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Why Can Not We Give Up Complex Functions in the Education of Automation Engineers?*

Abstract. Today, universities have a lot of freedom in designing both study programs and proposing the content of courses carried out during them. Only the generally formulated learning outcomes contained in the Level Characteristics of the Polish Qualifications Framework, which reflect Polish qualifications in relation to the European Qualifications Framework, should be met. Hence the questions: *"What should we teach in the subject...?"* Taking into account the basic concepts of automation, automatic control, and regulation, which students should be proficient in when solving engineering problems at a given level of qualification, the need to introduce complex functions into the curriculum content in the subject of mathematics is analyzed. Currently, topics related to this are often overlooked in the education of automation engineers.

1. Introduction

As is well known, mathematics is a basic subject in engineering studies. But it is also a subject that raises many concerns among candidates and students, especially in the first year. So let us think about what mathematics really is and why we teach it today. When answering this question, it is not possible to focus only on the schematic tasks and problems known from high school or from the basic course of calculus and linear algebra carried out in engineering studies. Why? Well, mathematics is a universal language, a tool that allows for the analysis and understanding of problems, including engineering problems, and for the search for reliable solutions. Mathematics is the subject that develops the ability to think critically, to perceive relationships between variables and between objects, and the ability to analyze and synthesize. Questions often arise about what should be

taught in mathematics to automation and robotics students, given the widespread availability of computational packages and the popularity of language models such as ChatGPT, as well as the often limited number of hours allocated to this subject by technical university authorities. On the other hand, it can be said that a certain canon of topics that should be covered in mathematics courses is widely known. The problem is much broader and more complex. To bring it closer, we will first turn to the history of faculties currently related to automation and robotics; then we will analyze selected basic concepts in the field of automation and related mathematical content. This is the starting point for answering the question of whether complex functions should be included in the mathematics curriculum. And if so, what should be given up?

2. A bit of history of educating students in automation-related fields

The beginnings of education in the field of automation and robotics at Polish technical universities date back to the sixties of the twentieth century. In the first years, it was most often a specialization in the field of electrical engineering (Jezierski, 2011), but already in the seventies, a new field of study of *automation and robotics* appeared in technical studies. Education in this field not only exists today at most technical universities but is also developing dynamically.

It is worth delving into history and citing data regarding the automation specialization program for the year of study beginning in the 1966/67 academic year. According to a study prepared for the General Council of Higher Education by Professor Edward Jezierski in 2011 (Jezierski, 2011), *the total number of classes was 4,605 mandatory contact hours, 450 hours related to thesis preparation, 27 weeks of professional internships, and 10 weeks of military training during the summer. Classes at the university included: lectures (2,070 hours), tutorials (1,680 hours), laboratories (765 hours), and projects (90 hours). The core subjects also included 480 hours of mathematics and 315 hours of physics.* Both the subjects that should be taught at universities, the topics that should be taught there, and the number of hours allocated to a given subject have been specified for many years by the Ministry of Higher Education and Science in the form of so-called minimum curricula.

In 2006, work began on the development and implementation of the Polish Qualifications Framework. Broadly speaking, it is a list of skills, knowledge, and social competencies that graduates of a given field of study should possess to successfully perform tasks specific to a given profession. As a result, the Act of 22 December 2015 (Act 2015, Act 2016) on the Integrated Qualifications System introduced the Polish Qualifications Framework. The Polish Qualifications Framework, like the European Qualifications Framework, consists of eight qualification levels. Each of them is described by very general statements characterizing the learning outcomes that a given qualification must demonstrate. The Polish Accreditation Committee requires that the curricular content of undergraduate and graduate programs be consistent with the intended learning outcomes and the current state of knowledge and research methodology in the discipline(s) to which the

field of study is assigned, as well as with the scope of the university's research activity in that discipline(s). Hence, frequent questions arise about what one should teach future automation and robotics engineers in mathematics. As mentioned, a certain canon is known. This canon covers topics in Calculus I (differential and integral calculus of functions of one variable), Calculus II (differential and integral calculus of functions of several variables), Linear Algebra with Geometry, and the Basics of Differential Equations (see, for example, McMullen (2018), Spivak (2006), Trajdos (2017), Żakowski and Dacewicz (2013), Żakowski and Kołodziej (2023), Żakowski and Leksiński (2023)). It also occasionally covers the basics of probability and statistics. Is the implementation of these issues really sufficient in the education of an automation engineer? To answer this question, we will analyze two of the basic concepts of automation and show their connections with mathematical content.

3. Basic issues in automation

Below, only the key automation concepts commonly used by engineers to analyze the behavior of objects will be presented, without providing introductions. Interested readers can refer to works such as, for example, Dębowski (2017), Dorf et al. (2021), Kowal (2018), Ogata (2009), and others.

3.1. Transition function

The *transfer function* is a rational complex function

$$G(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$

where a_i $(i = 0, 1, 2, \ldots, n)$ and b_k $(k = 0, 1, 2, \ldots, m)$ are real coefficients, $U(s) = L[u(t)]$ is the Laplace transform of the input signal $u(t)$, $t > 0$, $Y(s) = L[y(t)]$ is the Laplace transform of the output signal $y(t)$. As it can be seen, function $G(s)$ describes a relation between input and output signals, but in the complex domain. Based on the positions of the zeros and poles of this function, one can analyze the behavior of the considered input-output system. For example, a necessary condition for the stability of the considered system is, roughly speaking, that in the continuous-time case (i.e., t is a real variable), all pools have negative real parts. In order to check this condition, the student must solve the equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{2}$$

in the set of complex numbers and know the geometric interpretation of a complex number.

The problem becomes more complicated when one is dealing with a discrete system. In this case, we are still operating with a rational function

$$G(z) := \frac{Y(z)}{U(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0},$$

but with the difference that $U(z) = Z[u(t)]$ is the Z-transform of the input signal $u(t)$, t is a positive integer, $Y(z) = L[y(t)]$ is the Z-transform of the output signal $y(t)$. Now, a necessary condition for the stability of the input-output system is that the poles of this function lie on the unit circle centered at the origin of the coordinate system.

Remark: Although the transition function is commonly used in automation, it is important to be aware of the limitations in its use. Namely, it is used only for the analysis of linear control systems and with zero initial conditions or linearized systems in the vicinity of the operating point $(0, 0)$ (Dębowski, 2017; Kowal, 2018; Ogata, 2009).

3.2. State space description of control system

Linear control systems with one input $u(\cdot) \in \mathbb{R}^m$ and one output $y(\cdot) \in \mathbb{R}^p$ (SISO systems) can also be described by a system of first-order differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3}$$

$$y(t) = Cx(t) + Du(t), \tag{4}$$

where $t > 0$, $x(t) = [x_1(t) \ \ldots \ x_n(t)] \in \mathbb{R}^n$ is a state vector. Most often, it is assumed that A, called a state matrix, is a square matrix of dimension $n \times n$ with constant real or complex terms, B is a control (input) matrix of dimension $n \times m$, C is a output (response) matrix of dimension $p \times n$, D is a transition matrix of dimension $p \times m$. All of these matrices are matrices with real or complex terms. However, it should be borne in mind that in the general case, these can be non-stationary matrices (with t-dependent elements), and the matrix A can be a rectangular matrix. Using the Laplace transform, it can be shown that, under zero initial conditions,

$$G(s) = C[sI - A]^{-1}B + D,$$

where I denotes the unit matrix of the dimension $n \times n$. And, again, the necessary condition for the stability of the system, roughly speaking, that all roots of the characteristic equation of the matrix A should have negative real parts. Moreover, the system (3)-(4) is observable (i.e., on the basis of the knowledge of the input and output, we are able to reconstruct the state $x(t)$ of the system) if and only if

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

The system (3)-(4) is controllable (i.e., roughly speaking, from a given initial state one can reach the origin of the coordinate system) if and only if

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

3.3. Frequency transfer function

It can be introduced in two ways: starting from the transfer function $G(s)$ given by (1) or by using the Fourier transform. In each case, the *frequency transfer function* is obtained as a function of pulsation ω

$$G(s)|_{s=j\omega} = G(j\omega),$$

where j denotes the imaginary unit, i.e., $j^2 = -1$. So, frequency transfer function is a complex function dependent on the parameters of the system and the pulsation of the input signal. Then, it can be written as

$$G(j\omega) = P(\omega) + jQ(\omega)$$

or, taking into account the exponential form of a complex function, also as

$$G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}.$$

In applications, spectral analysis of this function is used. Thus, the so-called a frequency response is

$$A(\omega) = |G(j\omega)|$$

and the phase response is

$$\varphi(\omega) = \arg G(j\omega).$$

The task of analytically determining the frequency and phase characteristics is reduced to the complex functions properties.

$$A(\omega) = \sqrt{(P(\omega))^2 + (Q(\omega))^2},$$

$$\varphi(\omega) = \arctan\frac{Q(\omega)}{P(\omega)}.$$

So, students should know how to determine the real and the imaginary parts of a function of a complex variable, its modulus, and its argument. They should also know the basic functions of a complex variable, including rational functions and exponential functions, and their properties.

4. Mathematics content enabling students to understand basic issues in automation

As one could see, complex numbers and complex functions are essential for understanding the fundamental concepts of automation discussed above. However, it is also important to remember that their scope of application in engineering, as well as in subjects that naturally fit into the education process of automation engineers, is much broader. Subjects such as circuit theory, electrotechnics, electronics, and signal theory should not be overlooked.

When introducing complex numbers to future automation engineers and students in related fields, special emphasis should be placed on developing skills in

determining the real and imaginary parts of a complex number, the modulus and argument of a complex number, and their geometric interpretation. Therefore, problems such as:

On the complex plane, draw a set of points that satisfy the inequality

$$\{z \in \mathbb{C} : |z - 1 + 3j| \leq 2\}$$

or

On the complex plane, draw a set of points that satisfy the inequality

$$\left\{z \in \mathbb{C} : 0 \leq \arg(jz) \leq \frac{\pi}{3}\right\}$$

cannot be omitted. Another very important topic is the trigonometric form and the exponential form of the complex number and the relations between them. For example,

Represent the complex number $z = \sqrt{3} - j$ in trigonometric and exponential form.

or

Represent the complex number $z = 3e^{j\frac{5}{6}\pi}$ in trigonometric and algebraic form.

However, as equation (2) shows, solving complex polynomial equations cannot be omitted. It should be emphasized that solving second-degree equations is not enough, even though they are the easiest for both the academic teacher and the student. Special attention should be paid to the consequences of the fundamental theorem of algebra (Białynicki-Birula, 2017): *Every complex polynomial of degree n has exactly n complex roots, which can be multiple roots.* Therefore, problems such as:

Determine all complex numbers that satisfy the equation:

$$z^3 + 5z^2 + 4z - 10 = 0.$$

and

Determine the eigenvalues and eigenvectors of the matrices

$$\text{a) } A_1 = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}, \qquad \text{b) } A_2 = \begin{bmatrix} 1 & 5 \\ -2 & 4 \end{bmatrix}.$$

Give the characteristic polynomial of these matrices and determine their roots.

cannot be omitted.

Taking into account the conditions of controllability and observability of linear control systems described in Section 3.2, in the course of teaching linear algebra, it

is also good to pay attention to the multiplication, exponentiation, and inversion of complex matrices.

The types of problems discussed above, combined with the geometric interpretation of complex numbers and sets in core subjects such as fundamentals of automation, control and regulation, and control theory, will provide a foundation for analyzing and understanding the stability conditions of control and regulation objects and their applications in automatization and robotization processes. Knowledge of properties of complex numbers will also influence the ability to analyze the basic properties of complex variable functions, and consequently, among others, the spectral analysis of signals (signal theory) and the frequency analysis of control and regulation systems (automation, robotics, mechatronics, operation of automation devices).

And what about complex functions? As many interested parties likely know, complex analysis is not only a broad but also challenging area of mathematics. So how should it be taught to automation engineers, especially in their first semesters? First, we need to ask: *Why do we want to cover topics related to complex functions?* Next: *To what extent might an automation engineer need these functions?*

Analyzing the information presented in Section 3.3, it can be seen that students should be able to distinguish between the real and imaginary parts of a function of a complex variable, determine its modulus and argument, and graphically represent the resulting functions. The example of a task involving this knowledge is:

Determine the phase spectrum and amplitude spectrum of the function

$$f(t) = \begin{cases} 0 & \text{if } |t| > A, \\ a & \text{if } |t| \leq A. \end{cases}$$

Continuing the analysis of the information presented in Section 3.1, one encounters the concept of the Laplace transform of a function of a real variable and the concept of poles and zeros of a function. These are essential tools for a future automation engineer. Continuing our analysis presented in Section 3.1, we encounter the concept of the Laplace transform of a function of a real variable and the concepts of poles and zeros. To define the Laplace transform, we need at least familiarity with the concept of an improper integral and the ability to evaluate it. Furthermore, to properly define this transform, one should introduce the concept of a holomorphic function, i.e., a function which is differentiable at a point (x_0, y_0) and its neighborhood (Pawłuszewicz and Piekarski, 2024, Żakowski and Leksiński, 2023). This leads to the Cauchy-Riemann conditions.

Taking also into account the fact that the positions of the zeros and poles of the transfer function are used to analyze the behavior of an input-output system, the definition of single and multiple poles cannot be omitted during mathematics classes for students of automation and robotics and related topics.

Again, when considering the use of the Laplace transform in the training of automation engineers, one cannot ignore developing the ability to determine the inverse Laplace transform. Here, due to possible time constraints, the emphasis can be placed on the method of decomposition into simple fractions, although for reasons of simplicity, among other things, it would be advisable to also include

Heaviside's decomposition theorem (Kowal, 2018, Pawłuszewicz, and Piekarski, 2024):

If the following conditions are met: 1. $F(s) = \frac{P(s)}{Q(s)}$ *is a proper rational function; 2.* F *has only isolated poles* $s_1, \ldots, s_n$*; 3. the polynomials* P *and* Q *are irreducible then*

$$L^{-1}[F(s)] = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} e^{s_k t}.$$

Therefore, problems such as:

Using Heaviside's theorem find the inverse Laplace transform of the function

$$F(s) = \frac{2s + 1}{2s^3 - 9s^2 + 10s - 3}.$$

are welcome.

To sum up, in the field of complex functions, in the education of automation engineers, due to the implementation of topics and problems in core subjects typical for this educational profile, it would be necessary to cover topics such as decomposition of a function of a complex variable into real and imaginary parts, determination of its modulus and argument, a review of elementary functions of a complex variable, with particular attention paid to rational functions and the function e^z, conditions of differentiability and analyticity of complex functions, and the Laplace transform and its properties. So, problems such as:

Is function

$$f(t) = \begin{cases} t - c & \text{for } c \leq t < 2c, \\ -t + 3c & \text{for } 2c \leq t \leq 3c, \ c > 0, \\ 0 & \text{otherwise} \end{cases}$$

the original? Justify your answer. Find the Laplace transform of this function.

cannot be ignored. Also included are methods of determining the inverse Laplace transform and the use of the operator method to solve ordinary differential equations, for example,

Using the operator method, solve the equation

$$y''(\tau) - y(\tau) = \mathbf{1}(\tau) - \mathbf{1}(\tau - \pi), \qquad if \quad y(0) = 0, \quad y'(0) = 0.$$

5. Summary

It should be clearly emphasized that mathematics in engineering education is not an art in itself. For engineers, mathematics is a tool for problem-solving. It is also important to remember that mathematics is a subject that develops skills in logical thinking, analysis, and synthesis. Therefore, it is important to combine the content covered in this subject with that taught in other subjects, especially at the

basic level. Above, we attempted to answer the question why complex functions cannot be omitted in the education of automation engineers at the higher level. An attempt to analyze what mathematical content should be included in the education of automation engineers was undertaken in Pawłuszewicz (2025).

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