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Matrix analysis: method simplifications*

Abstract. This paper is devoted to explanations of calculation techniques in matrix theory under its teaching and learning, focusing on the notion of matrices and algebraic operations on matrices. Some attention is given to recommendations for teaching of determinants. A purpose of the main part of the article is to provide recommendations for lecturers to help them in teaching matrix theory, in a way that will enable for students to understand the content studied in a short time. Auxiliary schemes, short notations, and recommendations are given.

1. Introduction and Theoretical Backgrounds

Learning matrix analysis is an essential part in studying of linear algebra. Really (for example, see (Edelman, Wang, 2013; MacDuffee, 1950; Potter, Lessing, Aboufadel, 2019; Searle, 1966; Wikipedia, 2024; Steinbuch, Piske, 1963; Serbenyuk, 2020, 2023, 2023a), matrices are applied in finite-dimensional vector spaces and linear maps, probability theory and statistics, as well as analysis and geometry, graph theory, etc.

So, it is correct to begin teaching of linear algebra or higher mathematics with basics of matrix analysis.

The main problem of this research is to learn the basic theoretical backgrounds of matrix theory under short time (1-2 hours). Such situation exists in the case of the part-time or evening education, as well as of shorten programs.

There are many researches about teaching linear algebra, as well as matrix analysis. One can note the survey (Stewart, Andrews-Larson, Zandieh, 2019), in which the importance of linear algebra teaching, as well as the existence of discussions at international conferences (including the 13th International Congress in Mathematics Education (ICME 13) and the special issues in journals (for example,

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a special issue of ZDM Mathematics Education (2019) such that was devoted to teaching and learning of linear algebra)), were described.

Researches in the field of linear algebra teaching and learning consist of the following topics:

- General educational topics including the actuality and importance of studying, curriculums, and historical aspects, etc.
- Techniques under teaching and learning.
- Other researches (in this paper, such researches are devoted to only teaching and learning of matrix theory).

In the first item, one can note that in the paper (Carlson, Johnson, Lay, et al., 1993), the attention is given to the importance of linear algebra. Difficulties that students have in linear algebra course are considered in (Britton, Henderson, 2009; Carlson, Johnson, Lay, et al., 1993; Dubinsky, 1997) are devoted to improving the undergraduate linear algebra curriculum, and the paper (Dorier, Sierpinska, 2001) contains the consideration of "curricular reform actions, analyzing the sources of students' difficulties and their nature, as well as research based and controlled teaching experiments".

The following researches are devoted to learning and teaching of linear algebra:

(Trigueros, Wawro, 2020), which is a survey on approaches and using realistic problems and models under teaching; the paper (Harel, 2017) is about teaching experiment in the theory of systems of linear equations; (Gueudet, 2006) deals with geometry and "geometry intuition" under teaching and learning linear algebra; (Possani, Trigueros, Preciado, et al., 2010) and (Trigueros, Possani, 2013) are devoted to using mathematics education theories such as Models and Modeling and APOS theory in the teaching of linear algebra, etc.

Finally, different techniques of teaching matrix theory are considered in (Andrews-Larson, Wawro, Zandieh, 2017) and (Wati, Destiniar, Fuadiah, 2020). The papers (Figueroa, Possani, Trigueros, 2018) and (Daugulis, Sondore, 2018) deal with visualization techniques for matrix multiplication: in the second paper, the main attention is given to "a visual definition for matrix multiplication that is easier for many students to learn", and the first is about APOS approach. In addition, one can note the paper (Trigueros, Possani, 2013) which was devoted to using economics model (problems) in teaching (similar models were also considered under consideration of the examination of students' knowledge in (Serbenyuk, 2021).

We do not give a full survey on teaching techniques but refer the reader to (Stewart, Andrews-Larson, Zandieh, 2019) and (Trigueros, Wawro, 2020) (see also reference therein), where the main attention is given to synthesize recent researches and to new directions and questions related to the problem on learning and teaching linear algebra.

Research objective, Techniques, and methodology

The main goal of this research is to help students and lecturers in learning and teaching of matrix theory as an essential part in studying of linear algebra. This also includes techniques such that help to understand material under the time of one lecture.

The following techniques are used for obtaining results: observation, analysis of solving standard tasks by students, modelling educational tasks and problem questions, individual features of students of different forms of education, etc.

Discussion and Results

Basic notions

We must recall the main notions of matrix theory. It includes the notions of a matrices, its dimension, the main type of matrices. In this section, results and presented techniques are given by the italic font. Let us begin with the notion of a matrix:

Let us write a table of numbers. For example,

5	$\frac{\sqrt{2}}{3}$	$\frac{1}{4}$	8
188	-1	0	-$\sqrt{3}$

This is a matrix of real numbers. Let us note that each number is in a certain separate cell.

So, <u>a matrix of numbers</u> is a (rectangular) table of numbers, each of which has a fixed and clear defined location in the table. Here one can give descriptions of analogies with Excel cells and post accounts for print letters, etc.

However, in mathematics, the last notation is not correct. One can write by the following way:

$$A = \begin{pmatrix} 5 & \frac{\sqrt{2}}{3} & \frac{1}{4} & 8 \\ 188 & -1 & 0 & -\sqrt{3} \end{pmatrix} \text{ or } A = \begin{bmatrix} 5 & \frac{\sqrt{2}}{3} & \frac{1}{4} & 8 \\ 188 & -1 & 0 & -\sqrt{3} \end{bmatrix}.$$

Matrices are notated by symbols A, B, C, D, ...

One can note that in Wikipedia Contributors (Matrix (mathematics)), the following definition is given: "In mathematics, a matrix (plural matrices) is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to represent a mathematical object or a property of such an object." The last definition is more complex. One can give some remarks on using matrices for representations of various non-number data.

One can give recommendations which need to be highlighted separately in the notepad. *This is important in future under learning of basic operations.*

The first characteristic of a matrix is its size or dimension. Our matrix has 2 rows and 4 columns. This is notated by 2×4*and is called the dimension of a matrix. Note that always the first number is a number of rows and the second number is a number of columns:*

$m \times n$ *or m-by-n* - *(size) dimension of matrix;* ***m-rows*** *and* ***n-columns***.

It is important to consider few tasks on calculating of various dimensions of matrices.

The first type of matrices: <u>*rectangular matrices*</u> *and* <u>*square matrices*</u> *(when* $m = n$*).*

In the next step, we must give explanations about the arrangement of numbers in a matrix. Our goal is also to teach clear notations in full and shortened forms.

Elements of matrices are notated by little symbols with number characteristics of their place in matrix. That is, in a general case

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2\,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{m\,n-1} & a_{mn} \end{pmatrix}$$

- *full form* (this is useful for understanding)

Writing on the classroom blackboard must include explanations about coefficients in notations of elements and their connections with placements of elements. For example,

The first row: The first element is a_{11} *(which situated in the first row and the first column), the second element is* a_{12} *(the first row and the first column),* $\dots$*, the nth element* a_{1n} *is the last in this row because our matrix has the size* $m \times n$ *(mrows and ncolumns), etc., the last row is the mth row with elements* $\dots$. It is useful to propose students to help in writing notations after the first row.

One can note the following recommendations:

1. For the clearness, one can use the notation with the size of matrix (with a remark that the size is not separately notated under full writing of matrices):

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2\,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{m\,n-1} & a_{mn} \end{pmatrix}_{m \times n}$$

2. To reduce time, one can use a notation only for a square matrix with explanations about changes in the notation for the case when our matrix is a rectangular matrix. That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2\,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n\;n-1} & a_{nn} \end{pmatrix}_{n \times n}$$

Now we must consider the notion of the main diagonal and note that this notion is only for square matrices. One can give the following figure:

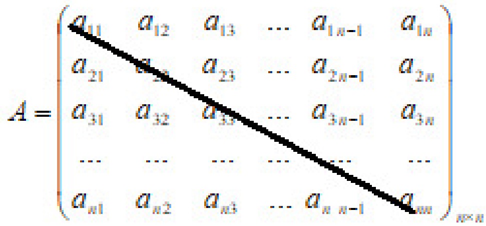

Figure 1: The main diagonal.

The main diagonal is a diagonal that is from the top left angle to the bottom right angle and this diagonal consists of elements with equal two indexes, i.e., elements of the main diagonal are following:

$$a_{11},\ a_{22},\ a_{33},\ \ldots,\ a_{nn}.$$

After the consideration of the notion of the main diagonal, the following types of matrices can be called: zero matrix, the identity matrix, as well as diagonal and triangular matrices.

Finally, the following notation is used in definitions of the basic operations.

The shortened form of a notation of a matrix
(this is useful for the compactness of notations):

$A = (a_{ij})_{m\times n}$, where $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ...n$.

Basic operations

1. One can begin with *the scalar multiplication*. Explanations must begin with an example. For example, a good form of explanations is given in Wikipedia Contributors (Matrix (mathematics)). In addition, we must note that *this operation is correct for any matrix.*

2*. The addition and the subtraction*. In this case, we must also begin with examples and note that *these operations are defined for matrices with equal sizes (dimensions). The following example of explanation is very good:*

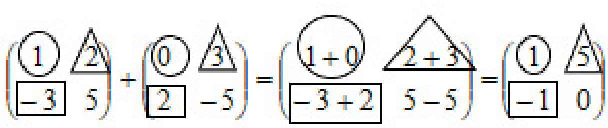

Figure 2: A schema for explanations of the matrix addition.

Auxiliary formulas can be written in terms of shortened forms of matrix notations. The attention is also must be given to cases of negative numbers (including, for the subtraction, the case when numbers of the second matrix are negative). The widespread mistakes in arithmetic operations can be noted for students.

3.*The matrix multiplication*. In this case, teaching (learning) consists of two parts. The first is about the well-posedness, and the second is about the multiplication process. One can describe by the following way:

We can multiply only two matrices such that the number of columns of the first matrix equals the number of rows of the second matrix, i.e., only matrices with the following dimensions: $A_{m\times n}$ *and* $B_{n\times p}$*. Then* $A_{m\times n}\, B_{n\times p} = C_{m\times p}$*. That is, values* n *are shrinking.* It is recommended to consider several examples in the form of discussion with students.

The second part is more complicated. As for the practice, the known formula for matrix multiplying is complicated for students. In this paper, the following technique is proposed for teaching and learning of the matrix multiplication:

Step 1. A description of the notion of the "scalar product" (or the "dot product"). *One can begin with example (here two equal-length sequences of numbers or coordinate vectors are used). That is, suppose* $\bar{a} = (1, -2, 3)$ *and* $\bar{b} = (5, -1, 0)$*; then*

$$\bar{a}\bar{b} = 1\cdot 5 + (-2)\cdot(-1) + 3\cdot 0 = 5 + 2 + 0 = 7.$$

The rule by other words:

Scalar product = First-by-first+second-by-second+third-by-third +. . . +last-by-last

Step 2. Suppose we have two matrices $A = (a_{ij})_{m\times n}$ *and* $B = (b_{jk})_{n\times p}$*. Then*

$$A_{m\times n}\, B_{n\times p} = C_{m\times p} = (c_{ik})_{m\times p},$$

where $i = 1, 2, 3, ..., m$*,* $j = 1, 2, 3, ...n$*, and* $k = 1,\ 2,\ 3,\ ...,\ p.$*. The matrix multiplication can be calculated by the following rule:*

$c_{i\,k}$= **SCALAR PRODUCT** of **ROW i** of **A** and **COLUMN k** of **B** (1)

Whence we deal with rows only for the first matrix and with columns only for the second matrix.

Remark. Relationship (1) describes a complicated formula (for understanding by students) of a rule for the calculation of the matrix product (for example, see (Rife, 1985, p. 50; Cohn, 1982, p. 99), etc.). According to this formula, we must consider only row vectors of the matrix A (the first matrix) and only column vectors of the matrix B (the second matrix). In addition, row vectors of the matrix A have a dimension such that equals $1\times n$, and column vectors of the matrix B have a dimension, that equals $n\times 1$. So, multiplying we obtain a number. Finally, one can say about lack of consistency, because the mentioned scalar products is of row and column vectors. However, this is for the simplification of notations, because we can consider scalar product of a row vector and the transpose of a column vector, or scalar product of the transpose of row vector and column vector.

So, we begin with an example. Suppose $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$. *Then* $A_{2\times 2}\, B_{2\times 2} = C_{2\times 2} = C$*. Writing elements of the matrix C as unknown* *variables is useful for the future calculations. That is,*

$$C_{2\times 2} = C = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

Let us calculate all values:
x_{11}: *we use the first row of A and the first column of B:*

$$x_{11} = \begin{pmatrix} 1 & 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot 1 + 2 \cdot (-1) = 1 - 2 = -1.$$

By analogy, we obtain

$$x_{12} = \begin{pmatrix} 1 & 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1 \cdot (-1) + 2 \cdot 0 = -1,$$

$$x_{21} = \begin{pmatrix} 3 & 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 \cdot 1 + 4 \cdot (-1) = -1,$$

$$x_{22} = \begin{pmatrix} 3 & 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 3 \cdot (-1) + 4 \cdot 0 = -3.$$

So, $C = \begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix}$.

4. *The transposition*. Here a good technique is to explain on few examples. *This operation can be formulated as following: we write the first row of our matrix as the first column, the second row as the second column, ..., the last row as the last column. So we obtain the result. The second method is vice versa: the first column changes into the first row, the second column changes into the second row, etc.* Also, one can use descriptions from (Cohn, 1982, p. 91).

Determinants

Describing definitions of the determinants of a 2×2 and 3×3 matrix, one can use a formula in the case of a 2×2 matrix, and the Rule of Sarrus (see (Cohn, 1982, p. 189-190)) for a 3×3 matrix. Such rule is more simplest for students. In addition, one can consider the triangles rule for 3×3 matrices.

Future explanations need more time that 1,5 hours. The Laplace expansion (or cofactor expansion) is very useful notion for the simplification of calculations of the determinant. This notion can be considered at the next lectures because students need more time for understanding peculiarities of applications of this technique. For the first, this technique is applicable for calculations of determinants of $n \times n$ matrix with $n > 3$. For the second, it is useful for matrices containing zeros (here the attention on 3×3 matrices). For the third, the Laplace expansion formula along the ith row (jth column) must be explained, focusing only on examples. For the fourth, a lecturer should pay the attention to the change of signs in the formula. One can use auxiliary basic descriptions from (Cohn, 1982, p. 195).

Conclusions

In the present paper, for quality students' knowledge in some topics of linear algebra, a certain system of explanations is modeled and described. The attention is given to matrix theory. The present techniques are focused on the clearness and the simplification under learning, as well as on the optimization of educational process.

The presented techniques require less time for explanations of basics of matrix theory and help students to demonstrate the knowledge in the mentioned topic of linear algebra.

Statements and Declarations

Competing Interests

The author states that there is no conflict of interest

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Study-specific approval by the appropriate ethics committee for research involving humans and/or animals, informed consent if the research involved human participants, and a statement on welfare of animals if the research involved animals (as appropriate)

There are not suitable for this research

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