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Three Alternative Proofs of the Banach Contraction Principle*

Abstract. This study presents three alternative proofs of the Banach contraction principle (BCP). These proofs are derived from proofs of theorems that generalized the BCP, established in the existing studies. By simplifying the proofs of the generalized theorems, the alternative proofs of the BCP are obtained. The author thinks that studying the alternative proofs provided in this article helps students understand the general theorems easily.

1. Introduction

The Banach contraction principle (BCP) is as follows:

Theorem 1 (Banach,1922)
Let X be a complete metric space and let $T : X \to X$ be an r-contraction, that is, there exists $r \in (0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then, T has a unique fixed point $x^* = Tx^*$ and $\{T^n x\}$ converges to x^* for any $x \in X$.

For a standard proof of this theorem, refer to Theorem 9.23 in Rudin (Rudin, 1964), for instance.

The BCP is a very useful tool in various fields of mathematics and other applied mathematical sciences. It is applicable to the proof of the inverse function theorem (see Theorem 9.24 in Rudin (1964)). Barcz (2020) used the BCP to approximate the golden number (see also Barcz (2019)). It is also effective to show the existence of the solution to the variational inequality problems (see Section 4

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in Kondo (2023)). Due to its usefulness, the BCP has been extended to various directions.

The purpose of this article is to provide three alternative proofs of the BCP. The proofs are derived from those of generalized theorems of the BCP. Generalization of the BCP using a φ-contraction is found in Boyd and Wong (1969) and others (Barcz, 1983; Berinde, 2007; Rus, 2001). Referring to Theorem 2.7 in Berinde (2007), the author simplifies the proof and presents the alternative proof 1 presented in the next section (Section 2). Similarly, from a proof in Wardowski (2012), the alternative proof 2 is deduced; (see Section 3). The alternative proof 3 in Section 4 is a by-product from the work of Ćirić (1974). The author thinks that studying these alternative proofs, which are relatively reader-friendly, would help students understand the works with φ-contraction and articles by Wardowski (2012) and Ćirić (1974).

2. Alternative proof 1

This section presents the first alternative proof of the BCP (Theorem 1). The author obtained the proof by simplifying Theorem 2.7 from Berinde (2007). We start with the following lemma:

Lemma 1 *Let X be a metric space and let $T : X \to X$ be an r-contraction, where $0 < r < 1$. Let $x \in X$ and $\varepsilon > 0$ that satisfy*

$$d(x, Tx) \leq (1 - r)\,\varepsilon. \tag{2.1}$$

Then, the ε-ball $B_\varepsilon(x) \equiv \{z \in X : d(x, z) < \varepsilon\}$ around x is T-invariant, that is, $y \in B_\varepsilon(x) \Longrightarrow Ty \in B_\varepsilon(x)$.

Proof. Let $y \in B_\varepsilon(x)$. We aim to show that $Ty \in B_\varepsilon(x)$. Using conditions (2.1) and $d(x, y) < \varepsilon$, we obtain

$$\begin{aligned} d(x, Ty) &\leq d(x, Tx) + d(Tx, Ty) \\ &\leq (1 - r)\,\varepsilon + rd(x, y) \\ &< (1 - r)\,\varepsilon + r\varepsilon = \varepsilon. \end{aligned}$$

This completes the proof. □

Now, we present the first alternative proof of the BCP:

Alternative proof 1. Let $x \in X$ and define $x_n = T^n x$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $x_0 = x$. Observe that $\{x_n\}$ is a Cauchy sequence. Indeed, we have the following:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq rd(x_{n-1}, x_n) \leq r^2 d(x_{n-2}, x_{n-1}) \\ &\leq \cdots \leq r^n d(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.2}$$

Choose $\varepsilon > 0$ arbitrarily and define $\delta = (1-r)\,\varepsilon/2 > 0$. From (2.2), for $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\left(x_{n_0}, x_{n_0+1}\right) = d\left(x_{n_0}, Tx_{n_0}\right) < \delta = (1-r)\,\frac{\varepsilon}{2}.$$

According to Lemma 1, $B_{\frac{\varepsilon}{2}}\left(x_{n_0}\right)$ is T-invariant. As $x_{n_0} \in B_{\frac{\varepsilon}{2}}\left(x_{n_0}\right)$, we have that $Tx_{n_0} = x_{n_0+1} \in B_{\frac{\varepsilon}{2}}\left(x_{n_0}\right)$. Similarly,

$$\left\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \cdots\right\} \subset B_{\frac{\varepsilon}{2}}\left(x_{n_0}\right).$$

Let $m, n \in \mathbb{N}$ with $m, n \geq n_0$. Then, $x_m, x_n \in B_{\frac{\varepsilon}{2}}\left(x_{n_0}\right)$ and we obtain

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{n_0}\right) + d\left(x_{n_0}, x_n\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have demonstrated that

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } m, n \geq n_0 \Longrightarrow d\left(x_m, x_n\right) < \varepsilon.$$

This indicates that $\{x_n\}$ is a Cauchy sequence, as claimed.

As X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. We verify that $x^* = Tx^*$. As T is continuous, it follows that

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Therefore, T has a fixed point.

We prove the uniqueness. Suppose that $u = Tu$ and $v = Tv$. It holds that

$$d\left(u, v\right) = d\left(Tu, Tv\right) \leq rd\left(u, v\right).$$

This implies that $(1-r)\,d\left(u, v\right) \leq 0$. As $0 < r < 1$, we obtain $u = v$. This completes the proof. □

3. Alternative proof 2

In this section, we provide the second alternative proof of the BCP (Theorem 1). Remind the result

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \begin{cases} = \infty & \text{if } s \leq 1; \\ \in \mathbb{R} & \text{if } s > 1; \end{cases} \tag{3.1}$$

from standard calculus (Theorem 3.28 in Rudin (1964)), for instance). Consequently, if $s > 1$, then

$$\sum_{k=n}^{\infty} \frac{1}{k^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=1}^{n-1} \frac{1}{k^s} \to 0$$

as $n \to \infty$.

Alternative proof 2. Choose $x \in X$ arbitrarily and define $x_n = T^n x$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $x_0 = x$. We prove that $\{x_n\}$ is a Cauchy sequence. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then it results that $x_{n+2} = Tx_{n+1} = Tx_n = x_{n+1}$. Similarly, it holds that

$$x_n = x_{n+1} = x_{n+2} = x_{n+3} = \cdots ,$$

which implies that $\{x_n\}$ is a Cauchy sequence. (Furthermore, x_n is a fixed point of T as $x_n = x_{n+1} = Tx_n$.) Thus, without loss of generality, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Consequently, it holds that $d(x_n, x_{n+1}) > 0$ and $\log d(x_n, x_{n+1})\ (\in \mathbb{R})$ can be considered.

It follows that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq rd(x_{n-1}, x_n).$$

By taking the logarithm value, we obtain the following:

$$\begin{aligned} \log d(x_n, x_{n+1}) &\leq \log r + \log d(x_{n-1}, x_n) \\ &\leq 2\log r + \log d(x_{n-2}, x_{n-1}) \\ &\leq \cdots \\ &\leq n \log r + \log d(x_0, x_1). \end{aligned} \tag{3.2}$$

As $0 < r < 1$, it holds that $\log r < 0$. Therefore, the rightmost side diverges to $-\infty$ as n tends to infinity. According to (3.2), we obtain $\log d(x_n, x_{n+1}) \to -\infty$, equivalently,

$$d(x_n, x_{n+1}) \to 0. \tag{3.3}$$

The inequality (3.2) also implies that

$$\begin{aligned} &\sqrt{d(x_n, x_{n+1})}\left[\log d(x_n, x_{n+1}) - \log d(x_0, x_1)\right] \\ &\leq \sqrt{d(x_n, x_{n+1})}n \log r < 0. \end{aligned} \tag{3.4}$$

From (3.3), it holds that $\sqrt{d(x_n, x_{n+1})} \log d(x_n, x_{n+1}) \to 0$. Using the squeeze theorem, we have from (3.4) that $\sqrt{d(x_n, x_{n+1})}n \to 0$ in the limit as $n \to \infty$.

As $\sqrt{d(x_n, x_{n+1})}n \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \sqrt{d(x_n, x_{n+1})}n < 1.$$

Consequently, for $n \geq n_0$, the following inequality holds true:

$$d(x_n, x_{n+1}) < \frac{1}{n^2}.$$

Let $m, n \in \mathbb{N}$ with $m > n \geq n_0$. According to (3.1), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \sum_{k=n}^{\infty} \frac{1}{k^2} < \infty. \end{aligned}$$

As $\sum_{k=n}^{\infty} \frac{1}{k^2} \to 0$ as $n \to \infty$, we obtain $d(x_n, x_m) \to 0$ as $m, n \to \infty$. This indicates that $\{x_n\}$ is a Cauchy sequence. The rest of the proof is same as the corresponding part of the alternative proof 1. $\square$

It is noteworthy that the log-function $F(x) = \log x$ has the following properties:

(F1) F is strictly increasing;
(F2) $x \to 0 \Longleftrightarrow F(x) \to -\infty$;
(F3) For $s \in (0,1)$, $x^s F(x) \to 0$ as $x \to 0$.

As already mentioned in the Introduction, the second alternative proof is deduced from a proof in Wardowski (2012). Wardowski (2012) called the mapping $F : (0,\infty) \to \mathbb{R}$ an *F-mapping* if it satisfies (F1)–(F3), and generalized the BCP using the concept F-mapping. For further extensions, see Wardowski and Dung (2014) and a survey article by Karapinar et al. (2020).

4. Alternative proof 3

To present the third alternative proof of the BCP (Theorem 1), we prepare a notation. Let $x \in X$, $T : X \to X$, and $n \in \mathbb{N}$. Define

$$M(x,n) = \max\left\{d(x,Tx),\ d\left(x,T^2x\right),\cdots,\ d(x,T^nx)\right\} \geq 0.$$

For all $x \in X$ and $n \in \mathbb{N}$, it holds that

- $M(x,n) = d\left(x,T^kx\right)$ for some $k \in \{1,\cdots,n\}$;
- $d\left(x,T^kx\right) \leq M(x,n)$ for all $k \in \{1,\cdots,n\}$.

Using these facts, we obtain the following lemma:

LEMMA 2 *Let X be a metric space and let $T : X \to X$ be an r-contraction, where $0 < r < 1$. Then, it holds that*

$$M(x,n) \leq \frac{1}{1-r} d(x,Tx)$$

for any $x \in X$ and $n \in \mathbb{N}$.

Proof. It follows that

$$\begin{aligned} M(x,n) &= \max\left\{d(x,Tx),\ d\left(x,T^2x\right),\cdots,\ d(x,T^nx)\right\} \\ &= d\left(x,T^kx\right) \quad \text{for some } k \in \{1,\cdots,n\} \\ &\leq d(x,Tx) + d\left(Tx,T^kx\right) \\ &\leq d(x,Tx) + rd\left(x,T^{k-1}x\right) \\ &\leq d(x,Tx) + rM(x,n). \end{aligned}$$

This yields $(1-r)M(x,n) \leq d(x,Tx)$. Thus, we obtain the desired result. $\square$

Now, we can establish the third alternative proof of the BCP:

Alternative proof 3. Select $x \in X$ arbitrarily and define $x_n = T^n x$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $x_0 = x$. We demonstrate that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$. It follows that

$$\begin{aligned} d(x_n, x_m) &= d\left(x_n, T^{m-n} x_n\right) \\ &\leq M(x_n, m-n) \qquad (4.1) \\ &= d\left(x_n, T^k x_n\right) \quad \text{for some } k \in \{1, 2, \cdots, m-n\} \\ &= d\left(T x_{n-1}, T^{k+1} x_{n-1}\right) \\ &\leq r d\left(x_{n-1}, T^k x_{n-1}\right) \\ &\leq r M(x_{n-1}, m-n). \qquad (4.2) \end{aligned}$$

Using (4.1), (4.2), and Lemma 2, we have

$$\begin{aligned} d(x_n, x_m) &\leq M(x_n, m-n) \leq r M(x_{n-1}, m-n) \\ &\leq r^2 M(x_{n-2}, m-n) \leq \cdots \leq r^n M(x, m-n) \\ &\leq \frac{r^n}{1-r} d(x, Tx) \to 0 \end{aligned}$$

as $m, n \to \infty$. This demonstrates that $\{x_n\}$ is a Cauchy sequence. The rest of the proof is same as the corresponding part of the alternative proof 1. □

The author obtained the above proof by scrutinizing a proof in Ćirić (1974), which may appear challenging due to the general type of contraction mapping it addresses. After studying the alternative proof in this section, one may understand the proof in Ćirić (1974) relatively easily.

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