFD , then it will surely either fall inside it, outside it, or it will miss, like CGD .”

The cases in which the segment falls inside or outside contradict the preceding proposition. The third case involves the intersection of circles, which contradicts Proposition III.10.

Propositions I.4 and III.24 can thus be summarized as follows: two points determine a line segment; three non-collinear points determine a circle.

Proposition **III.25** amounts to the following: given an arc of a circle, find the center of the circle to which it belongs.

Euclid joins the endpoints A and C of the arc, finds the midpoint D of the segment AC , draws a perpendicular to AC through D , and locates the point B as the intersection of this perpendicular with the arc.

Through this construction, he obtains the triangle $\triangle ADB$ and writes: “Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD ” (see Fig. 21).

Accordingly, in both the first and third cases, he transports the angle $\angle DBA$ onto the line BA at the point A (see Fig. 21, left and middle).

Here, in Proposition III.25, for the first time in the *Elements*, Euclid constructs an isosceles triangle with a given angle at the base. If we treat the Parallel Postulate as a constructive tool, then in each case the center E of the circle appears as

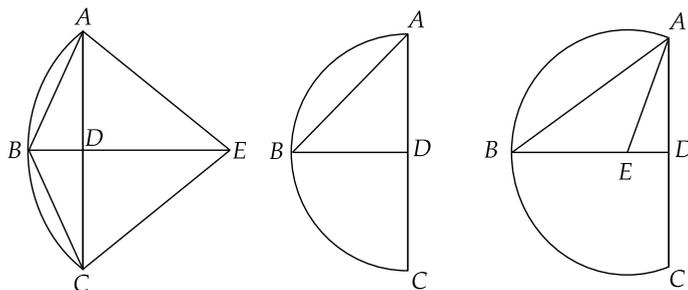


Figure 21: Proposition III.25

the intersection point of two lines whose existence is guaranteed by the Postulate (see Fig. 21, left and middle).

For comparison, let us see how Robin Hartshorne (Hartshorne, 2000, p. 100) introduces the existence of an isosceles triangle with base AB within the Hilbert system (see Fig. 21, right).

Let C be a point not on the line AB —this is guaranteed by Axiom I3. Regarding the angles $\angle A$ and $\angle B$, one of the following holds: $\angle A < \angle B$, $\angle A = \angle B$, or $\angle A > \angle B$.

In the first case, we transport the angle $\angle A$ to the point B . By Pasch's axiom, the line intersects side AC at D . By Euclid's Proposition I.6, the triangle $\triangle ADB$ is isosceles.

In Euclid's system, the existence of an isosceles triangle follows from the concept of a circle.

In Proposition III.25, Euclid constructs an isosceles triangle with a given base angle. Hartshorne's proof, however, is not constructive; it proceeds within absolute geometry to support arguments that, in Euclid's system, rely on properties of isosceles triangles.

7.2. Equal angles on equal circumferences

Proposition **III.26** reads: "Equal angles stand upon equal circumferences in equal circles, whether they are standing at the center or at the circumference."

In the respective circles, triangles $\triangle BGC$ and $\triangle EHF$, by Proposition I.4, have equal bases, BC and EF , respectively. By III.11, the segments of circles BAC and EDF are similar. Since they stand on equal segments BC and EF , they are also equal. Thus the circles ABC and EDF are equal. By Common Notion 3, the circumferences BKC and ELF are equal. In Euclid's words: "Thus, the remaining circumference BKC is equal to the (remaining) circumference ELF ."

However, by subtracting the circular segment BAC from the circle $ABCK$, we obtain the circular segment BKC , not merely the arc (circumference) BKC .

Proposition **III.27** reads: "Angles standing upon equal circumferences in equal circles are equal to one another, whether they are standing at the center or at the circumference".

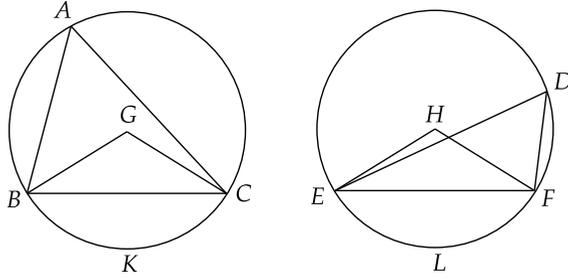


Figure 22: Proposition III.26

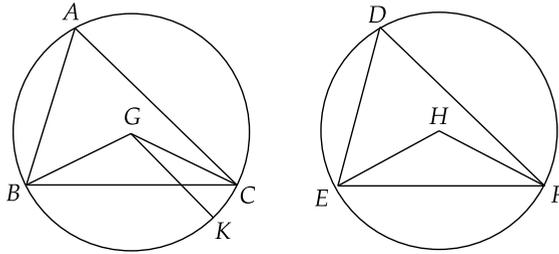


Figure 23: Proposition III.27

Let angles $\angle BGC$ and $\angle EHF$ stand upon equal arcs (circumferences) BKC and EF , respectively (see Fig. 23). Suppose $\angle BGC > \angle EHF$ and let the angle $\angle BGK$ be equal to the angle $\angle EHF$. Then Euclid writes: “Thus, circumference BK (is) equal to circumference EF . But EF is equal to BC . Thus, BK is also equal to BC , the lesser to the greater. The very thing is impossible.”

Thus, equal central angles imply equal arcs (circumferences), while the observation that K lies between B and C (on the arc) follows from the diagram. In sum, the concept of circumference does not add anything to this argument.

When, instead of equal arcs (circumferences), we refer to equal chords BC and EF , the proof can proceed as follows. Suppose $\angle BGC > \angle EHF$, and let $\angle BGK$ be equal to $\angle EHF$; then we obtain a network of isosceles triangles (see Fig. 24). In triangle $\triangle BKC$, we have $\angle BKC = \angle BCK$, and in triangle $\triangle GKC$, we have $\angle GKC = \angle GCK$. On the one hand, $\angle BKC > \angle GKC$; on the other, $\angle GCK > \angle BCK$ —a contradiction.

Indeed, in the following propositions, Euclid shows that equal chords determine equal arcs, and vice versa.

Proposition **III.28** reads: “Equal straight-lines cut off equal circumferences in equal circles.”

In fact, a chord cuts off two arcs—as Euclid observes—a greater and a lesser. It is also assumed that a diameter is a special case of a chord. Let us focus on the case of the lesser arcs.

The equality of chords $AB = DE$ determines the equality of central angles $\angle AKB = \angle DLE$. By III.26, equal central angles determine equal arcs.

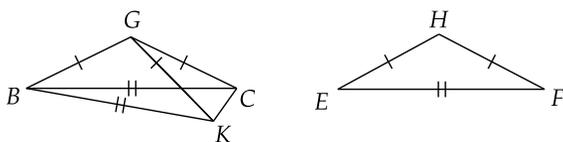


Figure 24: Alternative proof of Proposition III.27

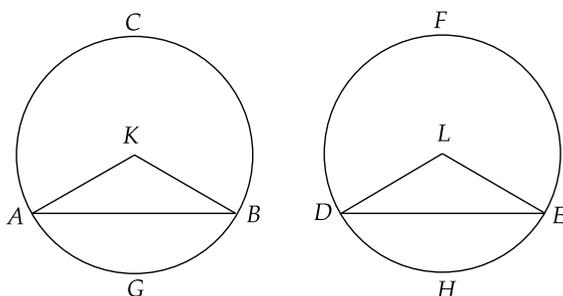


Figure 25: Proposition III.28

Proposition **III.29** reads: “Equal straight-lines subtend equal circumferences in equal circles.”

Euclid writes: “since the circumference BGC is equal to the circumference EHF, the angle BKC is also equal to (angle) ELF”. In this argument, he relies on Proposition III.27. Thus – as we have already observed – the equality of an arc reduces to the equality of the respective central angles. Since angles $\angle AKB$ and $\angle DLE$ are equal, the equality of chords AB and DE follows by Proposition I.4.

For the sake of the proof – and for the clarity of the diagrams – Euclid considers equal chords in equal circles. However, the same argument applies to equal chords in the same circle. Indeed, Ptolemy and modern mathematics apply Euclid’s results developed in Propositions III.20–29 to angles, chords, and arcs of a given circle.

As we have observed, the concept of an arc, as used by Euclid, does not contribute anything beyond the concept of a central angle; therefore, one of them can be abandoned.

Indeed, in modern mathematics, central angles and arcs are identified through the concept of radian measure.⁵

Proposition **III.30**: “To cut a given circumference in half.”

It is analogous to Proposition I.10, which concerns bisecting a segment. Euclid finds the midpoint C of the chord AB , which is determined by the arc. He then constructs two equal triangles, $\triangle ACD$ and $\triangle BCD$. Since the chords AD and BD – the respective sides of these triangles—are equal, it follows by III.28 that the arcs AD and BD are also equal.

⁵See Błaszczyk (2025).

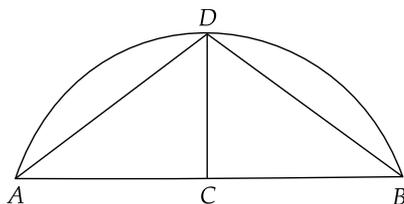


Figure 26: Proposition III.30

A chord determines two arcs, and the next proposition specifies that relationship.

8. Circumscribing and inscribing an angle in a circle. III.31–34

Proposition **III.31** reads: “In a circle, the angle in a semi-circle is a right-angle, that in a greater segment is less than a right-angle, and that in a lesser segment is greater than a right-angle.”

The proof relies on the theory of triangles developed in Book I and on Proposition III.22, applied to the quadrilateral $BADC$.

Let us examine how this works in the case of an angle in a semicircle, using a simplified, modern stylization. The angles at the vertex E of the isosceles triangles $\triangle AEB$ and $\triangle AEC$ sum to π (see Fig. 27, left). Therefore,

$$\angle BAC = \frac{1}{2}(\pi - \angle AEB) + \frac{1}{2}(\pi - \angle AEC) = \pi/2.$$

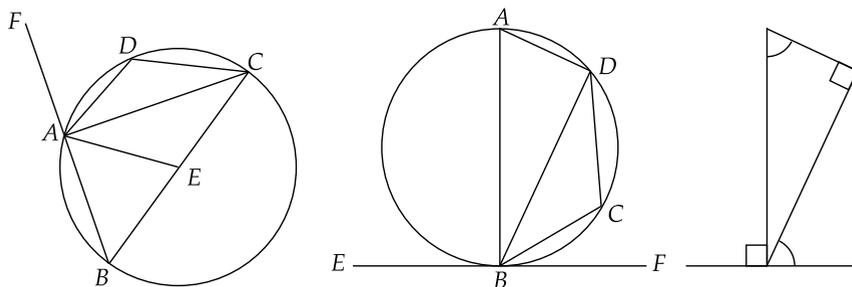


Figure 27: Propositions III.31 (left), III.32 and the scheme of its proof (middle and right)

In Proposition **III.32**, the diameter AB is perpendicular to the tangent EF (see Fig. 27, middle). Euclid shows that the angles $\angle BAD$ and $\angle DBF$ are equal. Although the observation is insightful, the proof is straightforward. In modern terms:

$$\angle BAD = \pi/2 - \angle ABD = \angle DBF.$$

Both proofs clearly rely on Proposition I.32 concerning the sum of the angles in a triangle.

Propositions III.33–34 include constructions such as “To draw a segment of a circle, subtending an angle equal to a given rectilinear angle” and “To cut off a segment, subtending an angle equal to a given rectilinear angle”. In effect, these propositions amount to constructing the circumcircle of a triangle and inscribing in a given circle a triangle equiangular to a given one.

Proposition **III.33** requires drawing a segment of a circle that stands on a chord subtending a given angle. Since a circular segment determines the entire circle, this amounts to finding the center of the circle. Throughout the construction, Euclid assumes that the side subtending the angle is given. This assumption allows us to extend his method to constructively prove the ASA and RASS triangle congruence criteria – that is, to construct a triangle when a side and two angles are given.

The proof consists of three cases: when the given angle is acute, a right angle, or obtuse. We shall focus on the first (see Fig. 28, left).

Euclid transports the angle C to line DA at A and draws a right angle with arm DA . He then writes: “And let AB have been cut in half at F . And let FG have been drawn from point F , at right angles to AB . And let GB have been joined.”

Euclid assumes that a perpendicular to AB drawn through the midpoint F meets the line AE . Since G is the center of the circle, FG is perpendicular to AB . Thus, Euclid obtains a chord AB of the circle that subtends an angle equal to C .

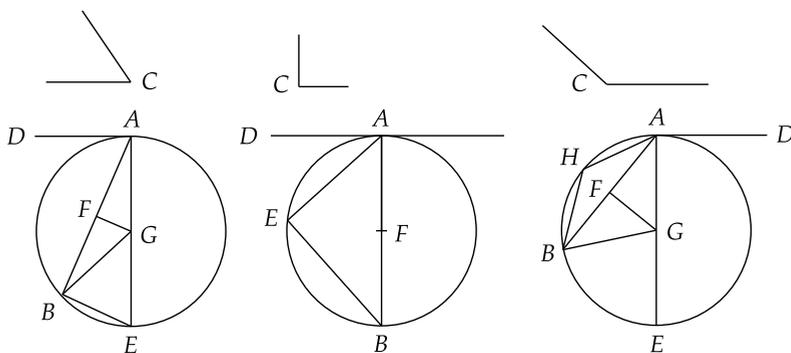


Figure 28: Proposition III.33

Proposition **III.33a**. Now, let us assume that C is an angle in a triangle with a given side and another given angle (see Fig. 29). Using Euclid’s construction, we find a circle with a chord subtending an angle equal to $\angle C$. Transporting the second angle to the existing chord AB at point B determines the third vertex of the triangle.

The RASS congruence criterion applies to the case in which the hypotenuse and one side of a right-angled triangle are given. In this construction, it means

that the hypotenuse coincides with the diameter of the circle.

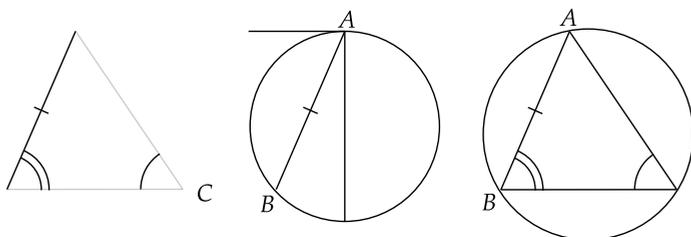


Figure 29: Proposition III.33a

Proposition **III.34** concerns inscribing an angle in a circle; however, it effectively amounts to inscribing in a circle a triangle that is equiangular – and hence similar (from the perspective of Book VI) – to a given one.

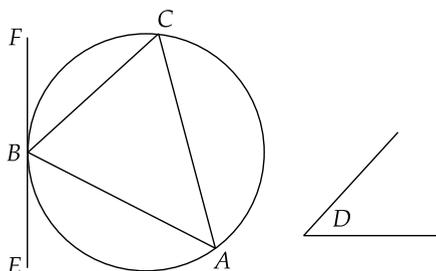


Figure 30: Proposition III.34

Since the circle is given, Euclid draws a radius and a tangent to the circle – line EBF (see Fig. 30). He then transports the angle $\angle D$ to line BF at point B and by III.32 obtains a chord BC subtending an angle $\angle BAC$, $\angle BAC = \angle D$. In fact, A is an arbitrary point on the corresponding arc of the circle.

We can extend Euclid’s construction to inscribe in a given circle a triangle equiangular to a given one (see Fig. 31). By transporting the angle α , we obtain the chord BC ; similarly, the chord BA is determined. By Proposition I.32, it follows that $\angle ABC = \gamma$.

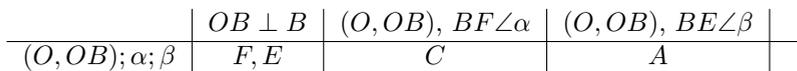


Figure 32 recalls Euclid’s proof of Proposition I.32, as schematized in the same figure. Indeed, there is a deep connection between I.32 and III.32, which forms the basis of Proposition III.34. This connection will be discussed in the next section.

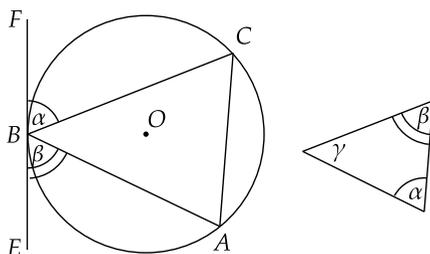


Figure 31: From Proposition III.34 to inscribing a triangle in a circle.

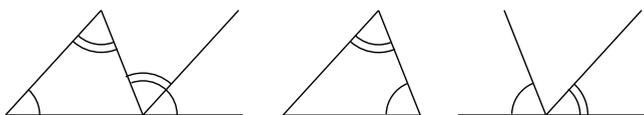


Figure 32: Scheme of the proofs I.32 (left) and III.34 (middle and right)

9. An alternative to the Parallel Postulate

Assuming the Archimedean axiom, Proposition I.32 is equivalent to the Parallel Postulate.⁶ By adopting the thesis of III.32 as an axiom, we can derive Proposition I.32, giving it foundational significance.

9.1. Axiom T

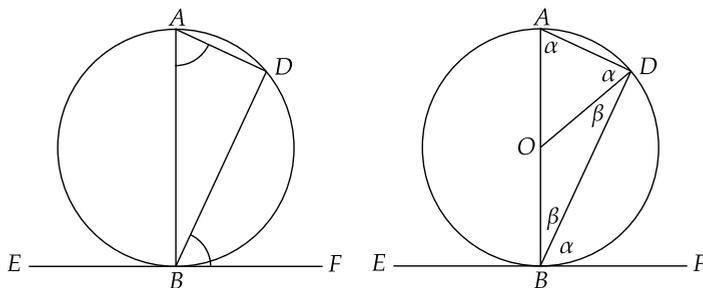


Figure 33: Axiom T (left) and deriving a Euclidean triangle from it (right)

Axiom (T). Let EBF be a tangent to a circle with diameter AB and chord BD . Then $\angle BAD = \angle DBF$ (see Fig. 33, left).

Given axiom (T), we can show that any triangle is Euclidean, i.e., its angles sum to π .⁷

⁶See § 10 below.

⁷In (Błaszczuk and Petiurenko, 2021), we show that axiom (T) does not entail the Parallel

Consider the central triangles $\triangle AOD$ and $\triangle BOD$ (Fig. 33, right). By axiom (T), $\alpha + \beta$ at vertex B equals $\pi/2$, so $\angle D = \pi/2$. Consequently, the sum of angles in $\triangle ADB$ is π , since $2\alpha + 2\beta = \pi$.

Since one Euclidean triangle exists, Saccheri's theory implies that every triangle in the plane is Euclidean.⁸

Even without Saccheri's theory, the existence of one right-angled Euclidean triangle implies the existence of a rectangle – on the assumption of the Archimedean axiom – of arbitrary size. Then every right-angled triangle is Euclidean. Next, by Proposition I.17, any triangle can be divided by an altitude into two right-angled triangles; hence, all triangles are Euclidean.

Now we proceed to explore applications of axiom (T) to tangents and circumscribed circles.

9.2. From Axiom T to Proposition I.32

We now give an alternative proof of Proposition I.32 relying only on axiom (T) and absolute geometry.

Lemma. Let $\triangle ABC$ be right-angled at C , with hypotenuse AB , and let O be the midpoint of AB . Then the circle with center O and radius OB circumscribes $\triangle ABC$ (Fig. 34).

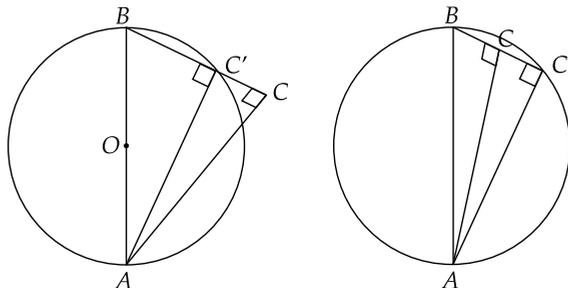


Figure 34: A circle circumscribing a right-angled triangle using axiom (T)

Proof. Suppose C does not lie on (O, OB) , and let the circle intersect ray BC at C' . By Axiom T, $\angle AC'B$ is right, contradicting Proposition I.16.

If (O, OB) does not intersect the triangle legs (Fig. 35, left), an arbitrary point C' on the semicircle yields $\angle AC'B$ right, but by Proposition I.21, $\angle C' > \angle C$. If there is only one intersection point (Fig. 35, right), Proposition I.16 is again contradicted. \square

Now, consider any triangle $\triangle ABC$ with angles α, β, γ , and let $AD \perp BC$ (by Proposition I.17; see Fig. 36). In the right-angled triangles $\triangle ADB$ and $\triangle ADC$, the angles sum to π , so $\alpha + \beta + \gamma = \pi$.

Postulate. In more detail, in that model all triangles are Euclidean, axiom (T) is satisfied, yet the Parallel Postulate fails.

⁸See § 10 below.

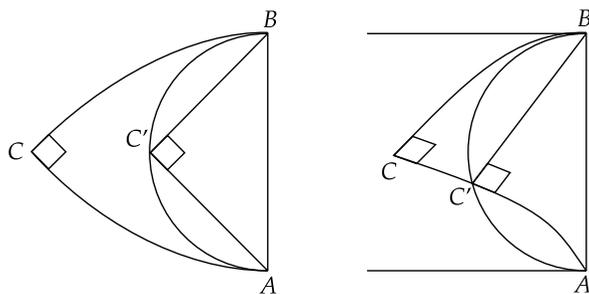


Figure 35: Cases where the legs of $\triangle ABC$ do not intersect the circumscribing circle

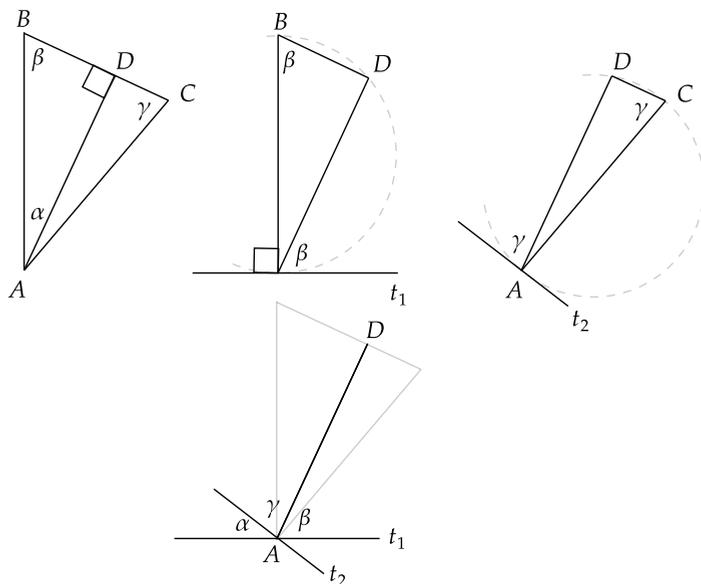


Figure 36: Using Axiom T to show that the sum of angles in any triangle is π

A more visual argument uses tangents: Let $t_1 \perp AB$ and $t_2 \perp AC$. Triangle $\triangle ADB$ is circumscribed by a circle, with β formed by chord AD and tangent t_1 (Axiom T). Similarly, $\triangle ADC$ has angle γ formed by AD and tangent t_2 . The angle between t_1 and t_2 is α . Since $\alpha + \beta + \gamma = \pi$, any triangle is Euclidean.

Thus, adopting Axiom T or assuming all triangles are Euclidean, every right-angled triangle is circumscribable. Therefore, the thesis that any right-angled triangle can be circumscribed is equivalent to the Parallel Postulate, assuming the Archimedean axiom.⁹ By contrast, Euclid's Proposition IV.5 shows that any

⁹See § 10 below.

triangle can be circumscribed, which is also equivalent to the Parallel Postulate.

9.3. From Axiom T to Proposition III.16

Proposition III.16 asserts: “Another straight line cannot be inserted between the tangent and the circle,” i.e., no line lies in the region between a tangent and the circle. Euclid’s proof assumes that distinct lines cannot share the same perpendicular. Here is an alternative proof using axiom (T).

Suppose line BC lies between t (the tangent at B) and the circle, forming angle β with the tangent (Fig. 37). Transporting this angle to AB at A , Axiom T gives a right-angled triangle $\triangle AC'B$ with $\alpha + \beta = \pi/2$. However, at B , $\alpha + \beta < \pi/2$, because CB does not intersect the semicircle, leading to a contradiction.

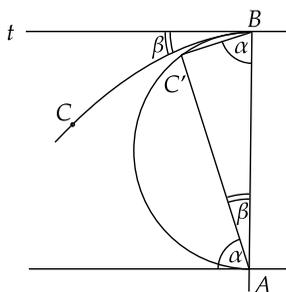


Figure 37: Using axiom (T) to prove that no line lies between a tangent and a circle

10. Tangent to a circle – an iff criterion. III.35–37

We present Euclid’s Propositions III.35–III.37 in modern notation for clarity.¹⁰

Proposition **III.35** states that for intersecting chords, the following equality holds:

$$AE \cdot EC = DE \cdot EB \quad (\text{Fig. 38}).$$

Euclid’s proof is somewhat elaborate, as he does not use differences. We provide a modern presentation:

In the circle ABC , let $FB = r = FC$. Consider the right-angled triangles $\triangle FGC$ and $\triangle FHB$:

$$GC^2 + GF^2 = BH^2 + FH^2.$$

Determining GF and FH in the right-angled triangles $\triangle FGE$ and $\triangle FHE$, we obtain

$$GC^2 + FE^2 - GE^2 = BH^2 + FE^2 - HE^2,$$

¹⁰In Euclid’s terms, the equality of products $AE \cdot EC = DE \cdot EB$ is expressed as the equality of rectangles with sides AE, EC and DE, EB respectively. The same applies to Propositions III.36 and III.37.

or equivalently,

$$GC^2 - GE^2 = BH^2 - HE^2.$$

By Proposition II.5,

$$AE \cdot EC = GC^2 - GE^2, \quad DE \cdot EB = BH^2 - HE^2,$$

so that

$$AE \cdot EC = DE \cdot EB.$$

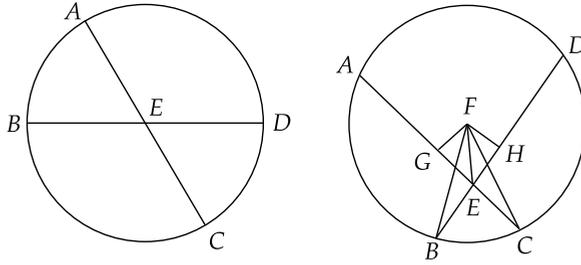


Figure 38: Proposition III.35

Proposition **III.36**: If BD is tangent to the circle ABC , then

$$BD^2 = AD \cdot DC \quad (\text{Fig. 39}).$$

If the line AD passes through the center of the circle,

$$BD^2 = BF^2 + DF^2 \quad \text{or} \quad BD^2 = CF^2 + DF^2.$$

By II.6,

$$DA \cdot DC = CF^2 + DF^2,$$

so

$$BD^2 = DA \cdot DC.$$

In the general case,

$$DB^2 = DE^2 - CE^2 = (DF^2 - EF^2) - (CF^2 - EF^2) = DF^2 - CF^2.$$

By II.6 again,

$$DA \cdot DC = DF^2 - CF^2,$$

and therefore,

$$DB^2 = DA \cdot DC.$$

Proposition III.37 provides a criterion for a line to be tangent to a circle

III.37: If $DB^2 = DC \cdot DA$, then $DB \perp FB$, where F is the center of the circle and FB is the radius.

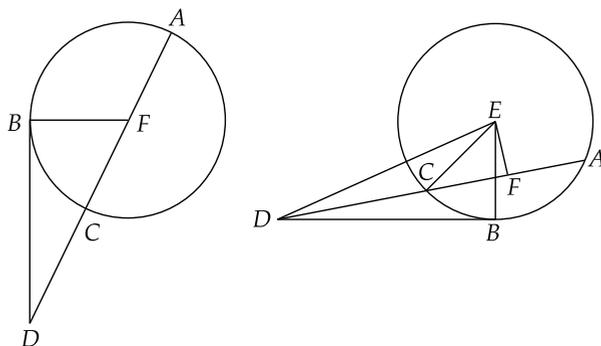


Figure 39: Proposition III.36

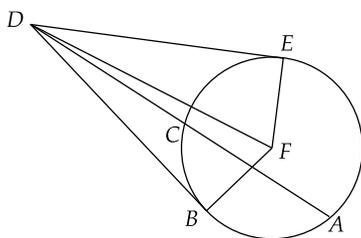


Figure 40: Proposition III.37

Let $DE \perp FE$, where FE is the radius. Then

$$DE^2 + EF^2 = DF^2.$$

By III.36, we have

$$DE^2 = DC \cdot DA,$$

so that $DB = DE$ and

$$DB^2 + BF^2 = DF^2.$$

By I.48, triangle $\triangle DBF$ is right-angled, which implies that DB is tangent to the circle ABC .¹¹

11. Saccheri's theory

We reconstruct the theory developed by Gerolamo Saccheri in his 1733 treatise *Euclides vindicatus*, Propositions 1–11.¹²

By definition, a Saccheri quadrilateral (hereafter abbreviated S-quadrilateral) has two right angles at the base, $\angle A = \frac{\pi}{2} = \angle B$, and equal sides, $AC = BD$ (see

¹¹In (Błaszczyk, Petiurenko, 2025), we present Poincaré's disk model of hyperbolic geometry based on Propositions III.17 and III.37.

¹²See (Saccheri, 2014).

Fig. 41, left). By the SAS criterion, its diagonals are equal, $AD = CB$; then, by SSS, triangles $\triangle ACD$ and $\triangle BDC$ are congruent, yielding equality of the summit angles: $\angle C = \angle D$.¹³

Since the summit angles are equal, Saccheri developed a trichotomy: they are acute, right, or obtuse. His proofs rely systematically on this classification. A key theorem asserts that if there exists one S-quadrilateral with acute summit angles, then all S-quadrilaterals have summit angles of the same type; similarly for right or obtuse angles.

We focus here on S-quadrilaterals with right summit angles.

Saccheri shows that if such an S-quadrilateral exists – i.e., if a rectangle exists – then a Euclidean triangle exists; it follows that all triangles are Euclidean. Moreover, the existence of at least one Euclidean triangle implies the Parallel Postulate, assuming the Archimedean axiom.¹⁴

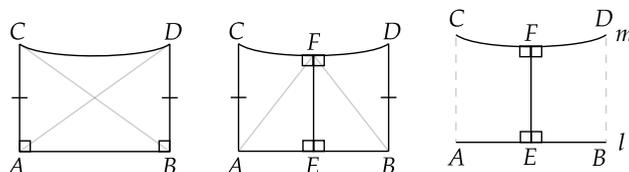


Figure 41: Saccheri's Propositions 1 and 2 (left and middle); view from the midline (right)

Proposition 2. The line EF joining the midpoints of AB and CD – called the midline – intersects AB and CD at right angles (see Fig. 41, middle).¹⁵

By SAS, triangles $\triangle ACF$ and $\triangle BDF$ are congruent, since F is the midpoint of CD , hence $CF = FD$. The congruence of triangles $\triangle AEF$ and $\triangle BEF$ implies $\angle AEF = \angle BEF = \frac{\pi}{2}$, and similarly, $\angle CFE = \angle DFE = \frac{\pi}{2}$. □

For Saccheri, the midline is an auxiliary tool; in hyperbolic geometry, it takes on a more fundamental role. Let m and l be non-intersecting, non-limiting parallel lines. There exists a unique common perpendicular EF (Fig. 41, right).¹⁶

Choose points C, D on m such that $CF = FD$, and drop perpendiculars to l , obtaining A and B . Applying SAS and AAS, $ABCD$ is an S-quadrilateral. The midline determines the class of S-quadrilaterals with bases on l . Without using the Archimedean axiom, S-quadrilaterals sharing the midline EF have summit angles of the same type: if one has acute summit angles, all do, and similarly for right or obtuse angles.¹⁷

Proposition 3. In an S-quadrilateral $ABCD$:

- If the summit angles are right, then $CD = AB$;

¹³Single-letter notation for angles is used when no confusion arises.

¹⁴See (Błaszczyk and Petiurenko, 2021) for a model in which all triangles are Euclidean but the Parallel Postulate fails.

¹⁵Numbering of propositions follows (Saccheri, 2014).

¹⁶See (Saccheri, 2014, Proposition I.30).

¹⁷See (Hartshorne, 2000, pp. 307–309).

- If obtuse, then $CD < AB$;
- If acute, then $CD > AB$ (Fig. 42).

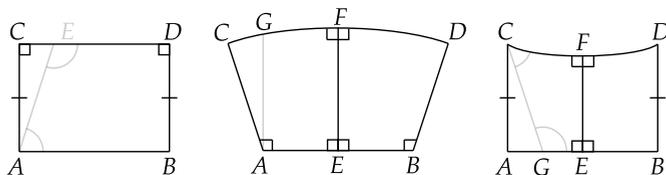


Figure 42: Saccheri's Proposition 3: right (left), obtuse (middle), and acute (right) summit angles

Proof (outline).

Case i. Suppose $AB < CD$ and let E satisfy $ED = AB$. Then $BDAE$ is an S-quadrilateral with base BD , so $\angle BAE = \angle AED$. By the exterior angle theorem (I.16), $\angle AED > \pi/2$ while $\angle BAE < \pi/2$, a contradiction. Similarly, $AB > CD$ leads to a contradiction.

Case ii. Let EF be the midline of $ABCD$. If $CF = AE$, then in S-quadrilateral $EFAC$ with base EF , $\angle ACF = \pi/2$, contradicting the assumption. Suppose $CF > AE$ and let G satisfy $AE = GF$. In the S-quadrilateral with base EF , $\angle EAG$ and $\angle FGA$ are acute, so by the exterior angle theorem, $\angle C < \angle FGA < \pi/2$, contradicting the obtuse assumption.

Case iii. If $CF < AE$, let G satisfy $CF = GE$. In S-quadrilateral $EFGC$, the summit angles are equal, but by the exterior angle theorem, $\angle EGC > \pi/2$, hence $\angle GCF > \pi/2$, so $\angle ACF > \pi/2$, contrary to the assumption. \square

11.1. Clavius's Axiom and the Lambert Quadrilateral

In the next proposition, Saccheri proves the converse of Proposition 3: if $CD = AB$, then the summit angles of the S-quadrilateral $ABCD$ are right; if $CD > AB$, they are acute; and if $CD < AB$, they are obtuse.

The first part of this converse allows us to reject an alternative form of the Parallel Postulate based on the notion of an *equidistant line*, commonly attributed to Christoph Clavius.

Clavius's Axiom. The locus of points equidistant from l is a straight line (see Fig. 43, left).

Let l be a straight line and A a point such that $A \notin l$.

Indeed, if the set of points equidistant from l forms a line p , then $ABCD$ is an S-quadrilateral (see Fig. 43, middle). According to Clavius's axiom, its midline EF – being perpendicular to CD – is equal to both AC and BD . By the SAS congruence criterion, $AE = CF$; and by the RASS criterion, applied to triangles $\triangle FEC$ and $\triangle AFE$, we obtain $AF = CE$. It follows that the S-quadrilateral $ABCD$ is a rectangle. Hence, by Proposition 11 (under the Archimedean assumption), the Parallel Postulate follows. However, this argument – inasmuch as it invokes Proposition 11 – depends on the Archimedean axiom.

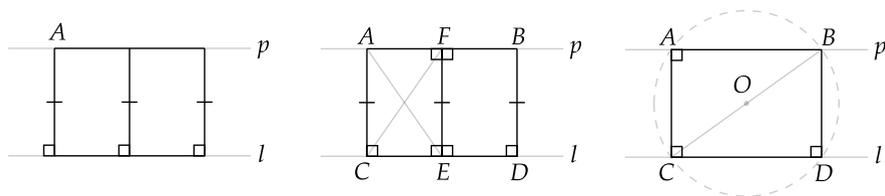


Figure 43: Clavius's axiom (left), its relation to S-quadrilaterals and Axiom T (middle and right)

On the other hand, let A be a point not on l . Dropping from A a perpendicular n to l , and then drawing through A a perpendicular to n – denoted p – we obtain a line parallel to l ; that is, p and l do not meet. Assuming the Parallel Postulate in Playfair's form, p is the unique line parallel to l through A .

Take a point B on p and let D be its perpendicular projection onto l . By construction, the quadrilateral $ABCD$ has three right angles, forming a *Lambert quadrilateral* (see Fig. 43, right). If O is the midpoint of CB , then, by the argument presented in § 8.2 – essentially, axiom (T) – we can circumscribe a circle about $ABCD$; consequently, by Euclid's Proposition III.22, the figure is a rectangle, and hence $AC = BD$.

Using the model of the semi-Euclidean plane presented in (Błaszczyk and Petiurenko, 2021), we can show that Clavius's axiom is not equivalent to the Parallel Postulate. Nevertheless, a slight modification of the above argument shows that Clavius's axiom is equivalent to the statement that all triangles in the plane are Euclidean.

11.2. From a Euclidean triangle to the Parallel Postulate

In Propositions 5–7, Saccheri establishes his well-known results: if there exists one S-quadrilateral with right summit angles (i.e., a rectangle), then all S-quadrilaterals are rectangles. The same holds for S-quadrilaterals with obtuse or acute summit angles. We present the proof for the rectangular case.

Proposition 5. If there exists one rectangle, then every S-quadrilateral is a rectangle.

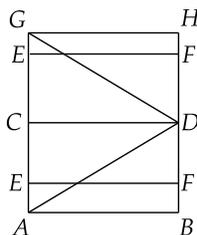


Figure 44: Saccheri's Proposition 5

Let $ABCD$ be a rectangle. On the extensions of the lines AC and BD , choose points G and H such that $CG = AC$ and $DH = BD$. Then $ABGH$ is an S-quadrilateral with base AB (see Fig. 44).

Applying the SAS criterion to the triangles $\triangle ACD$ and $\triangle DCG$, we obtain $AD = DG$ and $\angle ADC = \angle CDG$. Hence, $\angle ADB = \angle GDH$, and the triangles $\triangle ABD$ and $\triangle GHD$ are congruent. Consequently, the summit angles at G and H are right angles, $AB = GH$, and the S-quadrilateral $ABGH$ is a rectangle.

Thus, given the rectangle $ABCD$, we obtain a rectangle with twice the height. Saccheri argues that in this way the perpendiculars to AB can be “increased indefinitely” – an assumption that implicitly relies on the Archimedean axiom.¹⁸

We now supplement Saccheri’s argument by showing that a smaller rectangle with base AB also exists. Starting with the rectangle $ABGH$, whose midline is CD ,¹⁹ we observe that $ABCD$ is also a rectangle.

In $ABCD$, choose points E and F such that $AE = BF$. Then $ABEF$ and $CDEF$ are S-quadrilaterals with bases AB and CD , respectively.

If the summit angles of $ABEF$ are not right, they must be either obtuse or acute. In the first case, $AB > EF$. Then, in the S-quadrilateral $CDEF$, the summit angles are acute, and hence $CD < EF$. Since $CD = AB$, this yields a contradiction.

Similarly, one shows that the summit angles of $ABEF$ cannot be obtuse. Therefore, $ABEF$ is a rectangle. By the same reasoning, $GHEF$ is also a rectangle.

Finally, $EFEF$ is a rectangle with base $EF = AB$ but with smaller height, since $EE < AG$.

By the same argument, we can increase or decrease the base of a given rectangle, since in $ABCD$ we may also take CD as the base.

Thus, by varying the size while preserving the base, every S-quadrilateral can be reduced to the given case.

In summary, assuming the Archimedean axiom, if there exists one rectangle, then all S-quadrilaterals are rectangles. □

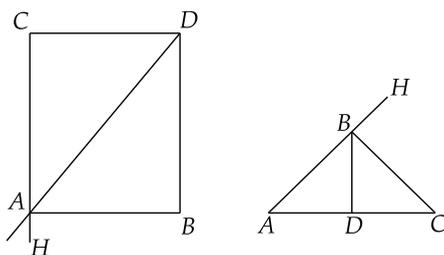


Figure 45: Saccheri’s Propositions 9 and 15

In Proposition 9, Saccheri shows that if a rectangle exists, then every right-

¹⁸See (Saccheri, 2014, p. 77).

¹⁹Strictly speaking, CD is the midline of the rectangle $GAHB$.

angled triangle is Euclidean. Indeed, given a right-angled triangle $\triangle ABD$, we construct the S-quadrilateral $ABCD$ (with base AB and sides AC, BD), which – by the preceding proposition – is a rectangle (see Fig. 45, left). It follows that the acute angles of $\triangle ABD$ sum to $\frac{\pi}{2}$.

In Proposition 15, Saccheri proves that the existence of a rectangle implies that the angles of any triangle sum to π . This follows from the fact that all right-angled triangles are Euclidean and that any triangle can be divided into two right-angled triangles (see Fig. 45, right).

Proposition 11. The existence of a Euclidean right-angled triangle implies the Parallel Postulate.

Saccheri's original argument is quite involved; we use the fact (established above) that the existence of one Euclidean right-angled triangle implies that all triangles are Euclidean, and we give a concise reconstruction.

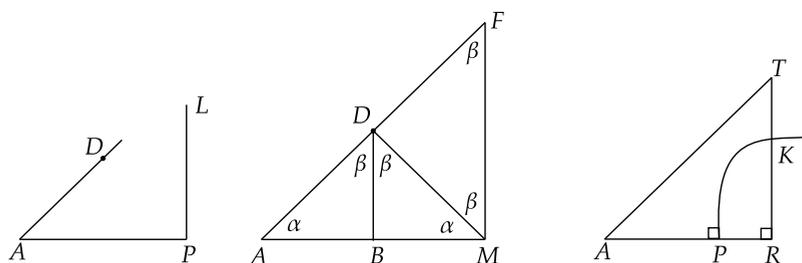


Figure 46: Scheme of Saccheri's Proposition 11

Given lines AD and PL forming, respectively, an acute and a right angle with the transversal AP , we aim to show that AD intersects PL (see Fig. 46, left).

Dropping a perpendicular from D to AP , we obtain point B . Let F and M be points on AD and AP , respectively, such that $AD = DF = DM$ (see Fig. 46, middle).²⁰

By assumption, the right-angled triangle $\triangle ABD$ is Euclidean. By construction, the triangles $\triangle ADM$ and $\triangle DFM$ are isosceles. The triangles $\triangle ABD$ and $\triangle DBF$ are congruent, and $\alpha + \beta = \frac{\pi}{2}$. Since $\triangle DFM$ is Euclidean, its base angles are equal to β , and hence $\triangle AMF$ is right-angled.

Repeating this construction, we obtain right-angled triangles whose dimensions grow by successive duplication. Eventually, we obtain a right-angled triangle $\triangle ART$ whose base exceeds AP .

As Saccheri writes:

It is manifest that this duplication of the preceding interval can be so many times repeated, that thus in AD continued we attain to a certain point T , from which the perpendicular let fall upon AP prolonged cuts off a certain AR greater than the finite AP , however great. (Saccheri, 2014, p. 87)

²⁰In the original proof, M is the perpendicular projection of F onto AP , and Saccheri proves that $DM = DF$.

This argument clearly relies on the Archimedean axiom.

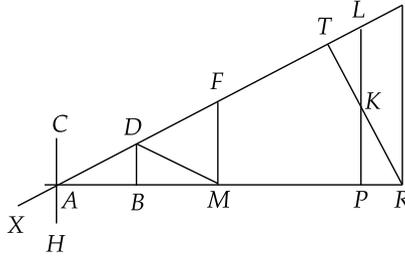


Figure 47: Diagram accompanying Saccheri’s Proposition 11

In the final step, Saccheri shows that the line PL must intersect AT rather than RT .²¹ If PL were to intersect RT , then triangle $\triangle PKR$ would contain two right angles, contradicting Proposition I.17 (see Fig. 46, right).

□

For completeness, we add the standard argument showing that the existence of a single Euclidean triangle implies the existence of a rectangle.

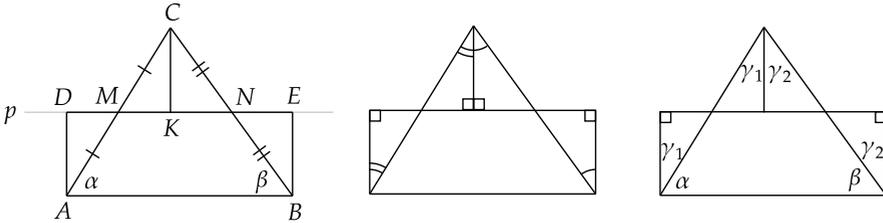


Figure 48: From a Euclidean triangle to the existence of a rectangle

Let $\triangle ABC$ be Euclidean, with angles α, β, γ . Let p be a line through the midpoints M and N of AC and BC . Dropping perpendiculars from A , B , and C to p , we obtain points D , E , and K .

The triangles $\triangle MDA$ and $\triangle MKC$ are congruent, and similarly $\triangle NEB$ and $\triangle NKC$ are congruent. Hence, $DA = CK = EB$, and $DEAB$ is an S-quadrilateral with base DE .

The line CK divides γ into γ_1 and γ_2 such that $\alpha + \gamma_1$ and $\beta + \gamma_2$ are the summit angles of this S-quadrilateral. Since $\triangle ABC$ is Euclidean, $\alpha + \beta + (\gamma_1 + \gamma_2) = \pi$, and therefore $\alpha + \gamma_1 = \beta + \gamma_2 = \frac{\pi}{2}$. Hence, $DEAB$ is a rectangle.²²

²¹That it must intersect one of the sides of $\triangle ART$ is assumed; in modern terms, this follows from Pasch’s axiom.

²²The case $\alpha > \frac{\pi}{2}$ can be treated analogously.

11.3. Saccheri and Lobachevsky

Lobachevsky proves the following result: if $a < b$, then $\Pi(a) > \Pi(b)$. This expresses the relationship between the length of a segment perpendicular to a given line and the so-called angle of parallelism. In the course of the proof, he establishes an auxiliary lemma to the effect that if $\angle AE_1E$ is a right angle and AB is the limiting parallel with respect to E_1E (see Fig. 49), then any ray such as AG meets EE_1 (see Fig. 50, left). We call attention to this lemma because its proof bears a striking similarity to Saccheri's Proposition 11.

Assuming implicitly the Archimedean axiom and working within absolute geometry, Lobachevsky shows that the sum of the angles in a triangle is not greater than π . He further proves that if there exists even one Euclidean right-angled triangle, then all triangles are Euclidean, and in particular gives a new proof that Euclid's Parallel Postulate implies this conclusion.

He then adopts the contrary (non-Euclidean) hypothesis, namely that all right-angled triangles are non-Euclidean, i.e., their angles α, β, γ satisfy

$$\alpha + \beta + \gamma < \pi,$$

and on this basis proves the theorem stated above.

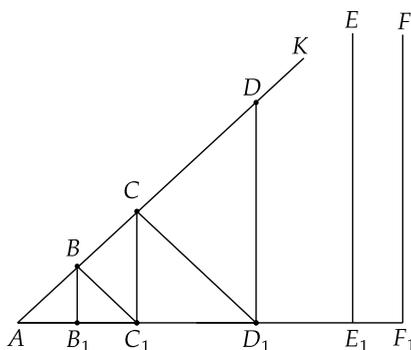


Figure 49: Diagram accompanying Lobachevsky's theorem

He first shows that some perpendiculars to AF_1 meet AB , while others do not. This argument closely parallels Saccheri's reasoning in Proposition 11.

Dropping a perpendicular from B to AF_1 , he obtains the point B_1 . He then chooses a point C_1 on AF_1 such that $AB_1 = B_1C_1$. The perpendicular through C_1 meets AB – if it meets it at all – at a point C . Next, he selects a point D_1 on AF_1 such that $AC_1 = C_1D_1$. The perpendicular through D_1 meets AB – provided it meets it – at a point D , and so on.

In this way, he constructs a sequence of right-angled triangles, each having a base twice as long as that of the preceding one.

Lobachevsky then concludes that there exists a perpendicular such as F_1F that does not meet AB . The argument is as follows. The right-angled triangle $\triangle ABB_1$ is non-Euclidean, with angle sum $\pi - \varepsilon$.

By the congruence

$$\triangle AB_1B = \triangle BB_1C_1,$$

the triangle $\triangle ABC_1$ has angle sum $\pi - 2\varepsilon$. Since the triangle $\triangle BC_1C$ has angle sum $\leq \pi$, it follows that the triangle $\triangle ACC_1$ has angle sum $\leq \pi - 2\varepsilon$.

Iterating this construction, the triangle $\triangle ADD_1$ has angle sum $\leq \pi - 2^2\varepsilon$, and, in general, we obtain a sequence of triangles whose angle sums are $\pi - 2^n\varepsilon$.

For sufficiently large n , we have $2^n\varepsilon > \pi$ – an implicit use of the Archimedean axiom – so this construction cannot continue indefinitely. Therefore, there exists a perpendicular such as F_1F that does not meet AB .

Thus, since some perpendiculars to AC_1 meet AB while others do not, there must exist a boundary line separating these two classes – namely, E_1E .

Это требует, чтобы, наконец, перпендикулы EE_1, FF_1 к одному боку угла A не сходились более с другим. Пусть из них EE_1 – тот перпендикул к AE_1 , который сам не сходится с AD , но, в одну сторону которого к острию A все прочие пересекают бок AD , тогда как по другую сторону все перпендикулы FF_1 , сколько бы ни продолжались, не встречаются с AD . В таком случае бок AD параллелен с перпендикулом EE_1 , (Лобачевский, 1949, р. 276)²³

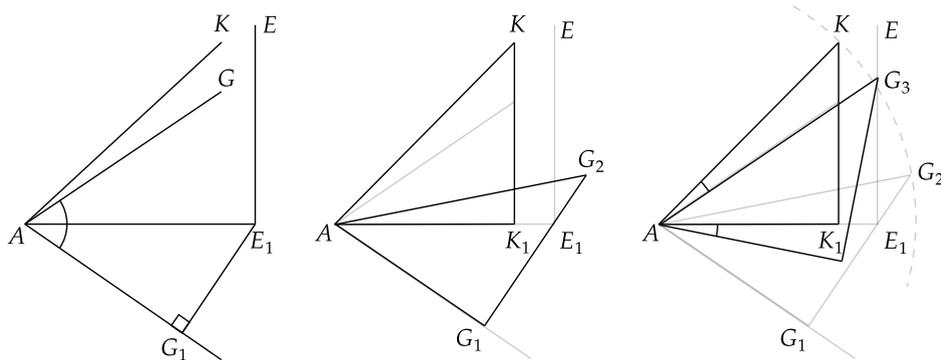


Figure 50: Scheme of the second part of Lobachevsky's theorem

In the next step, Lobachevsky shows that E_1F is parallel – in his sense – to AB , meaning that any ray such as AG , lying inside the angle $\angle GAC_1$, meets E_1E . We briefly sketch the argument.

Choose a ray AG_1 such that $\angle GAE_1 = \angle G_1AE_1$, and let $AK_1 = AG_1$, with $\angle AG_1E_1 = \frac{\pi}{2}$. Since $AK_1 < AE_1$, the perpendicular through K_1 meets AB at a point K (see Fig. 50, left).

²³“This requires that, eventually, the perpendiculars EE_1, FF_1 dropped to one side of the angle A no longer meet the other side. Let EE_1 be that perpendicular to AE_1 which itself does not meet AD , while on one side of it – toward the vertex A – all others intersect AD , whereas on the other side all perpendiculars FF_1 , however far extended, do not meet AD . In this case, the side AD is parallel to the perpendicular EE_1 .”

Let $\triangle AG_1G_2$ be congruent to $\triangle AK_1K$ (see Fig. 50, middle). By rotating $\triangle AG_1G_2$, we obtain a point G_3 at which the ray AG meets E_1E (see Fig. 50, right).

□

The existence of the line E_1E expresses the existence of a limiting parallel, which is a central component of Lobachevsky's axiom.

According to Lobachevsky, rays emanating from a point C can be divided into two classes: those that meet a given line AB , such as CD , and those that do not, such as CE (see Fig. 51). If CE were the only ray not meeting AB , then all triangles would be Euclidean.²⁴ Hence, he assumes the existence of additional non-intersecting rays, such as CK , and introduces a limiting ray CF , which separates the two classes. It follows that any ray lying inside the angle $\angle FCD$ intersects AB .

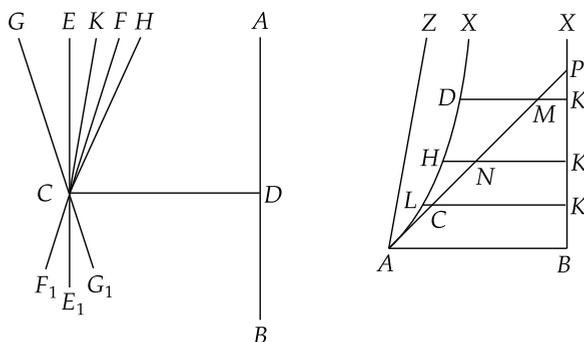


Figure 51: Lobachevsky's axiom (left) and Saccheri's Proposition 31 (right)

Линии, выходя из одной точки, либо пересекают данную прямую в той же плоскости, либо никогда с ней не встречаются, сколько бы ни продолжались. Надобно, следовательно, различать между такими линиями в отношении к одной данной: встречающиеся или сводные, и не встречающиеся или несводные, к которым принадлежат параллельные, составляя переход от одних к другим – разводным, (Лобачевский, 1949, p. 266)²⁵

In Proposition I.32 (under the hypothesis of acute summit angles), Saccheri similarly observes that rays emanating from a point split into those that meet a given line and those that do not.

And hence follows thirdly, that (in this hypothesis) there must be a certain determinate acute angle BAX , drawn under which AX so approaches ever more to BX , that only at an infinite distance does it meet it, (Saccheri, 2014, p. 153)

²⁴This corresponds to Proposition 11.

²⁵“Lines drawn from a single point in the same plane either intersect a given straight line or never meet it, however far extended. One must therefore distinguish such lines as intersecting and non-intersecting; parallel lines form the boundary between these two classes.”

Despite these similarities, there is no evidence that Nikolai Lobachevsky was acquainted – directly or indirectly – with Girolamo Saccheri's *Euclides vindicatus*.

References

- Błaszczyk, P.: Reading Newton's *Principia* through Euclidean proportion and Nonstandard Analysis. *Foundations of Science* 31, 55–95, 2026.
- Błaszczyk, P., Petiurenko, A.: *Interpreting Euclid*, WN UKEN, Kraków, 2025.
- Błaszczyk, P., Petiurenko, A.: Commentary to Book I of the *Elements*. *AUPC* 13, 43–93, 2021.
- Fitzpatrick, R.: *Euclid's Elements of Geometry translated by R. Fitzpatrick*, 2007; <http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf>
- Hartshorne, R.: *Geometry: Euclid and Beyond*. Springer, New York, 2000.
- Saccheri, G.: *Euclid Vindicated from Every Blemish*. Edited and annotated by V. De Risi. Translated by G. B. Halsted and L. Allegri. Birkhäuser, Cham, 2014.
- Лобачевский, Н. И.: Сочинения по геометрии. Геометрия : Новые начала геометрии с полной теорией параллельных. Т. 2. Гос. издво технико-теоретической литературы, Москва–Ленинград, 1949.

Piotr Błaszczyk
University of the National Education
Commission in Krakow, Poland
e-mail: piotr.blaszczyk@up.krakow.pl

Anna Petiurenko
University of the National Education
Commission in Krakow, Poland
e-mail: anna.petiurenko@up.krakow.pl