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Approaches, difficulties, and types of 'graph as picture' misconceptions in using function to describe movement – a case study of stone throwing*

Abstract. The paper presents the study focused on the use of function graphs to represent familiar movements from practice. It examines how participants solve a simple task that requires identifying the correct graph describing changes in speed over time during vertical projection. From a mathematical perspective, the task requires only the ability to recognise the correct monotonicity of a function. A research method employed in the study was eye-tracking, combined with other approaches, such as interviews and questionnaires.

The task was completed by 345 participants with varying levels of mathematical expertise. It was challenging, with an overall success rate of 0.43. The paper presents and categorises various strategies used to solve the task, also examining participants' visual attention. It analyses the difficulties and misconceptions that emerged during problem-solving. Two types of categorisations are introduced: nine mathematical strategies (with 21 subcategories) and eight distinct types of 'graph as picture' misconceptions revealed during the task.

Initial pedagogical implications for further research and teaching practice are formulated, emphasising the need to address the identified difficulties and misconceptions in schools to prepare students to use function graphs in simple contexts.

1. Introduction

This article examines students' ability to engage in elementary mathematical modelling of the motion of a stone during vertical projection, with a focus on applying covariational reasoning and distinguishing between increasing and decreasing

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functions and their graphs. It analyses participants' capacity to choose the correct function graph based on its monotonicity and everyday knowledge, and it identifies the approaches, difficulties, and misconceptions that arise in this process. The theoretical background section outlines the educational role of modelling, the importance of *covariational reasoning* in the context of *functional thinking*, and the need to address persistent graph-related misconceptions.

2. Theoretical Background

2.1. Modelling at school

Many scientists claim that mathematical modelling fosters metacognitive skills, conceptual understanding, competencies, creative abilities, and an innovative attitude in students, as well as stress the socio-cultural role of mathematics (Asempapa, 2015).

The ability to perform mathematical modelling by students at the level of primary and secondary schools, as well as at the university level, is listed among the key skills and objectives of mathematics education alltogether. Research shows that modelling promotes students' understanding of a wide range of key mathematical concepts.

> Mathematical modelling should be an essential part of mathematics education for all students, at all levels, in order to foster the ability to apply mathematics meaningfully in real-world contexts. (Blum, Galbraith, Henn, & Niss, 2007, p. 3).

Berry and Houston stress that "when using mathematics to solve real world problems one of our aims is to obtain a mathematical model that will **describe or represent** some aspect[1] of the real situation" (Berry & Houston, 2004, p. 10).

This point can be elaborated. For example, Borda (2024) emphasised the role of the process of simplification:

> (...) we do not require to represent the problem in its entirety, only the most relevant aspects of it, in such a way that we will be able to analyse the most important aspects of the real-world problem, without having to analyse every single small detail. The step of simplifying a complex problem into a simpler model is crucial for students, as it helps them not only express complex situations through mathematical models and structures but also develop the ability to identify the most relevant aspects of a real-world situation that should be analysed. (Borda, 2024, p. 139)

The level of simplification and sophistication chosen in the modelling is also important in the context of this research.

Mathematical modelling is a difficult activity for students and even pre-service mathematics teachers, as analysed, among others, by Pyzara in her works (Pyzara, 2014, 2017, 2018).

[1]Emphasis added.

Various tools are used in mathematical modelling. Differential calculus, functions, and equations are used when it is necessary to build a model of processes or changes occurring over time. It is because the need to highlight the relationships between variables and distinguish the parameters influencing the described process. Therefore, it is important to develop a didactic approach to reasoning involving such relationships, i.e., *covariational reasoning*, and to explore of *functional thinking.*

2.2. Covariational reasoning

Covariational reasoning, rooted in the work of Confrey and Smith (1995), involves analysing, manipulating, and understanding the relationships between quantities that change in relation to one another. Carlson et al. (2002) and Thompson and Carlson (2017) offer a nuanced perspective on this approach, further elaborating its theoretical and pedagogical dimensions. Some perspectives are presented in the remainder.

2.2.1. Mental actions in covariational reasoning according to Carlson et al. (2002)

Carlson et al. (2002) characterise how individuals reason about dynamic relationships between two quantities and describe a hierarchy of *mental actions* within the covariation framework. These mental actions reflect the increasing sophistication in understanding and representing relationships. Below is a summary of the five mental actions:

Mental Action 1 (MA1): Coordinating dependence. Learners recognise that the value of one variable depends on the value of another. This is typically shown by labelling axes or explicitly relating variable changes, such as "y changes when x changes."

Mental Action 2 (MA2): Coordinating direction of change. Individuals identify whether a variable increases or decreases in response to changes in another variable. They may represent this relationship through verbal descriptions or by constructing a monotonic (always increasing or decreasing) graph.

Mental Action 3 (MA3): Coordinating amount of change. Learners not only identify the direction but also attend to how much one variable changes in response to another. This is often demonstrated by plotting points, constructing secant lines, or discussing relative magnitudes of change.

Mental Action 4 (MA4): Coordinating average rate of change. Learners understand and describe how a function changes at a constant rate over uniform intervals of the input. They can construct secant lines over these intervals and verbalise the average rate of change over the domain.

Mental Action 5 (MA5): Coordinating instantaneous rate of change. At the most advanced level, individuals comprehend how a function's rate of change varies continuously across its domain. They can construct smooth curves with accurate concavity and inflection points, and articulate how the instantaneous rate of change evolves throughout the function.

These mental actions provide a framework for analysing students' functional reasoning and support instructions that move learners from basic dependence recognition toward a calculus-ready understanding of change.

2.2.2. Levels of covariational reasoning according to Thompson and Carlson (2017)

Castillo-Garsow (2012) and Castillo-Garsow et al. (2013) observed differences in the ways students reason about continuous phenomena. While some students interpreted changes in variable values only as distinct, separate increments, others understood change as a fluid and continuous process. To describe these differing conceptions, Castillo-Garsow introduced the terms "chunky" reasoning, referring to stepwise interpretations of change, and "smooth" reasoning, referring to perceptions of change as continuous. This foundational work on covariational reasoning has been further developed by other researchers. Thompson and Carlson (2017) conceptualised levels of *covariational reasoning* to describe how individuals mentally coordinate the variation of two quantities. These levels reflect the increasing sophistication in understanding how quantities co-vary. The levels can be described as follows:

1. *No coordination,* when individuals are unable to coordinate two varying quantities. They focus on only one variable at a time without recognising the relationship between them.

2. *Precoordination of values,* when individuals recognise that two variables vary, but perceive this variation asynchronously. They may notice the variable differences separately, but do not anticipate a coordinated relationship or form (x, y) value pairs.

3. *Gross coordination of values.* There is a recognition that two quantities vary in a general sense (e.g., "as x increases, y increases"). However, individuals do not recognise specific value pairs or the nature of their relationship. The reasoning is qualitative rather than quantitative or multiplicative.

4. *Coordination of values*, when individuals can coordinate specific values of one variable with the corresponding values of another, anticipating or forming ordered pairs $(x,\ y)$. This reflects an understanding of covariation at the level of discrete data points.

5. *Chunky continuous covariation*, when individuals conceptualise both variables as changing in perceptible chunks or intervals. Although variation is envisioned as simultaneous, it is not considered continuous. The idea of co-varying over intervals, rather than smoothly, is dominant.

6. *Smooth continuous covariation* is the most sophisticated level, where individuals perceive both variables as varying smoothly and continuously. There is a dynamic image of simultaneous variation, and the reasoning supports continuous functional relationships.

These levels represent a developmental trajectory in understanding relationships between varying quantities, essential for interpreting functions and modelling real-world phenomena.

2.3. Functional thinking

Functional thinking can be understood as the "process of building, describing, and reasoning with and about functions" (Stephens et al., 2017, p. 144). It encompasses a range of conceptions and perspectives on functions (e.g., Carlson, 1998; Doorman et al., 2012; Even, 1990; Pittalis et al., 2020; Sierpińska, 1992; Thompson & Carlson, 2017; Vinner & Dreyfus, 1989), each of which plays a vital role in the comprehensive development of the function concept in students and in fostering the ability to use it meaningfully. Learners are expected to engage with various conceptualisations of functions, including:

(a) the *input-output assignment*, emphasising the operational and computational nature of the concept of a function, often serving as a preliminary structure (e.g., Doorman et al., 2012; Even, 1990; Sfard, 1991);

(b) the *dynamic process of covariation*, focusing on the covariation between two variables (e.g., Thompson & Carlson, 2017);

(c) the *correspondence view*, emphasising the specific relationship between independent and dependent variables (e.g., Doorman et al., 2012; Sfard, 1991; Sierpinska, 1992);

(d) and the structural view of functions as *mathematical objects* that can be examined, compared, and linked to other mathematical notions (e.g., Even, 1990; Sajka, 2003; Sfard, 1991; Sierpinska, 1992).

2.4. Mathematics misconceptions in graph interpretation

Leinhardt, Zaslavsky, and Stein (1990) in their well-known review, identify misconceptions concerning function graphs. These categories reflect conceptual misunderstandings and difficulties linked to representational practices. The classification highlights recurring challenges in students' understanding of graphs and their connection to functions.

1. What is and is not a function. Students often have an overly restricted view of what graphs of functions can be. They tend to classify only regular, patterned, or linear graphs as functions, neglecting the less-familiar representations, even when they conform to the formal definition.

2. Correspondence. Students struggle with understanding that a function involves a consistent assignment between elements of two sets. Errors include expecting one-to-one mappings and not recognising that multiple values of the domain can share a value in the range.

3. Linearity. There is a strong bias toward linear functions. Students frequently assume that graphs of functions must be straight lines or resemble linear patterns, leading to misclassification or inappropriate interpretation of nonlinear functions.

4. Continuous vs. discrete graphs. Students often default to drawing continuous curves, even when discrete points are more appropriate (e.g., whole-number data). They may not understand the distinction or rationale for using discrete versus continuous representations.

5. Representations of functions. Difficulties arise in translating between symbolic, graphical, and tabular representations. Students may not see these as equivalent, and often consider only one direction (e.g., equation to graph) or depend heavily on familiar formats like $y=mx+b$.

6. Relative reading and interpretation

Students tend to interpret graphs pointwise rather than globally. They may focus on individual data points and fail to consider broader patterns or intervals. This issue can be divided into three misconceptions:

- *Interval/Point confusion:* Students give single-point answers when the task requires interpreting an interval.
- *Slope/Height confusion:* They confuse steepness (rate) with vertical position (value).
- *Iconic interpretations:* Students treat graphs as literal pictures of events.

7. Concept of variable. Students may misunderstand the notion of a variable, seeing it only as a label or a static placeholder. They often struggle with dynamic interpretations of variables in relationships as they change, or in functions involving multiple variables.

8. Notation. Students can become confused due to symbolic conventions and coordinate systems. They may not understand various types of notation, such as ordered pairs, variable naming, or axis orientation, leading to errors in plotting or interpretation.

In this publication, special attention is given to a type of misconception called the ***iconic interpretation***, labelled also in literature as ***graph as picture*** (also spelled as 'graph-as-picture') misconception (e.g., Janvier 1981; Clement 1985; Garcia-Garcia & Cox, 2010; May, 2017), which is the name – the ***'Picture'*** misconception for short – will be used hereinafter in the paper.

Leinhardt, Zaslavsky, and Stein (1990) claim that this is one of the most persistent misconceptions students exhibit when interpreting graphs of real-world situations. This specific error refers to the tendency to treat a graph as a literal or pictorial representation of the physical situation it describes, rather than as a symbolic depiction of relationships between variables.

For instance, students often misread a distance-time graph as a map of a journey, interpreting its visual form as the path taken. In Kerslake's (1981) study, when shown a graph with vertical segments, students interpreted these as literal vertical climbs or directional movements like "going east, then north". In another case, a graph showing variations in speed was confused with the shape of an actual racetrack, with students counting bends or matching graphical curves to turns in the track. These interpretations reflect a confusion between symbolic representation and the depicted scenario's concrete features.

Leinhardt, Zaslavsky, and Stein (1990) stress that the *iconic interpretation* is particularly difficult to overcome because it is rooted in intuitive and perceptual reasoning. The visual nature of graphs invites associations with familiar physical experiences (e.g., movement, slope, and direction), and students often rely on prior real-world knowledge to interpret them. However, this reliance can hinder abstraction and mislead reasoning – especially when the graphical conventions (such as axes and slope) are symbolic rather than pictorial.

The authors emphasise that both *personal distractors* (based on past experience) and *pictorial distractors* (based on visual features of the graph) can interfere with proper interpretation. This dual source of interference complicates learning, as learners must suppress concrete associations and engage in abstract reasoning about variable relationships.

In summary, *iconic interpretation* exemplifies a robust and recurring challenge in graph comprehension, where students misinterpret symbolic graphs as literal images. Addressing this issue requires instructional strategies that explicitly distinguish between symbolic representations and the physical phenomena they model.

2.5. Polish curriculum context

The term *functional thinking* in Poland was not commonly used in scientific discourse in mathematics education until recently, as confirmed by preliminary research (e.g., Sajka, 2023). However, the notion of function itself is given a lot of attention in teaching, especially at the secondary school level.

In Poland, Grade 7 students (13–14 year old students) learn about different motions in detail during physics classes (uniform motion, uniformly accelerated or decelerated motion, variable motion), as well as various motion graphs (e.g., distance-time, speed-time, acceleration-time), also in the context of projectile motion. During mathematics classes in Poland, Grade 7–8 students learn how to read simple graphs without knowing the definition of a function. The notion of function is introduced with its formal definition only at secondary school level (Grade 9). We described the struggle with mathematics and physics curricula in the context of the notion of function and motion graphs in one of our previous papers (Sajka & Rosiek, 2019). Despite ongoing reforms, the situation has not changed to this day. Due to the uncorrelated curriculum, students must use linear and quadratic function formulas and graphs in the context of mechanics during physics lessons (Grade 7) without yet knowing functions and their graphs from mathematics lessons. This situation causes significant difficulties for students and negative associations with physics as a school subject (Rosiek & Sajka, 2019).

Covariational reasoning is not explicitly included in mathematics curricula at Grades 7–9 and for this reason, Polish textbooks contain very few content addressing the development of the covariational aspect of function at the stage of shaping the notion of function, being mainly focused on reading simple graphs. What is interesting, some Slovak textbooks, despite a similar educational model contain such tasks, as mentioned by Slabý, Semanišinová, and Climent (2022) in the context of research on specialised knowledge of middle school teachers concerning the concept of function.

However, at secondary school level, mathematics requires an understanding of covariation and the ability to apply covariational reasoning in many different contexts. This is not only important for modelling, but also for pure mathematics, such as differential calculus, where we use derivatives to analyse rates of change.

3. Research aim and methodological approaches

3.1. Aim

The study presented in this paper is an excerpt of a wider study which aimed to investigate using functions by the participants, at different levels of mathematical expertise, as a tool to describe a basic[2] real-life situation in the context of movement analysis.

Specifically, the aims are to reveal, understand, and categorise:

A1. approaches and strategies of reasoning of the participants, at different levels of mathematical expertise, when solving a problem regarding choosing the mathematical model for a basic real-life situation described in words, with graphs provided,

A2. mistakes and misconceptions, as well as doubts when using functions to describe motion, and to diagnose their possible causes.

This article presents a fragment of research conducted to achieve the mentioned objective, narrowed down to the analysis of mathematical modelling of the motion of a stone during a vertical throw.

3.2. Methodologies

In order to achieve the objectives A1 and A2, a descriptive exploratory study methodology was used in combination with a case study, where the *Stone Problem* (in two versions) solutions are analysed. To ensure a multifaceted research approach, the research took several years to complete (2014–2022), and was carried out on different groups of subjects, with the use of varying methodologies: eye-tracking (using two types of eye-trackers) together with interviews or written questionnaires, and written worksheet answers to the task with the analysis of the accompanying open questions. The results presented in this paper involve, in total, 345 respondents. Table 1 presents the methodology and participants of the seven studies analysed in this publication from a case study perspective.

Mathematical modelling, implemented at the level of this task, is taught in Poland from Grade 7 to Grade 8 during physics lessons, and, occasionally, in Grades 8–9 in mathematics. All groups of respondents demonstrated a higher level of mathematical skill than the Grade 9. The purpose of selecting this research group was twofold. Firstly, it was important to ensure that the topics required to solve the task had been fully discussed at school in mathematics and physics lessons. The second objective of this selection was to diagnose the difficulties experienced by students from groups that were significantly more advanced in mathematics and physics than students in Grade 9 of average secondary schools. It can be assumed with a high probability that if difficulties were detected in more

[2] Concerning the mathematical expertise needed to solve it.

Table 1: Research methodology with groups and numbers of participants (ET – Eye-Tracking, WQ – Written Questionnaire, I – interviews with chosen participants)

Empirical research	**Participants**	**N**	**Methodology**
Study 1	1a. Experts (academics)	4	ET + I
	1b. University students (different kind of studies)	77	
	1c. Secondary school students	22	
Study 2	Mathematics and computer science university students	62	ET + WQ + I
Study 3	Pre-service mathematics teachers	20	ET + WQ + I
Study 4	Pre-service mathematics teachers	16	WQ
Study 5	Post-graduate students	9	WQ
Study 6	Freshmen studying to become math teachers	78	WQ
Study 7	Freshmen studying to become math teachers	57	WQ
TOTAL:		345	

advanced groups, they would be even more prevalent at lower levels of education, among less talented individuals, with less mathematical experience. Moreover, we wanted to check whether the use and understanding of graphs is relevant after finishing school, hence the *Stone Problem* being directed to university students from different fields of study. Finally, due to the use of eye-tracking, the answers of experts were required to gain insight into strategies for analysing visual tasks and solving them, as well as having reference points for the work of the other participants. Therefore, the study group included: (a) experts – academics in mathematics, physics, and computer science, (b) university students of different academic majors, as a group with different scientific interests, originating from different schools and different regions, and influenced by different teachers, (c) students starting university studies to become mathematics teachers, similarly to group (b), but with an additional interesting context of being future mathematics teachers, (d) secondary school students in Grade 10 from a school in a large city, completing an extended programme in mathematics and physics, who were highly motivated to learn and very talented, and (e) postgraduate students training to become mathematics or physics teachers.

The analysis involved all types of solutions, comments, and difficulties encountered by the respondents when working on this task, using various approaches.

Study 1 was carried out by the *Interdisciplinary Group of Cognitive Didactics*[3] of the *University of the National Education Commission* (formerly: *Pedagogical University of Krakow*). The *Stone Problem* results from Study 1 were published in fragments in different contexts, namely: analysing the usefulness of eye-tracking methodology for research in physics and mathematics education (e.g., Rosiek & Sajka, 2017), with further results published in several short reports in Polish, in the context of comparing the work of experts and novices (e.g., Wcisło et al., 2014),

[3] The author is a member of the Group.

concerning diagnosis of certain difficulties with interpreting the graphs of functions (e.g., Sajka & Rosiek, 2014) or in the context of the analysis of the relative change of pupil diameter in eye-tracking research participants as a determinant of the subjective assessment of the difficulty level of a task and making decisions regarding answer selection (Rosiek & Sajka, 2021; Rosiek, 2020), and in other contexts of physics education (Rosiek, 2020).

The initial results of the problem-solving strategies from Studies 1–5 were discussed at the 2019 DIDFYZ conference, where the preliminary categorisation was performed (Rosiek & Sajka, 2019).

The last two studies (Studies 6–7) had a wider research scope, conducted on a special sample – students beginning their studies in mathematics to become teachers. These two studies were conducted in 2021 and 2022 at a certain university in Poland on the first day of their studies. The research aimed to diagnose the knowledge and skills related to the same content that is assessed in the *Matura exam*, which is the external, obligatory exam after secondary school in Poland. The structure of both *Research Worksheets* was identical and most of the tasks were the same, including the *Stone Problem*. We write more extensively about Study 6 in Sajka and Przybyło (2025).

The current study presents and analyses the participants' approaches in Studies 1–7 towards modelling at this level of difficulty alongside the problems and misconceptions and presents their categorisation with examples. The aim was to analyse the usage of the properties of functions and their graphs when solving the *Stone Problem* as part of Studies 1–7. Analysing all the studies constitutes a new approach which allows to provide insight and an overview of the reasonings. It also provides general results within the scope of Niemierko's (1999) classification of task difficulty, commonly used in the analysis and assessment of Polish external exams – in the context of singular tasks and particular examination worksheets and exam editions. The classification is described in Table 2.

Table 2: Difficulty of task or set of tasks according to Niemierko (1999)

Success rate	0.00–0.19	**0.20–0.49**	0.50–0.69	0.70–0.89	0.90–1.00
Worksheet/Task	Very difficult	**Difficult**	Moderately difficult	Easy	Very easy

3.3. Research tools

In the seven research approaches, two versions of the *Stone Problem* were used.

Stone Problem 1

The first version of the *Stone Problem* was initially used in Study 1 for research using eye-tracking methodology by the mentioned *Interdisciplinary Group of Cognitive Didactics* of the *University of the National Education Commission,* then led by Władysław Błasiak, who provided this task for the study. The task was also used in Studies 2–5 (see Table 1).

Stone Problem 1 is formulated as a multiple-choice task to recognise a speed-time motion graph. Figure 1 presents the English translation of the *Stone Problem* in its first version.

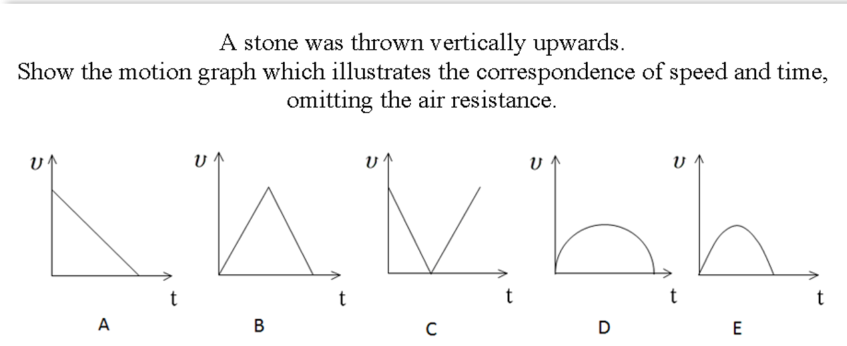


Figure 1: Translation of *Stone Problem 1* from Polish to English.

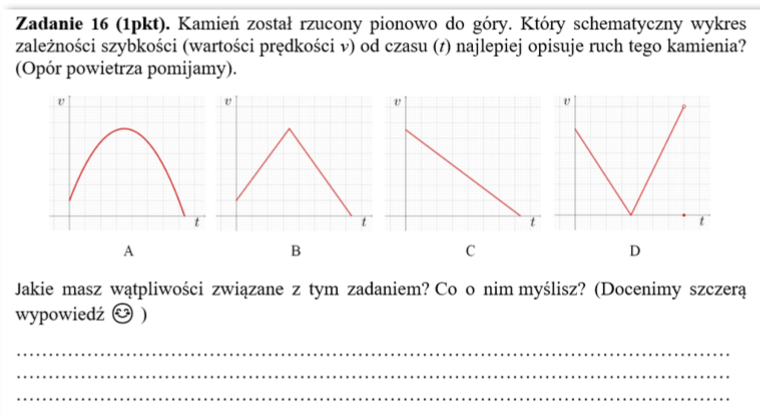
As emphasised in the introduction (Berry & Houston, 2004), a certain level of simplification must be adopted in mathematical modelling, as we are unable to present every detail of a real-world situation. Therefore, when formulating this task, we assumed a very high level of simplification, which is also reflected in the omission of scales on the axes, analysing the movement of the stone in a general manner only when it is moving (not after it landed). At this simplified level of accuracy provided in this task, the student's role is to identify the general properties of the function (monotonicity and minimal speed) that models this movement and to choose the expected answer C out of given five answers. The answer A can be also accepted as a proper graph if the student models only the first phase of the movement.

In order to solve this task in its first version, it is enough to imagine the movement of the stone and to use the common knowledge that the stone will fall, so the movement is two-phase, where first the stone rises upwards, slowing down, stops, i.e., reaches a minimum speed of 0 at the highest point of its trajectory, and then falls downwards, accelerating. If the student does not have sufficient knowledge of physics to know that this movement is uniformly variable, they can still use the process of elimination to identify the correct answer C as the only one in which the graph is biphasic and starts with a decreasing function, reaches a minimum value of 0, and then changes its monotonicity to an increasing function. However, the task contains fundamental difficulties, which we will refer to in the presentation and analysis of the results.

The task in its presented form is possible to solve by Grade 7 students in Poland (see paragraph 2.5) and in the majority of educational systems in Europe. This task is perceived by physics teachers as simple.

Stone Problem 2

The second version of the task was prepared intentionally for freshmen studying to become mathematics teachers. The task was slightly modified. Firstly, it was simplified in terms of the number of distractors compared to the first version. The semi-circular graph was removed, as it was the least attractive distractor in the previous studies. Some minor modifications were made to the appearance of the other graphs, considering the doubts raised by some of the participants regarding the solution to the first version of the task, as well as addressing the boundary conditions they mentioned. A point was therefore added to the timeline to show a speed of 0 at the end of the movement, and the graphs were differentiated to consider the fact that the second phase of the movement lasts longer. In addition, the frequently selected piecewise linear function graph and the parabola from the original E graph started from a non-zero initial speed, as emphasised by multiple participants. Since the study concerned basic mathematical skills, no U-shaped graph was included, as analysing this graph would require knowledge of physics. In the second version of the task, it was still sufficient to recognise the correct monotonicity of the function modelling the phenomenon, as in the first version.


Zadanie 16 (1pkt). Kamień został rzucony pionowo do góry. Który schematyczny wykres zależności szybkości (wartości prędkości *v*) od czasu (*t*) najlepiej opisuje ruch tego kamienia? (Opór powietrza pomijamy).

Jakie masz wątpliwości związane z tym zadaniem? Co o nim myślisz? (Docenimy szczerą wypowiedź 😉)

....
....
....
....

Figure 2: *Stone Problem 2* designed as written diagnostic test for students starting university studies in mathematics to become teachers (in Polish).

Finally, to take into account what was stressed in the introduction about many different possibilities to model the situation according to the chosen level of precision – the wording of the task was changed into 'Show the motion graph which best illustrates', to allow students to treat the expected answer (D) as not ideal.

Furthermore, the wording of the task has slightly changed into the following:

> A stone was thrown vertically upwards. Show the motion graph which best illustrates its speed changes (v-values) in time (t). (Omitting air resistance)

The phrase "best illustrates" stresses the schematic nature of this graph. The encouragement for sharing any comments or doubts and a dedicated place for them was added:

> What doubts do you have about this task? What do you think about it? (We appreciate honest feedback)

Figure 2 presents the task in Polish, under the name of Task 16.

4. Research results overview

This section of the article briefly presents the quantitative results of Studies 1–7.

4.1. Answers to *Stone Problem 1*

Table 3 shows the results of our Studies 1 to 5. It presents research results for different groups of participants: experts (academics), post-graduate students in mathematics and physics, university students (different kinds of studies), pre-service mathematics teachers, and secondary school students (Grade 10), using different methodologies: eye-tracking (using HiSpeed or Gaze Point), written questionnaires, and interviews with chosen participants.

The success rate present in the last column is calculated by considering the number of expected answers C divided by the total number of participants. Answer A was excluded from the success rate because an analysis of the whole movement was expected, and additionally some participants chose it based on misconception (see *Picture 3* misconception). The participants who gave two answers usually indicated A and/or C or two out of B/D/E, so the provided method of calculation does not influence the overall success rate.

Table 3: Results of *Stone Problem 1* (expected answer C)

Research	N	Answers							Success rate
		A	B	**C**	D	E	NA	2 Answers	
Study 1a	4	0	0	**4**	0	0	0	0	1
Study 1b	77	5	17	**22**	3	30	0	0	0.29
Study 1c	22	0	3	**11**	3	5	0	0	0.50
Study 2	62	4	13	**25**	4	14	2	2	0.40
Study 3	20	2	4	**10**	0	4	0	0	0.50
Study 4	16	2	3	**11**	3	1	0	2	0.69
Study 5	9	1	1	**7**	0	1	1	2	0.78
TOTAL:	210	14	41	**90**	13	55	3	6	**0.43**

Analysing these results, it can be concluded that the overall success rate was 0.43, although it differed among the groups of subjects, as shown in the last column of Table 3.

4.2. Answers to *Stone Problem 2*

The distribution of answers for the next two studies is shown in Table 4. The success rate was counted in the same manner as for Studies 1-5 (number of D out of N). Only 30 people out of 78 chose the correct answer D in Study 6 and 28 participants out of 57 in Study 7. The overall success rate in this group of participants was also about 0.43 (0.4296).

Table 4: Results of *Stone Problem 2* provided by freshmen of mathematics teaching (expected answer D)

Research	N	Answer						Comments	Success rate
		A	B	C	**D**	Other	No Answer		
Study 6	78	23	7	5	**30**	2	11	22	0.38
Study 7	57	12	8	3	**28**	1	6	9	0.49
TOTAL	135	35	15	8	**58**	3	17	31	**0.43**

The answer 'other' was spontaneously provided by the participants in the space intended for them to share their comments or doubts and was not among the proposed answers. Two participants proposed completely wrong answers [see: P14(6) – MS1 strategy and P75(6) – MS2 strategy] and one proposed the U-shaped answer [see: P05(7) – MS6 strategy]. The vast majority of the respondents did not comment on the task (56 in Study 6 and 48 in Study 7).

4.3. Task difficulty

It is worth noting that the overall combined success rate in Studies 1–7 was about 0.43. According to Niemierko (1999) the *Stone Task* is generally difficult (see Table 2).

Another observation is the fact that in the group of freshmen studying mathematics teaching (Studies 6 and 7), the success rate is the same as in the entire sample. We cannot draw any reliable conclusions from this data because the sample is not representative, but it strengthens the motivation to look closely at the justifications, doubts, and answers in this group.

Furthermore, the success rate in our study did not prove to be directly dependent on our respondents' age or their school experience, which shows the increasing orderliness of this indicator presented in Table 5 for different groups of respondents. For example, the first three groups of students, including math & computer science university students and teaching majors, who had more years of experience in learning mathematics and physics, achieved worse results than the secondary school students (Grade 10). Of course, the study was not conducted on a representative sample, but the lack of a trend is evident among our respondents.

Table 5: Success rate in individual groups of respondents of Studies 1–7

Success rate	0.29	0.40	0.43	0.50	0.58	0.78	1
Partici-pants	University students (mixed kinds of studies)	Math & com-puter science univer-sity students	Math teaching majors	Secondary school students (ex-tended level, Grade 10)	Pre-service math teachers	Post-graduate students in math and physics	Experts (aca-demics)

The results demonstrate how challenging this task is, even in a diverse group of students, contrary to superficial opinions based on the physics curriculum for Grade 7.

5. Different mathematical strategies & misconceptions

From the perspective of the case study methodology, understanding any kind of reasoning used while solving the *Stone Problem* is worth mentioning, more so if it was unique or idiosyncratic and uncommon. It is important to reveal as many different approaches and types of reasoning as possible, as this should help design proper teaching instructions and address the subsequent long-term conclusions of this study regarding the diagnosed difficulties.

Therefore, the following two sections of the paper provide examples of different kinds of individual reasoning. It should be emphasised at this point that sometimes several types were observed together in selected statements, and such examples are also presented.

When quoting the participants' written works or statements, or oculographic data (such as scan paths and heat maps) they are coded as 'Pxy(z)', meaning data of Participant with the code 'xy' from Study number 'z'.

In the next two sections of this paper, all categories of solutions to the task identified in the seven studies are presented. Categorisations have been made from the perspective of the concept of function and its graph usage and understanding. Mathematical approaches for solving the task are grouped below according to the global or local properties of the function selected as the correct answer by the participants. The global properties taken into account included the monotonicity of the selected function and the type of covariation reasoning revealed in the selection of the graph shape. In this context, choosing a linear or piecewise linear function is assumed to demonstrate the participant's reasoning about the constant rate of change of a function with two given segments. In contrast, choosing a parabolic or semi-circular graph is assumed to demonstrate reasoning regarding a non-constant rate of change. The local properties, meanwhile, include the initial values and zero points of the function that models the described motion. Moreover, the case study revealed various *Picture* misconceptions, described and numbered below for identification purposes. Due to the task having two different versions, the shape of chosen graph is added in brackets to avoid confusion while presenting the students' answers[4].

Examples of the categories are presented first to describe the whole context. A summary of the different categories of reasonings is provided in Table 6 in the Discussion section of the paper, and a summary of the different types of the *Picture* misconception is provided in Table 7.

MS1. Vertical line as a graph strategy

Three students, on their first day of mathematics studies for teachers, pointed out such an answer during Study 6.

The first one, P14(6), decided not to mark any of the provided graphs and proposed a vertical line as a graph in the commentary space, demonstrating a strong association with the stone's movement trajectory:

> [P14(6)]: "The slope of the throw also seems important to me. But more reliably, when we throw something vertically, it usually falls the same way – we get hit on the head with this stone."

Similarly, two others chose answer A, but suggested that this was the answer closest to being correct, and that the graph should be vertical:

> [P41(6)]: "A stone thrown vertically upwards will fall vertically downwards (it is impossible to throw a stone in a perfectly straight, vertical line)."
>
> [P50(6)]: "There is no vertical line among the drawings."

This kind of answer represent the *Picture* misconception of treating the graph literally as a picture of trajectory: [|]. The vertical line will be named the *Picture 1* misconception. The participants who chose this answer demonstrated a level of *no coordination* concerning covariational reasoning (Thompson & Carlson, 2017), because they paid attention only to the change of one variable: the height or position of the stone (instead of its speed), ignoring the change in time.

Moreover, the afunctional graph provided by these participants could be interpreted as a manifestation of their lack of understanding graphs and, especially, functional relationships, as they indicated an infinite number of values at a single point in time. This approach was unexpected, as the participants were mathematics students. Leinhardt, Zaslavsky, and Stein (1990) stressed that *personal distractors* (based on past experiences) and *pictorial distractors* (based on the visual features of the graph) are accumulated in this misconception. However, the Discussion section of this paper provides another possible interpretation of this misconception appearing in group of mathematics students in the context of the *Dual Process Theory* (Kahneman, 2011), as an unconscious and strong activation of System 1 of fast thinking without overcoming it.

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[4] e.g parabola [∩] representing graph E of *Stone Problem 1* and graph a of *Stone Problem 2*, and for graph D of *Stone Problem 1* is used: [semicircle].

MS2. Increasing function strategy

The second out of the self-proposed answers in Study 6 emphasised the monophasic nature of the movement and, again, the strong association of the graph with the trajectory of the movement, presenting the next misconception: *Picture 2*. We read:

> [P75(6)]: "No graph is suitable. Over time the speed should increase."

This person demonstrated ability to think covariationally, as they took into account the time change and described the functional relation. However, we do not know precisely what type of increasing function this person was taking into account.

MS3. Decreasing linear function strategy

Choosing a decreasing linear function had two disclosed reasons.

MS3a. Stone goes down

Some respondents read the task carelessly and treated the decreasing function graph as the trajectory of the movement at first glance – this misconception will be referred to as *Picture 3*. The response of the graph shape [\ ] can be also the result of a modified trajectory of the stone [|]. An example of this reasoning is the statement:

> [P05(3)]: "The first graph that caught my eye was graph A [\ ], but then I realized the stone is thrown upward, not down from a certain height. (...)"

The picture misconception was strongly rooted in this person, as the participant changed the answer from A [\ ], revealing the *Picture 3* misconception, into the parabolic shape in answer E [∩], which indicates a different misconception, *Picture 6*.

MS3b. Did the stone fall? No

The most common reason for those who chose the graph of the decreasing linear function was the following question:

> [P29(6)]: "Did the stone fall?" [choosing \ ]

This dilemma occurred mostly for mathematics students and pre-service mathematics teachers when deciding between the shape \ or \/ as the answer. In Study 1, this type of argumentation appeared only among mathematics students who tend towards abstract reasoning; the other students, however, assumed, using common sense, that the stone must fall and that the movement should be analysed holistically.

However, one graduate physics student wrote the following comment for this task:

> "Lack of accurate description of motion in the task – should only the vertical throw be considered, or the fall as well."

The participant eventually decided on both answers: \ and \/ to be correct depending on the context.

As mentioned in the methodology section, the above question on whether we should analyse the second phase of the movement or not, answers such as: \ should be considered correct, as they depict the first phase of the stone's movement. The participant providing such an explanation overcame the fundamental difficulty of the task concerning the *Picture* misconception.

Moreover, the question of whether the second phase of the stone's movement should be included in the answer also arose among those who answered incorrectly:

> [P21(6)]: " It depends on the height <u>and whether we also observe how it falls</u>."
> [choosing /\ ]

The last statement also shows the participant's physics misconception, stating that the graph "depends on the height", revealing the *Picture 5* misconception (see category MS4e).

MS4. Piecewise linear function analysis strategy

MS4a. The piecewise linear functions analysis strategy based on physics knowledge

One of our experts from Study 1 was a mathematics scientist [P62(1)] who analysed graphs B and C exclusively after intently reading the wording of the problem, which is visible in the scan path, presented in Figure 3.

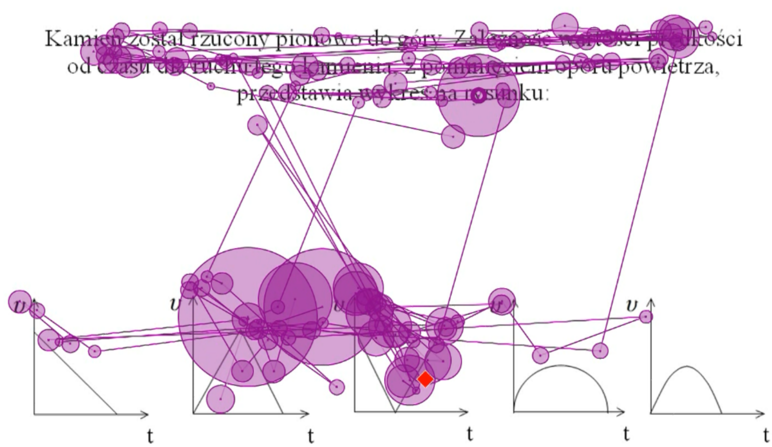

Figure 3: Linear functions analysis strategy by expert P62(1)

Only graphs with linear changes of speed (increase and decrease) were considered by the academic. It can be ascertained that the subject assumed that the changes in the velocity of the stone are described by the linear function $v(t) = gt$, which is a piecewise linear function. The strategy of combining physics-related knowledge regarding the relation of the speed of the stone to time during its movement, as well as the expert's knowledge of linear functions in segments resulted in the analysis of graphs B and C. The expert confirmed his way of thinking during the interview, it was obvious for him that the answer should be a piecewise linear function.

MS4b. Piecewise linear function analysis strategy based on typical school task

Respondents sometimes delivered the correct answer choosing – a bit randomly – between piecewise linear functions, e.g., as explicitly written here:

> [P16(3)]: "I don't know which answer is correct – but I think the graphs presented during physics lessons were mostly linear, so I discarded curves with curved arcs."

If not for the sincere written statement revealing the subject's strategy, the heat map of the visual attention of this student (see Figure 4) could only suggest choosing answer C.

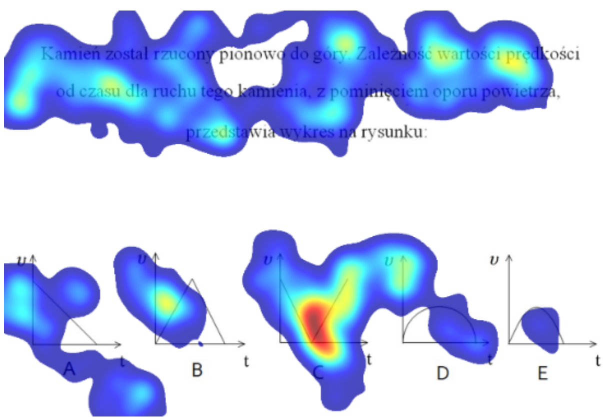

Figure 4: Heat map of visual attention of participant P16(3)

MS4c. Did the stone fall? Yes

Dilemmas related to the question were also resolved in favour of the correct answer and analysis of the entire movement. For example, a student wrote:

> "The motion of the stone could raise some doubts. (a) whether only the "upward" movement – answer A (b) or until the object falls to the ground ("upward and downward") – answer C [\/]."

Another mathematics student wrote:

> [P27(7)]: "In secondary school, I took physics as an advanced course (I was also planning to study physics at university, but in the end, I chose mathematics) – these types of problems were common, but I'm not entirely sure what type of movement was made by the stone. Did it fall? Do we calculate the speed upwards or downwards as well? I marked D [\/], but I'm not absolutely sure of the answer."

As mentioned in the methodology section – the last answer shows that both answers provided by this person are correct and that their doubts are fully justified.

MS4d. The stone will bounce back

A completely different reasoning was shown by P42(3), who initially chose answer A [\ ] and wrote the following comment:

> [P42(3)]: "At first, I considered whether the stone would bounce. I figured it wouldn't, because when falling in this way:  onto the ground, it would not bounce. Now that I'm writing this, I am having second thoughts, as its speed increased when falling, so it most likely will. I would have chosen C [\/] now."

It is worth to note that without the written statement revealing the subject's reasoning, the scan path of their visual attention (Figure 5) could only suggest a hesitation between selecting the answers A and C. We would have no concerns and would not question the correctness of this answer – the hesitation could be easily hypothetically explained by one-phase or two-phase movement consideration.

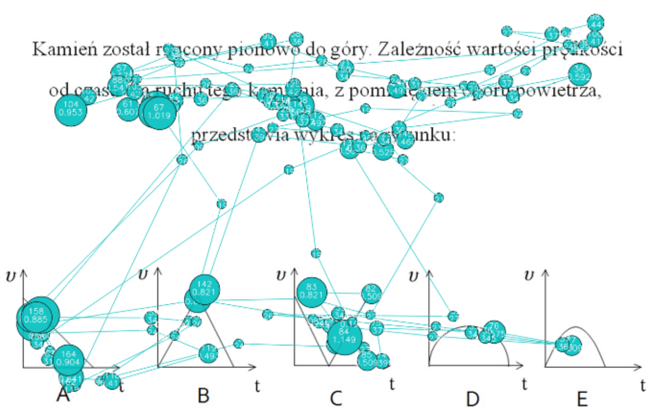

Figure 5: Scan path of P42(3)'s improper strategy regarding bounce of stone

This unexpected appearance of the *Picture* misconception is numbered as *Picture 4*.

MS4e. Piecewise linear function analysis strategy resulting in the wrong answer

It is worth noting that many respondents considered only linear functions. One example is the work of computer science student P27(1), whose scan path is provided in Figure 6. They provided the wrong answer B [/\], which is connected with the *Picture 5* misconception, considering the trajectory of a stone going "up and down" in a linear manner, which can be also perceived as combining *Picture 2* and *Picture 3*.

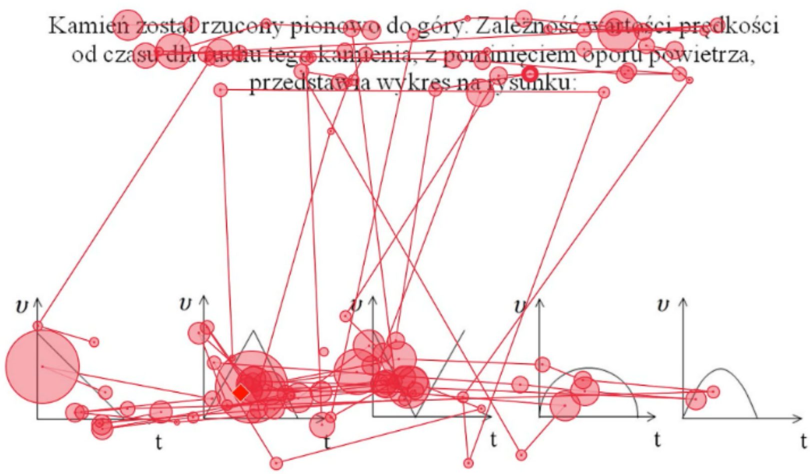

Figure 6: Linear function analysis strategy implemented by P27(1), resulting in wrong monotonicity.

MS4f. Pointy graph as no air resistance

The students attempted to provide a scientific explanation of the correctness of their answer B [/\], but ended up presenting pseudo-scientific explanations, such as in this example:

> [P02(3)]: "I think this is the correct answer, because if we are to not consider air resistance, the graph has to be pointy."

MS5. Strategy of searching for another function of proper monotonicity

This category includes responses by those who believed that the V shape was not correct at all or not entirely correct, and therefore searched for a different, correct shape for the function graph.

For example, the students taking part in Study 4 were asked to share their doubts in a form. An interesting and in-depth description of the mental work done in regard to *Stone Problem 1* was provided by a pre-service teacher of mathematics P14(4). The participant provided the self-analysis and retrospective descriptions of his struggles, in present tense, the translation of which is provided with drawings in the Appendix. As his description mentioned many different issues, over the course of this paper, fragments of his deliberations are provided when related to the described categories.

MS5a. Graph more sophisticated than \/

This category is illustrated by the mentioned student P14(4). After choosing the proper answer, he wrote:

> [P14(4)]: "(...) 2) I'm still thinking about the inaccuracy of the graph, whether the speed is not supposed to increase at first (something like this: ). By throwing the stone, we provide it with acceleration, which will be shortly negated by the force of gravity."

Providing a model of a real situation, as mentioned in the theoretical part, section 2.1 (Berry & Houston, 2004), we always need to decide the level of precision. His reasoning takes into account the beginning of the movement, which is good. However, he does not consider the final part of the movement, ending with speed equal to 0, which can be perceived as an oversight.

For Study 6 and 7, the graphs were purposely changed to not to start with 0 speed. This did not prevent the participants from having doubts concerning boundary conditions. One of the concerns provided in this context involved the initial speed issue, which, according to the respondent, should be 0:

> [P28(6)]: "Why do none of the answers start at zero speed when the stone is held right before being thrown at speed?"

However, not having conducted interviews during these studies, we cannot claim what kind of graph the participant expected. The fact that the participant chose correct answer D [\/] suggests that the initial conditions would only make the graph more sophisticated and detailed.

MS5b. Possibly U-shaped graph

Many respondents had doubts about the shape of the correct graph and considered it a parabola that opens upward with zero at its apex. Some of them shared their doubts. One example comes from the work of the mentioned participant:

> [P14(4)]: "(...) 1) (...) I wonder if this is what makes the graph a  and not a . (...)" (see Appendix for the context)

MS5c. U-shaped graph

Some respondents thought the proper graph should have a U shape – their strategy can be described as choosing \/ as being the most similar to the correct answer. Some examples are provided below.

> [P13(5)]: "I think a U-shaped graph would apply here, but there's no such answer, so I chose C [\/]."

Participant P63(6) gave the expected answer D [\/], but mentioned two concerns:

> [P63(6)]: "It is unknown to what height it was thrown, so in a certain case its fall speed would have reached maximum at some point. Also, this graph should be more parabolic."

> [P63(6)]: "The force of gravity is at work. It will be a graph that is a parabola opening to the top, because, as is especially visible when the stone is falling, its speed will keep increasing, and the stone will keep accelerating significantly. It will cross the 10 metres between floors 2 and 1 faster than between floors 5 and 4."

The last answer reflects the respondent's belief that the stone gaining speed means that the acceleration is increasing, leading them to the conclusion that the rate of change is not uniform. The respondent used "keep accelerating" interchangeably with "gaining speed". Indeed, in everyday language, to keep accelerating means to increase speed, and in everyday language, we do not usually concern ourselves with the rate of change of the acceleration itself, focusing our attention on speed. Thus, this reasoning may have its origins in everyday knowledge and language.

The next category (MS6 – Example 3) provides another example while showing the struggle with monotonicity.

In categories MS5b and MS5c, two types of misconceptions emerged: "Faster = increasing acceleration", as in the last example, and "Smooth graphs in nature" in other examples. It is also possible that the participants unconsciously confused the trajectory of the movement (the *Picture* misconception) or the type of graph – instead of a speed-time graph they could be thinking of distance-time motion graphs (see, e.g., MS6a).

MS6. Struggle with monotonicity

MS6a. Struggling and overcoming

Some respondents revealed their struggles with the intrusive, incorrect monotonicity – resulting in the *Picture* misconception being present despite the answer being correct. Four examples are provided to illustrate the diversity of reasoning and descriptions in this context.

Example 1 comes from the mentioned pre-service mathematics teacher P14(4)'s retrospection on solving *Stone Problem 1* (see Appendix for the context):

> [P14(4)]: "First, I discarded A, B, and C, and hesitated strongly between D and E. After 15–20 seconds of deliberation, I chose E [∩], as the speed was increasing too rapidly in D. After a while, I realized that I'm reading the graph incorrectly: I thought the speed increased on the graph where it actually decreased. Therefore, the only possible

solution had to be C, as when the stone flies upward, the speed will decrease, and when it starts to fall, the speed will increase. I am now convinced that it's supposed to be C, but I'm thinking about the fact of ignoring air resistance: (...)

3) Speed vs time dependency graphs can be subconsciously confused with a graph of the flight of the stone: [Figure 7]"

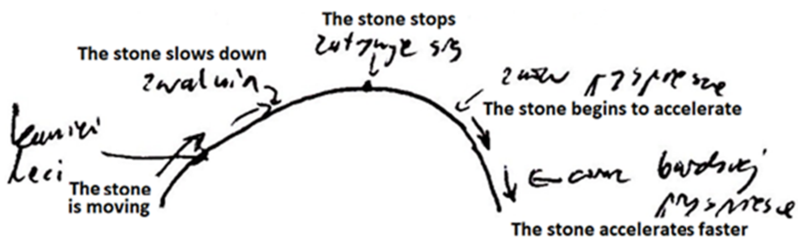

Figure 7: Stone trajectory analysis by P14(4)

This reasoning includes the *Picture 6* misconception manifesting as the conviction regarding the parabolic up-and-down shape [∩], with its two reasons already mentioned by the participant.

Example 2 shows a positive inference of common knowledge which was observed during the interviews given by some of the subjects who provided the correct answer C. Figure 8 shows the struggle of P32(3) related to discarding answer B [/\].

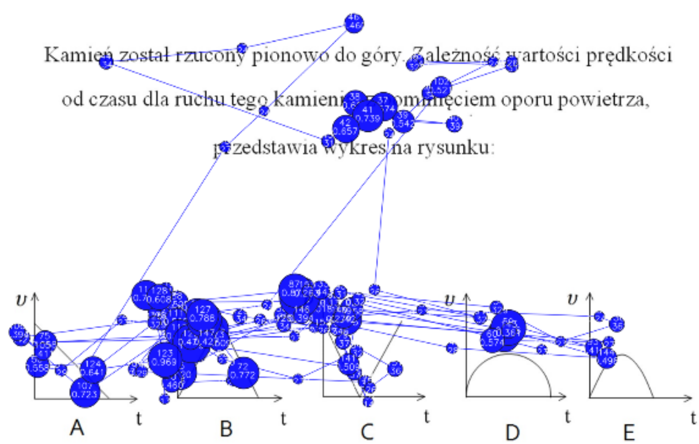

Figure 8: Visual attention of P32(3)

P32(3): described her struggle with the task as follows:

[P32(3)]: "I don't know why I spent so much time on this task – obviously the object had to stop, and then move again, which is only represented on one graph. You can see how much I'm being blocked by my aversion to physics. Even a task which imposes a specific association causes me to feel reluctant, even though the task is very easy."

Further examples of struggling with the monotonicity and overcoming difficulties are provided. Example 3 comes from Study 7, where only one participant proposed their own graph, provided in Figure 9. The student proposed the U shape, but marked a completely different graph: answer A, the parabola opening to the bottom.

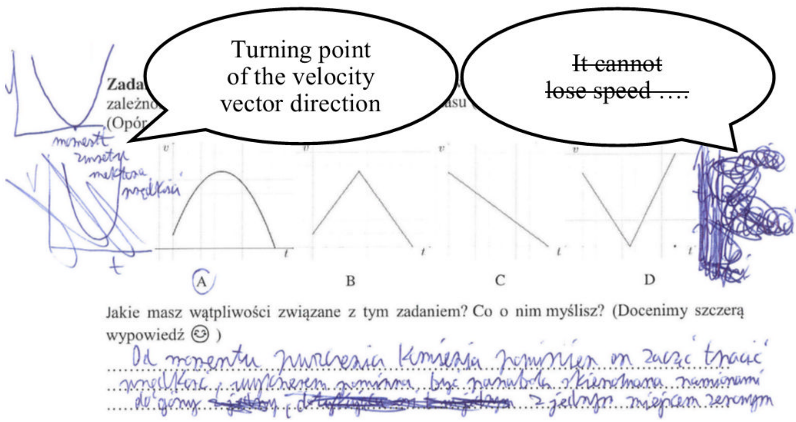

Figure 9: Answer to Stone Problem 2^{nd} version by P05(7).

The following comment was provided as part of the answer:

> [P05(7)]: "From the moment the stone is thrown, it should lose speed, the graph should be a parabola opening to the top with one zero-point."

The work likely shows the participants' struggle in overcoming the difficulty presented by the "up-down" direction of the trajectory. This is indicated by choosing answer A, with a comment stating "It cannot lose speed ... " [P05(7)] written next to answer D [\/], then very firmly crossed out (see Figure 9). P05(7) likely forgot to cross out the previous answer A. It is worth noting that they formulated an additional, sophisticated comment next to the zero-point of the graph, showing their expertise in physics: "Turning point of the velocity vector direction [sense][5]".

Example 4 shows that some participants could explicitly denote their struggles with the movement path of the graph:

> [P62(6)]: "I'm not sure if this concerns the trajectory of the flight or the changes in speed in relation to time"

[5] During physics lessons in Poland, emphasis is placed on distinguishing between a velocity vector and its value as well as a third feature, sense, apart from the magnitude (size) and direction of vectors, i.e., here, the participant mentioned that the opposite vectors have the same direction (defined by a straight line) but opposite senses.

MS6b. Struggling and not overcoming

Many responses revealed this approach – such doubts can be seen, for example, in the following works.

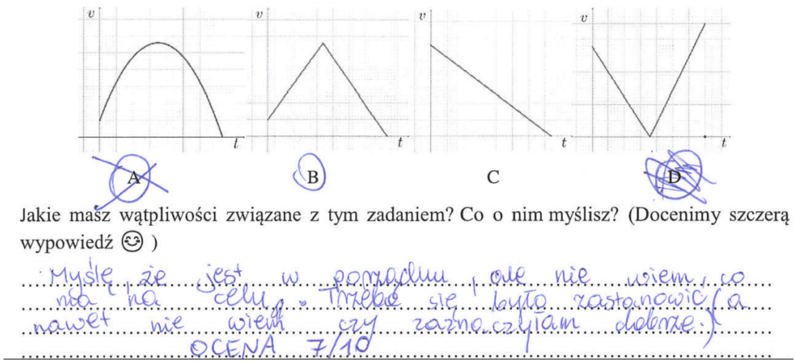

Figure 10: Struggling with the answer by P12(7)

In Figure 10, we can see that P12(7) marked and crossed out answers D and A, finally choosing answer C, writing the following in the comments:

> [P12(7)]: "I think it's fine [this task], but I don't know what the point of it is. I had to think about it (and I'm not even sure if I marked it correctly). RATING 7/10."

Another respondent wrote, for example:

> [P39(7)]: "Answer D [\/] is very clever and threw me off track. Part of me is still debating whether D is the correct answer. . . "

MS7. Non-linear functions analysis strategy

In this category, the participants exhibited the *Picture 6* misconception denoted earlier, expecting a parabolic answer [∩] or an answer which is not linear, i.e., a semi-circle shape. We can distinguish three different reasons for expecting this shape:

1. The *Picture 6* misconception itself, concerning the visualisation of the trajectory of the movement:
   a) a literal representation of such a trajectory (i.e. considering diagonal projection)
   b) Taking into account changes over time in the graph
2. Based on physics-related beliefs that natural shapes are not as 'pointy,'
3. Based on confusion between speed-time and distance-time motion graphs.

The reasons can be combined, not consciously decided, and we cannot be sure which were dominant in the provided examples. In category MS5, we provided P14(4)'s method of overcoming reasons 1 and 3. The following examples illustrate the MS7 category.

Student P11(1) imagined the shape of the graph as non-linear, usually parabolic and approximating the trajectory of a stone – the *Picture 6* misconception – with a visual analysis of only the graphs included in answers D and E, as can be seen in the form of numerous fixations at the area of interest containing these answers (see Figure 11).

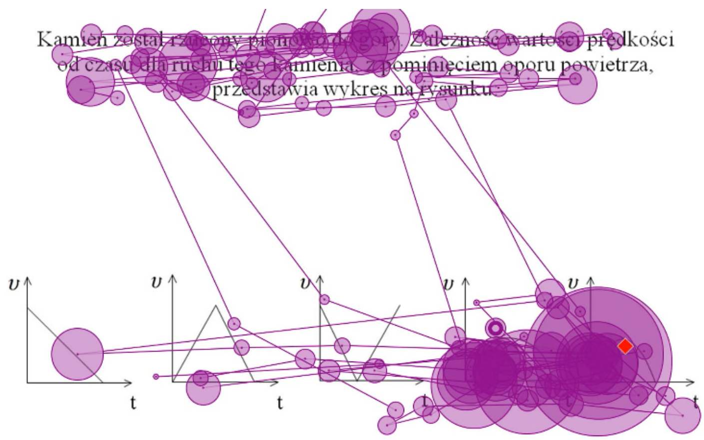

Figure 11: Misconception *Picture 6* [P11(1)]

The second part of participant P05(3)'s statement (the first part was analysed in MS3a) reveals a belief in the rounded shape of graphs describing nature when choosing answer E:

> [P05(3)]: "The first graph that caught my eye was graph A, but then I realized the stone is thrown upward, not down from a certain height. The other graphs seemed too sharp to me."

Figure 12 shows another example – the heat map of participant's P17(3) visual attention, showing deliberation between answers D and E.

This participant provided the following comment, choosing answer E:

> [P17(3)]: "This task was quite easy to visualise. The last two graphs seem similar, but a clear difference can be seen after further deliberation."

Yet another person states:

> [P19(3)]: "I don't like physics-related tasks, but I chose answer E because I consider it <u>the most natural</u> when imagining how an object behaves when thrown upward."

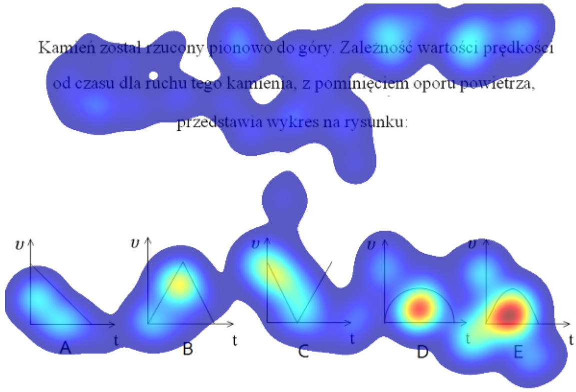

Figure 12: Visual attention of participant P17(3)

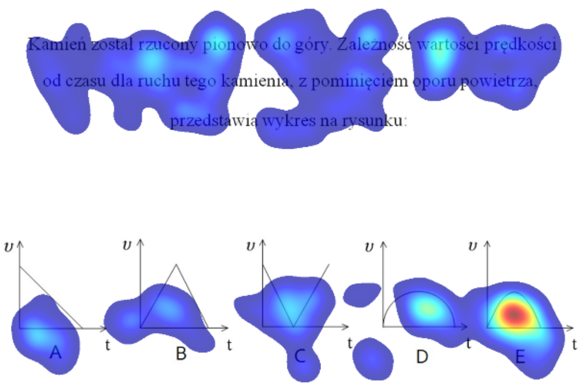

Figure 13: Visual attention of participant P21(3)

Figure 13 shows, in turn, the visual attention of participant P21(3) based on their intensive analysis of graphs D and E.

Choosing answer E, the student only wrote:

[P21(3)] “This seems to be a trick question (. . .)”

The next example shows slightly different reasons behind the same answer. Computer science student P24(1) most likely selected the graph they visualised beforehand, barely analysing the other graphs. The participants’ scan path is shown in Figure 14.

Another participant chose E [∩], and explained this choice as follows:

[P07(6)]: “When a stone is thrown upwards, (...) when it reaches its highest point, it is rather unlikely to fall rapidly, as if it had bounced off something.”

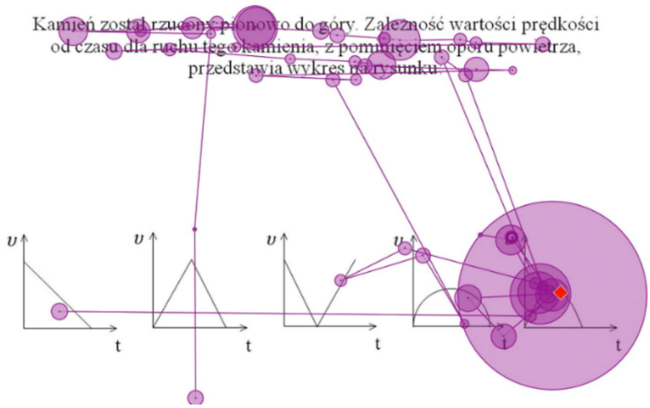

Figure 14: Visualising the expected parabolic graph, presenting the *Picture 8* misconception [P24(1)]

MS8. Up-down graph shape strategy

In this task, the 'up and down' association activates generally because of the intuitive visualisation of the trajectory of the movement. Such a response is furthermore compounded by experiences in the subjects' everyday life, as everyone has experienced performing and observing an upward vertical throw. This experience reinforces the temptation to choose the "up-down" shape of the graph. The consideration of any 'up and down' shapes will be referred to as the *Picture 7* misconception.

MS8a. Only the 'up and down' shape

As stated in our previous paper regarding Study 1 (Rosiek & Sajka, 2019), the most popular incorrect answers were those whose shape resembled the path of the stone; 'up and down' – answers B, D, and E. Figure 2 is the so-called Gridded Area of Interest for 99 students taking part in the first study. It shows the dwell time of their visual attention when working on the task. We can see that in the student group, these were the charts they focused on, which directly influenced their choice of answer. As many as 52% of all 210 subjects of the study gave one of those three answers.

In fact, also every hesitation between answers A [∩] and B [/\] in Study 7 is an example of this approach – such as the following, where the participant answered A, and wrote the following comment:

> [P20(7)] "A and B are too similar".

Another example can be the following:

> [P08(6)]: „I have doubts about the point at which the stone stops flying upwards and starts falling downwards, I am not sure about the relationship between speed and time."

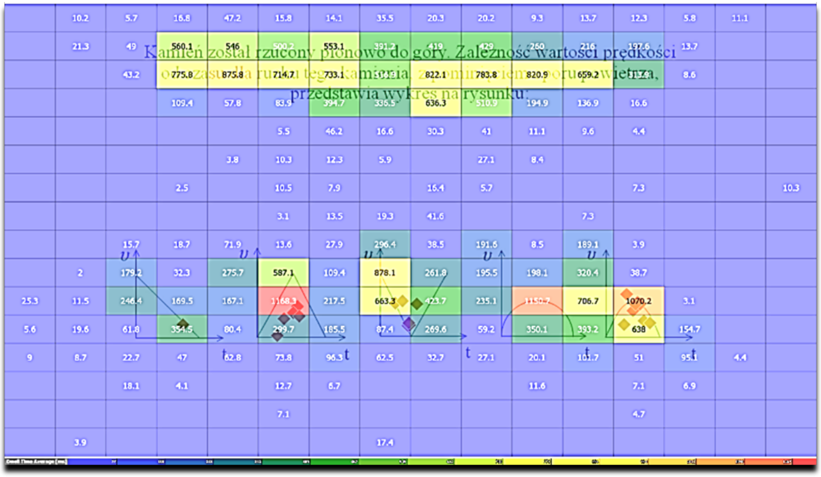

Figure 15: Gridded Area of Interests of dwell time spent on the slide for 99 high school and university students taking part in Study 1

MS8b. 'Up and down', also verbally

Most respondents in category MS8 unconsciously pointed to the "up and down" graphs, imitating the trajectory of movement, knowing that a stone flying upwards slows down and loses speed. However, there were people who were so strongly influenced by the *Picture* misconception that their verbal description was also distorted, and they expressed the incorrect monotonicity even in writing. This category is represented by participant P21(3), who described his hesitation between E [∩] and B [/\]:

> [P21(3)]: "The stone was thrown upward, therefore **it rose to a certain height at an increasing speed** up to a certain point. Then the stone **fell, so its speed was decreasing to zero**. Actually, if movement resistance were to be ignored, the stone would fly upward at an increasing speed up to a certain point. I don't know whether to choose E [∩] or B [/\] here."

MS8c. 'Up and down' asymmetrical graph

One of the examples of this category is the participant who, answering A [∩], revealed their misunderstanding of the standard acceleration of gravity, thinking that it is not the same in both phases of motion (uniformly decelerated and then accelerated), and expressed doubt about the pointed shape in the graph:

> [P07(6)]: "When a stone is thrown upwards, it seems to me that it is slower to rise than it is to fall, and when it reaches its highest point, it is not likely to fall abruptly, as if it had bounced off something".

This misconception is numbered as *Picture 8*.

MS8d. Shape more/less steep

Among the students who considered A [∩] or B [/\], another doubt was specified:

> [P23(6)]: “My doubt is what the mass of the stone is, because I think the bigger it is, the faster the stone will fall.”

In this case, the chosen answer was eventually A [∩]. The person likely considered another shape, similar to a parabola, but, due to the weight of the stone, more or less steep. This response revealed a physics misconception about “mass influence”. This is another way of expressing the *Picture 8* misconception.

MS9. Wrong strategy based merely on boundary conditions

Strategy MS9 is unique in that it is not based on the general properties of functions. In this case, the wrong answers (B, D, and E) were chosen as a result of a strategy based on the incorrect identification of boundary conditions. The participants implemented only a pointwise approach to function analysis, completely ignoring its general properties, such as monotonicity. The following comment provided by participant P9(5) is an example of this kind of reasoning:

> [P9(5)]: “At the moment of the throw, the speed was equal to 0, and when the stone fell to the ground (when it was already laying on the ground) the speed was also 0. So I discarded answers A [\] and C [\/].”

Then, the university student chose answer B [/\], and provided further insight:

> [P9(5)]: “I think this task is kind of related to physics, something related to uniformly accelerated motion, and it’s hard for me to answer this, because I’ve already forgotten physics stuff, so I’m not sure of this answer, but those are just my assumptions.”

We could describe the attitude as looking at the boundary conditions being enough to find the proper function (*Boundary enough* misconception).

6. Discussion

6.1. Mathematical Strategies Overview

In the previous section, the 9 mathematical strategies (MS) for solving the task were distinguished alongside their examples – 21 in total, including subcategories. This section provides their summary, starting from the overview provided in Table 6.

Table 6: Mathematical strategies of solving the *Stone Task* from Studies 1–7

Categories and subcategories		**Chosen function properties**		**Mathematical MISCON-CEPTION**
		Monotonicity	Rate of change	
MS1. Vertical line as a graph		No function	No covariation	*Picture 1*
MS2. Increasing function		Opposite tendency	Unknown	*Picture 2*
MS3. Decreasing linear function	MS3a. Stone goes down	Right tendency (by mistake)	Constant	*Picture 3*
	MS3b. Did the stone fall? No	Right tendency	Constant	–
MS4. Piecewise linear function analysis	MS4a. Based on physics knowledge	Correct	Constant	–
	MS4b. Based on typical school task	Correct	Constant	–
	MS4c. Did the stone fall? Yes	Correct	Constant	–
	MS4d. Stone will bounce back	Correct (by mistake)	Constant	*Picture 4*
	MS4e. Resulting in the wrong answer	Opposite	Constant	*Picture 5*
	MS4f. Pointy graph as no air resistance	Opposite	Constant	*Picture 5 + Pseudo-scientific*
MS5. Searching for another function of proper monotonicity	MS5a. Graph more sophisticated than V	Correct	Constant	–
	MS5b. Possibly U-shaped graph	Correct	Non-constant	–
	MS5c. U-shaped graph	Correct	Non-constant	–
MS6. Struggle with monotonicity	MS6a. Struggling and overcoming	Correct	Unknown	*Picture (various)*
	MS6b. Struggling and not overcoming	Opposite	Unknown	*Picture (various)*
MS7. Non-linear functions analysis		Opposite	Non-constant	*Picture 6*
MS8. Up-down graph shape	MS8a. Only the 'up and down' shape	Opposite	Unknown	*Picture 7*
	MS8b. 'Up and down', also verbally	Opposite	Unknown	–
	MS8c. 'Up and down' asymmetrical graph	Opposite	Changing rate, non-constant	*Picture 8*
	MS8d. Shape more/less steep	Opposite	Considered	–
MS9. Boundary condition		Local property		*Boundary enough*

6.2. Different 'graph as picture' misconceptions

The second overview draws attention to the specific difficulties associated with this task, i.e., misconceptions. These particularly include the eight types of iconic interpretation, the *graph as picture* misconception, which appeared in this task (see Table 7). All *Picture* misconceptions highlighted in Table 7 are described in section 5.

Table 7: Types of *graph as picture* misconceptions shown while solving the *Stone Problem* in Studies 1–7

Type of picture misconception	**Shape of chosen graph**	**Type of function**	**Explanation & hypothetical sources**
Picture 1	vertical line: |	no function	Exact trajectory of projective motion
Picture 2	up: /	linear increasing function	(a) Trajectory – the stone goes up in time – modified *Picture 1* (b) Distance in time
Picture 3	down: \	linear decreasing function	Trajectory – the stone goes down in time – modified *Picture 1*
Picture 4	down and up: ∨	piecewise linear function	Trajectory – because the stone will bounce back
Picture 5	up and down: ∧	piecewise linear function	(a) Trajectory up–down, shape never occurring in nature (b) Trajectory of a diagonal throw
Picture 6	up and down: ∩	non-linear piecewise function	(a) Trajectory with time (b) Trajectory of a diagonal throw (c) Distance–time graph confusion (d) Physics-related beliefs that natural shapes are not as 'pointy'
Picture 7	only up and down	piecewise function (any)	(a) Trajectory with time (b) Trajectory of a diagonal throw (c) Distance–time graph confusion
Picture 8	up and down, not symmetric	non-linear piecewise function	(a) 'Stone will go up slower' (b) 'Speed depends on mass' (shape more/less steep)

Some other mathematical misconceptions were revealed through the task, such as *Boundary enough* in the meaning that looking at the boundary conditions, ignoring the general function properties, is enough to find a proper graph. Other justifications revealed *pseudo-analytical* thought processes (Vinner, 1997), like through the category MS4f, where the incorrect pointy-shape graph [/\] was justified by no air resistance or through the category MS9 where the same answer was justified only by the boundary conditions (*Boundary enough* misconception).

Additionally, some physics misconceptions were revealed throughout the study, like in the case of the misconception *Picture 6* where the participants mentioned that natural shapes cannot have a pointy shape, but should be smooth, and in case of *Picture 8*, where we have two examples of physics misconceptions: (a) stone will go up slower (MS8c) and (b) speed depends on mass, so the shape can be more/less steep (MS8d).

The next misconception was connected with the belief that the stone gaining speed means that it is keeps accelerating. Therefore, using "gaining acceleration" interchangeably with "gaining speed". This problem is caused by everyday knowledge and language. To keep accelerating means to increase speed, and in everyday life we do not analyse the rate of change of the acceleration itself, focusing our attention on speed.

6.3. Result interpretation summary

The *graph as picture* misconception appeared in this task in 8 various ways. Leinhardt, Zaslavsky and Stein (1990) stress that this kind of misconception is particularly difficult to overcome, because it is rooted in intuitive and perceptual reasoning, and the visual nature of graphs invites associations with familiar physical experiences. From this perspective, the *Stone Problem* directly activated this real-world intuitive knowledge, activating both *personal distractors* (based on everyday experience) and *pictorial distractors* (based on visual features of the graph) which interfered with proper interpretation. This dual source of interference complicated the process of solving the task, as learners must suppress concrete associations and engage in abstract reasoning about variable relationships.

However, most of the respondents were mathematics students and people with a solid mathematical background. It is therefore worth considering how else can such notable difficulties with an elementary task be interpreted from the point of view of mathematics knowledge. The results of the study can certainly be interpreted in the context of Systems 1 and 2 according to Kahneman (2011) and their application in mathematics education (Leron & Hazzan, 2006; 2009). System 1 refers to a quick, intuitive response, without the activation of critical or analytical thinking. In this context, it can be concluded that the majority of respondents succumbed to System 1 – based on quick thinking, intuition, and initial associations when solving the task, caused by its formulation. This quick and effortless approach towards solving the task would explain the poor results.

Another reason for failure may have been the omission of important parts of the content of the task, as observed in the results of previous eye-tracking studies, where, in the context of the *Stone Problem*, the analysis of the type of relationship and the legend of the axis was omitted (Wcisło et al., 2014).

Another interpretation involves the term of *antisignal* (Hejný, 2014). Vondrová (2020) describes a task containing an *antisignal*, according to Hejný (2014, p. 51), as tasks in which a word or words signal an operation opposite to the one that leads to the correct solution. Budínová (2021) elaborates:

> While selecting the proper performance, **signal words** are often used to guide the pupil: "brought", "got", "has more than" to guide the pupil while adding (Adetula, 1990). If such a word introduces the solver to an operation other than proper, we call it an **anti-signal** (Hejný, 2014) or a **distractor** (Adetula, 1990; Nesher, 1976). Anti-signals can be another obstacle for many pupils in solving a problem. (Budínová, 2021, p. 122)

We can assume that the *Stone Problem* task included an *antisignal*, because the imposing 'up and down' image of the stone-throwing trajectory must be interpreted as an abstract speed-time motion graph with the opposite shape.

In the book by Vondrová et al. (2019), the authors proved *antisignal* tasks to be significantly more difficult for students than the original task variants. As expected, students most often confused the correct operation with its inverse. However, the expectation that the influence of the *antisignal* would decrease with the age of the students as they gained experience in solving more complex problems did not materialise. This demonstrates the power of the signal strategy (Vondrová, 2020, p. 74). The lack of age influence is in some sense confirmed by the approaches to the *Stone Problem* described in this study.

7. Limitations and follow-up

The reasoning strategies, as well as the wide range of misconceptions and difficulties identified in this study, emerged spontaneously from the participants – primarily from those who were invited to comment or who voluntarily shared their reflections. Notably, the majority of respondents across Studies 1–7 did not get interviewed, limiting the scope of insight into the prevalence of particular reasoning approaches or misconceptions. As a result, the current findings do not allow for quantifying the frequency or distribution of these phenomena.

Therefore, future studies should incorporate in-depth qualitative methods, such as interviews or think-aloud protocols, to examine the scale and variability of the observed reasoning strategies. Such research could also uncover additional misconceptions or cognitive obstacles not detected in the current investigation.

An important methodological implication lies in the educational background of the participants. The reasoning strategies were identified among students from educational stages more advanced than those at which the relevant content is typically introduced. Further research should focus on students at the level of initial instruction – particularly secondary school students in Grade 9 – to assess how such reasoning develops and to explore opportunities for early intervention or conceptual support.

Moreover, incorporating the explicit shape of the U-curve into future task designs may help determine its strength as a distractor and clarify the cognitive basis for its selection by students.

Another observation emerging from the data is that the accuracy of responses appears not to be correlated with the participants' age or even their level of mathematical experience. This finding warrants more systematic investigation to determine whether and how mathematical maturity influences performance when solving such tasks.

Additionally, the study raises interpretative questions concerning the theoretical framing of student responses – whether these are best understood through the lens of *antisignal* reasoning, dual-process theory (DPT), or embodied cognition. Future studies should explicitly compare these frameworks to evaluate their explanatory power in interpreting students' approaches to the task. Including students' subjective assessments of task difficulty could offer further insight into the role of cognitive load and intuitive reasoning.

Finally, a key area for future research is the development and evaluation of pedagogical strategies, that can effectively support students in improving their reasoning on tasks of this nature. Exploring the practical implications of these findings, particularly in terms of instructional design and intervention, will be essential for translating research into classroom practice.

8. Pedagogical conclusions

Current practices in secondary mathematics education in Poland insufficiently address the development of covariational reasoning in the context of functional relationships. This notable omission should be reconsidered, as it has significant implications for students' conceptual understanding. The findings related to the *Stone Problem* suggest that many students have not yet developed covariational reasoning to a satisfactory level. Furthermore, the interpretation of function graphs, particularly within the context of elementary mathematical modelling, remains a considerable challenge for learners.

These observations point to the need for a pedagogical shift in the way functions are introduced and taught. One promising direction involves incorporating function graphs as tools for describing motion at earlier stages of mathematical instruction, ideally beginning at the lower secondary level (Grades 7–8). Such an approach could be embedded within propaedeutic activities that draw upon students' experiences with natural or classroom-based experiments. Engaging students in the process of generating, observing, and interpreting movement data can provide a meaningful context for understanding the dynamic relationships that functions represent.

By encouraging students to describe motion through graphical representations, mathematics instruction can help them construct more robust mental models of covariational reasoning. This, in turn, may reduce their reliance on *iconic interpretations* of graphs. These early interventions could serve as a foundation for more sophisticated reasoning about functions in later educational stages. In recent years, researchers in mathematics education have conducted studies and developed learning environments designed to examine and promote this approach (e.g., Duijzer, 2020; Ferrara, 2014; Nemirovsky, Kelton, & Rhodehamel, 2013). However, these approaches have not yet become widely adopted in mathematics lessons in schools. More initiatives are needed to introduce the findings in schools. Such efforts have recently been implemented in selected learning environments designed within the *Enhancing functional thinking from primary to upper secondary school* (FunThink) project[6] and have been initiated in Polish and Slovak schools through the *Embodying Math & Physics Education* (EMPE) project[7].

Moreover, pedagogical strategies should include structured opportunities for students to reflect on their own problem-solving processes. Following the completion of modelling tasks, such as the *Stone Problem*, well-designed post-task interventions could prompt students to engage in metacognitive reflection, thereby fostering greater sensitivity to inconsistencies, reasoning shortcuts, or intuitive but incorrect conclusions. This practice can help cultivate critical and analytical thinking skills, essential not only in mathematics, but across disciplines.

Taken together, these pedagogical conclusions suggest that enhancing students' reasoning about functions requires both curricular adjustments and targeted instructional strategies. These changes should aim not only to improve procedural

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[6] Enhancing functional thinking from primary to upper secondary school (FunThink project) ID: 2020-1-DE01-KA203-005677, `https://www.funthink.eu/`

[7] Embodying Math & Physics Education (EMPE project), ID: 2023-1-PL01-KA210-SCH-000165829, `https://empe.uken.krakow.pl/`

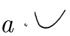
fluency but also to develop deeper conceptual understanding through meaningful contexts, early engagement, and reflective practice.

9. Appendix

The answer of the pre-service mathematics teacher P14(4).

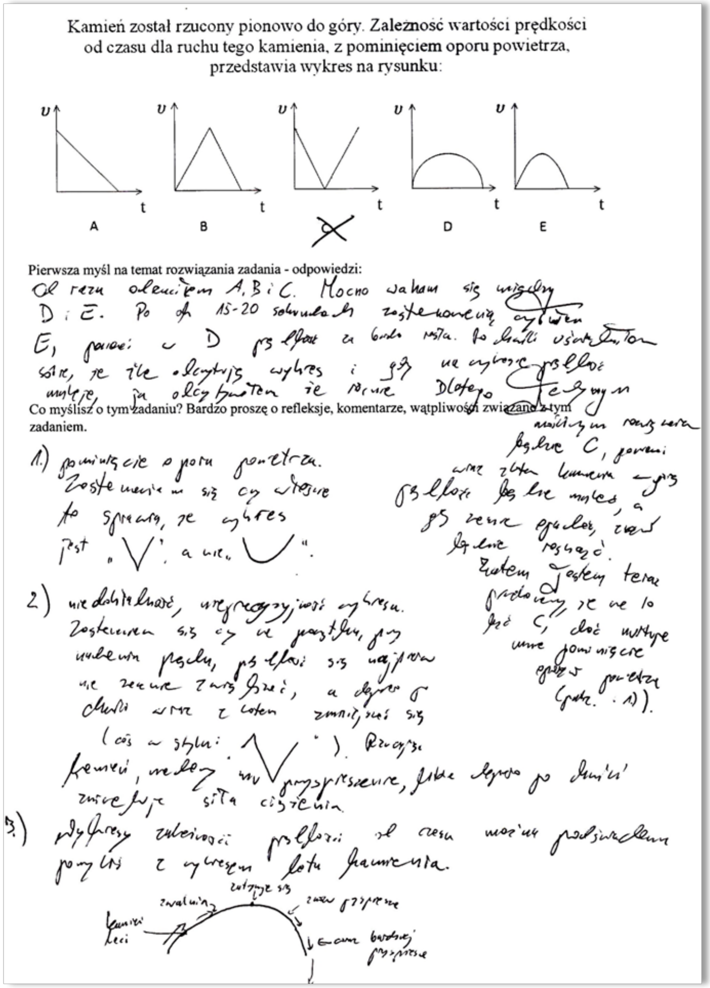
Translation of Figure 16:

"*First, I discard A, B, and C, and hesitate strongly between D and E. After 15–20 seconds of deliberation, I choose E, as the speed is increasing too rapidly in D. After a while, I realize that I'm reading the graph incorrectly: I thought the speed increased on the graph where it actually decreased. Therefore, the only possible solution has to be C, as when the stone flies upward, the speed is going to decrease, and when it starts to fall, the speed will increase. I am now convinced that it's supposed to be C, but I'm thinking about the fact of ignoring air resistance:*

1) (...) I wonder if this is what makes the graph a  *and not a* *.*

2) I'm still thinking about the inaccuracy of the graph, whether the speed is not supposed to increase at first (something like this:  *). By throwing the stone, we provide it with acceleration, which will be shortly negated by the force of gravity.*

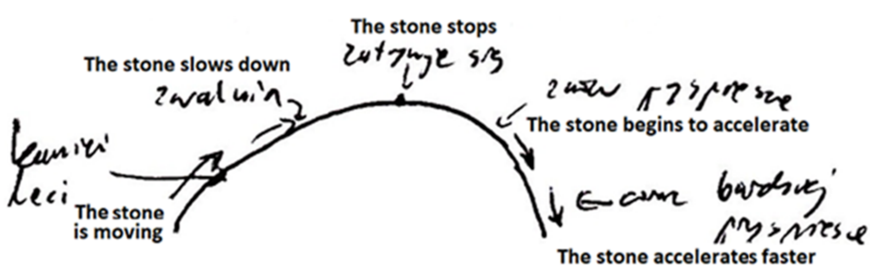
3) Speed vs time dependency graphs can be subconsciously confused with a graph of the flight of the stone:"


Kamień został rzucony pionowo do góry. Zależność wartości prędkości od czasu dla ruchu tego kamienia, z pominięciem oporu powietrza, przedstawia wykres na rysunku:

Pierwsza myśl na temat rozwiązania zadania - odpowiedzi:

Co myślisz o tym zadaniu? Bardzo proszę o refleksje, komentarze, wątpliwości związane z tym zadaniem.

Figure 16: Dilemmas concerning the proper shape of the graph written by a pre-service math teacher [P14(4)]



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