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Comparison of different types of reasoning & proof present in solved tasks from geometry in Slovak and Czech mathematics textbooks for lower secondary school*

Abstract. This article presents a comparative study of two series of mathematics textbooks for lower secondary schools from Slovakia and the Czech Republic. The analysis focuses on the presence of reasoning and proving (R&P) solved tasks in geometry. Different ways of reasoning are analysed as well. The data were analysed using descriptive statistics methods. Results show similarities in the ways of reasoning present in both textbook series. Nevertheless, there are very different numbers of solved tasks and solved tasks focusing on R&P. This might be caused by different approaches textbook authors took with involving pupils in tasks with instructions. Although the deductive way of reasoning prevails in both countries, other ways are also present. The majority of solved R&P tasks also provide insight and explanation of geometrical statements.

Introduction

This paper presents a comparative analysis of two textbook series for lower secondary levels (grades 6–9, ages 11–15), one from Slovakia and one from the Czech Republic. Research focuses on the presence of solved reasoning and proving (R&P) tasks in geometry. The analysis also inquires what different *ways* and *roles of reasoning* are involved in these tasks.

In the field of mathematics education, much attention has been paid to R&P (e.g., Hanna & De Villiers, 2012; Stylianides & Harel, 2018), and the topic remains an essential subject of study (e.g., Stylianides, 2014; Michal et al., 2022; Herbert & William, 2023). R&P is often challenging to teach, particularly at the primary and lower secondary levels (Fischbein, 1982; Balacheff, 1988; De Villiers, 1990).

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Slovakia and the Czech Republic have a long-shared history, including the educational systems, which are close to each other and developed independently only after 1989 (Slavíčková & Novotná, 2022). They also have similar languages, which results in Slovaks and Czechs understanding each other in their mother tongues (ibid.). This closeness leads to much cooperation between the two countries. Studies connected to R&P, including data from different series of Slovak and Czech textbooks, have been conducted recently (see, e.g., Novotná & Slavíčková, 2023; Slavíčková et al., 2023; Slavíčková & Novotná, 2023). These studies focused on various perspectives, mainly how the topic is introduced and processed from a discovery and investigation perspectives and what types of R&P tasks are present in selected textbooks used in Slovak and Czech lower secondary schools. This paper describes a comparative study of two selected textbook series. These were selected and analysed to answer the following question:

Are there any substantial differences in the modes of reasoning in the area of geometry between selected Slovak and Czech textbook series?

During the performed analyses, the applicability of the coding system from (Sevinc et al., 2022) was piloted. Results of this pilot testing are also shared in this paper.

Theoretical Framework

In mathematics, the proof is usually understood as follows: "A formal proof, in the sense of Hilbert (1928/1967), is a sequence of assertions, the last of which is the theorem that is proved and each of which is either an axiom or the result of applying a rule of inference to previous formulas in the sequence..." (Tall et al., 2012, p. 15). In mathematics education, however, what constitutes proof might be different. Thus, some authors differentiate between different levels of proof rigour in an educational context – for instance, Hanna (1990) differentiates between *teaching proof*, *acceptable proof,* and *formal proof;* similarly, Balacheff (1988) talks about *explanation, proof* and *mathematical proof* or Blum & Kirsch (1991) distinguish *experimental verifications, pre-formal proofs,* and *formal proofs.* This broader look, which might, for instance, include non-deductive approaches, is congruent with what Watson (2008, p. 1) says about school mathematics: "... school mathematics is not, and perhaps never can be, a subset of the recognised discipline of mathematics, because it has different warrants, authorities, forms of reasoning, core activities, purposes and unifying concepts..." Though different authors use different categories or understand notions such as proof, justification, or reasoning differently (Hanna, 2020), authors usually agree that it is essential to engage pupils with R&P in a classroom (e.g., Balacheff, 1988; Hanna, 1990; Schoenfeld, 1994; Stacey & Vincent, 2009). While discussing the beneficial aspects of proof, it is often conjoined with different purposes or roles of proof in mathematics and teaching. Hanna (1990) distinguishes between *proof that proves* and *proof that (also) explains.* Both types are valid mathematical proofs, yet only the second provides an understanding of "why" the statement is true. Hanna

(1990, p. 9) mentions the importance of including such tasks in classroom instruction: "...whenever possible, we should present to students the proofs that explain rather than ones that only prove." Similarly, Peled & Zaslavsky (1997) talk about *counter-examples that (only) prove* and *counter-examples that (also) explain.* Bell (1976) distinguishes three roles of proof: *verification, illumination,* and *systematisation.* De Villiers (1990) expands the list to *verification, explanation, systematisation, discovery,* and *communication.*

The Slovak national educational programme for the 2nd level of primary school (grades 5–9, ages 11–15) mentions, among other things, several goals of learning connected to R&P, such as developing pupils' logical and critical thinking or teaching how to formulate hypotheses and support statements with arguments. Looking closer at the characteristics of the subject of mathematics (ŠPÚ, 2014) we find that the subject is primarily focused on building basic mathematical literacy and developing cognitive areas, including reasoning – solving more complex problems that require a broader understanding of connections and relationships. Among the subject's goals is that pupils develop their logical and critical thinking, reasoning, communicating, and collaborating in a group to solve a problem. The educational standard is divided into individual thematic units, specifying what pupils should know at the end of each year of primary school.

The Czech national curriculum for primary education[1] (grades 1–9, ages 7–15) is a relatively short document which comprises several parts. Among other things, part of the *educational areas* describes general *learning objectives* and mentions different educational *content areas* in mathematics (e.g., *Numbers and Numerical Operations, Two- and Three-dimensional Geometry*). What goes into what content area is described using only so-called *expected outcomes.* These briefly mention what a pupil should learn in a given *content area.* Objectives with connection to R&P are *argumentation based on combinatorial and logical thinking*, *an ability to discuss using mathematical notions and relations to classify concepts or to develop the skill of making hypotheses based on experience* or *experiment and to test or to refute them with counterexamples.* In *expected outcomes*, R&P appears mostly in connection to reasoning in geometry, e.g., "apply theorems on congruent and similar triangles for argumentation and when calculating (FEP EE, 2007, p. 30)." Outside of geometry, there is an *outcome* of using logical reasoning and combinatorial reasoning while solving problems.

Textbooks are a critical element in the educational system (Eisenmann & Even, 2011; Haggarty & Pepin, 2002). Textbook style and content might influence in-class instructions, as Fan and Kaeley (2000) mention how different textbooks resulted in different teaching strategies (e.g., time assigned to group work) or how much different technologies were involved in lessons. Researchers also suggest a connection between what is included in the textbook and teachers' decisions on what and how to teach (e.g., Tarr et al., 2006). In contrast, Sosniak and Stodolsky (1993) found that teachers rely on textbooks less, have high flexibility in their use, and report frequent use of other resources. However, the research results by Michal and Kiss (2023) suggest that textbooks are the most common resource used by in-service mathematics teachers in Slovakia and the Czech Republic.

[1] Framework Education Programme for Elementary Education (FEP EE)

Requirements of the national curriculum are reflected in textbook content and should, therefore, provide opportunities for teachers to be involved in enough diverse R&P tasks. Even though appropriate R&P tasks in textbooks might not imply the appropriate use of these tasks by teachers in their lessons (Stylianides, 2014), several analyses focusing on R&P content took place to analyse what types of reasoning are present in textbooks.

Several mathematics education researchers have focused on identifying the different ways of reasoning conveyed by textbook tasks and on analysing the occurrence of R&P tasks. Dolev and Even (2013) report differences in different textbook assignments, where some require significantly more justification by pupils than others. They also found out that there were considerably more tasks requiring reasoning in the area of geometry. Fujita and Jones (2014) focused on textbook analysis of R&P tasks in geometry. They found out that Japanese eighth-grade textbooks in the area of geometry usually focus on direct proofs.

Stacey and Vincent (2009) developed a framework for textbook analysis called *modes of reasoning.* They based it upon the student-centred notion of *proof schemes* by Harel & Sowder (1998, 2007). By proof scheme, they understand: "A person's proof scheme consists of what constitutes ascertaining and persuading for that person (Harel & Sowder, 1998, p. 244)." As Stacey & Vincent (2009) explain, they had to adjust this framework as authors of textbooks might not find the reasoning, they include in explanations convincing but as didactically sound and appropriate for pupils of a given age. After analysing Australian textbooks, they came up with seven *modes of reasoning* present in textbooks. They noticed that different modes of reasoning are presented unevenly in different topics or that every textbook tries to provide at least some rule derivation or explanation, not only present it (ibid.). They also considered most explanations as didactically appropriate rather than scientific (ibid). The majority of explanations done in the three selected geometrical topics were deductive (23), some were empirical (7), and only a few used other *modes of reasoning* (3).

Silverman and Even (2016) used the *modes of reasoning* framework to analyse 7th-grade Israeli textbooks. Results show that most R&P tasks were present in explanatory texts rather than tasks for pupils to solve. They found every *mode of reasoning* among 200 selected R&P tasks. Most of the reasoning done in geometry was using *empirical* or *deductive* ways (algebra primarily deductive), and only three explanations were based on *external convictions* (Harel & Sowder, 2007). In geometry, there were also more empirical justifications than deductive ones, which authors find surprising as geometry is a suitable area for proof introduction based on the historical development of proving. Both Silverman & Even (2016) and Stacey & Vincent (2009) also note that analysed textbooks do not indicate in any way which proofs are valid mathematical proofs and which are "only" serving as a didactical explanation.

Another tool used for textbook analysis is an analytical framework developed by Stylianides (2008). This framework also serves as an instructional tool in teacher professional development sessions.

Sevinç et al. (2022) presented an integrated framework that can be used to analyse the ways of reasoning in mathematics textbooks. This framework, similar to the one by Stacey & Vincent (2009), was developed to analyse *solved tasks* in textbooks. The framework is briefly introduced in Table 1.

Table 1: Ways of reasoning (Sevinç et al., 2022, p. 2085)

	Different ways of reasoning		**Short characteristics**
1	**Appeal to authority**		no explanation or reasoning, e.g., Euclid, a textbook, etc. says it is so
2	**Simple (1-step) deductive reasoning**		A single deduction from one or more premises
3	**Mathematising**		the explanation/ justification of transformation/decontextualization of a word problem/a problem defined in the real world, to a strictly mathematical form
4	**Reasoning by analogy**		involves making a conjecture based on similarities between two cases, one well known (the source) and another, usually less well understood (the target).
5	**Reasoning with empirical arguments/specific cases**		reasoning begins with specific cases and produces a generalization from these cases; testing claims using evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, and so forth
	a	Making claims and generalizing	
	b	Justification of claim	
6	**Developing conclusions/ justifying/refuting through deductive reasoning**		conclusions are derived from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments
	a	Generic example	
	b	Counterexample	
	c	Systematic enumeration	
	d	Other	
7	**Other**		e.g., abductive reasoning

Methodology

As multiple textbook series are available in both countries, the choice was based on similarity in the teaching approach of both series. First, both sets of textbooks cover the full range of mathematics at the lower secondary level, which is very similar in both countries. Second, their philosophy has no striking differences – they both focus on using defined or introduced concepts and procedures. Both constructivist-based and problem-solving strategies are present. Both include many unsolved tasks asking for justification. Our study focuses on solved tasks (similar to Silverman & Even, 2016 or Stacey & Vincent, 2009). The reason is apparent: Unsolved tasks do not allow us to decide unambiguously what type of R&P they might include or if they involve reasoning. The only key for classifying them as R&P is the presence of a keyword (prove, show...) in the assignment. Under a solved task, we understand any task where the solution (or at least one part) is explicitly presented.

For Slovak textbooks, the selected series was from the author collective Šedivý et al. There are two textbooks in each grade – *Part 1* and *Part 2*. Even though these textbooks are not the most recent on the Slovak market, according to our experience from meetings with mathematics teachers in practice, these textbooks are currently used most often in lower secondary schools. Regarding the structure of Slovak textbooks, there are examples of tasks, solutions, formulas and summaries, assignments, exercises, tasks, notes, and curriculum extensions. From these categories, examples and some tasks were considered *solved tasks*. The distinction between solved and unsolved tasks was clear.

The Czech textbook series selected is by Odvárko and Kadleček. The series includes this characterisation:

> *The mathematics textbook for lower secondary school by Odvárko and Kadleček is a complete set of textbooks for pupils in grades 6-9. The authors stress that these textbooks are written for pupils. This is reflected in the form, language, and content. We want to teach pupils how to acquire, process, and evaluate information independently. For a start, at least those prepared for them in the textbook. Therefore, we would like the pupils to use our textbooks as independently as possible.*[2] (Odvárko, Kadleček, 1998, p. 6)

The series consists of three volumes per grade. In contrast to Slovak textbooks, the Czech series does not include purely *solved tasks*. Even the tasks that include solutions ask pupils some additional questions and try to involve them actively in reading. Thus, it was not always without issues to tell if the task should be considered as solved. It was decided that only tasks with complete solutions should be considered *solved tasks*. Most *solved tasks* were introductory tasks for a new chapter/notion.

Our investigation is also restricted to geometry as this area of mathematics is present in each grade of lower secondary school. From a historical point of view, it is also natural to begin learning about R&P in geometry, and thus, there should be R&P tasks present. Another reason for selecting geometry was the results of a pilot study conducted by Michal et al. (2022), where teachers were asked in which areas of school mathematics, they involved R&P the most. According to this study, participating teachers from the Czech Republic are involved in R&P the most in geometry (in accordance with the national curriculum), while Slovak teachers are involved in algebra. Algebra is, however, present only later in the curriculum and not across all the grades. The other reason for this choice is that

[2] Text of this citation as well as texts in figures showing tasks from textbooks were translated by paper authors.

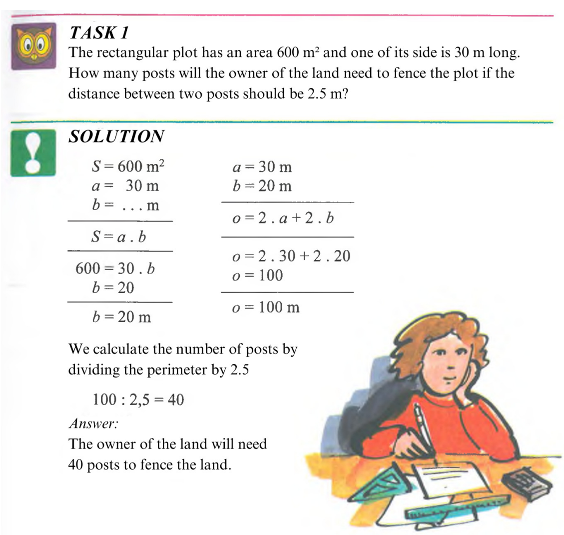
geometry offers a variety of representations (verbal, graphical, manipulative, ...) and a variety of solving strategies.

Together, eight textbooks focusing on geometry-related tasks were analysed in the Slovak series. Out of twelve volumes in the Czech series, seven include some topics from the area of geometry and thus were involved in the analysis.

This article uses the following terminology: An R&P task is a task where reasoning and/or justification is present, at least in a part of the solution. As reasoning, all types presented in Table 1 were considered.

Figure 1 shows an example of a task that we did not consider R&P, even though there is an explanatory comment in the solution that we get the number of fence posts as the perimeter of the fence divided by the distance between the posts. However, this comment does not explicitly state why this is so, so we do not consider it an argumentation. We, therefore, did not include such tasks in the analysis.

The textbooks were coded using the codes based on the framework of Sevinç et al. (2022). The codes for the role in which the R&P is present initially emerged from the roles of De Villiers (1990). However, this categorisation was somewhat subjective as it was not always possible to distinguish different roles based only on the task itself without further context of the learning situation in which the task would be used. Thus, it was agreed to determine and code "only" if the explanation involves conviction/verification (analogically to *proof that proves* (Hanna, 1990)) or also explains (analogically to *proof that (also) explains* (Hanna, 1990)).



TASK 1

The rectangular plot has an area 600 m² and one of its side is 30 m long. How many posts will the owner of the land need to fence the plot if the distance between two posts should be 2.5 m?

SOLUTION

$S = 600\ \text{m}^2$, $a = 30$ m, $b = \ldots$ m

$S = a\,.\,b$

$600 = 30\,.\,b$

$b = 20$

$b = 20$ m

$a = 30$ m, $b = 20$ m

$o = 2\,.\,a + 2\,.\,b$

$o = 2\,.\,30 + 2\,.\,20$

$o = 100$

$o = 100$ m

We calculate the number of posts by dividing the perimeter by 2.5

100 : 2,5 = 40

Answer:

The owner of the land will need 40 posts to fence the land.

Figure 1: Non-R&P task with explanation. (Šedivý et al., 1998, p. 25)

Coding was done using MaxQDA software using the inductive-deductive approach (Saldaña, 2013). Each country first coded their textbooks in pairs. After that, new codes were developed for coding the representations and two categories in which R&P tasks were later sorted. Two types of R&P tasks were considered.

The difference is in the formulation of the assignment:

1. **Solved R&P task (K1)** A task that aims to prove/demonstrate/show that some statement is valid. The statement might be only implicit in the task itself but is explicitly stated during the solution or after the task (see Fig. 2 and 3).

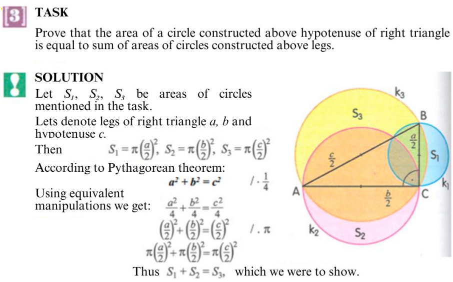


Figure 2: Solved R&P task (K1) from Slovak textbook (Solved R&P task (K1) from Czech textbook (Odvárko, Kadleček, 2011a, p. 41)et al., 2001b, p. 20)

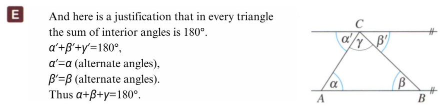


Figure 3: Solved R&P task (K1) from Czech textbook (Odvárko, Kadleček, 2011a, p. 41)

2. **Solved task involving R&P (K2)** – A task that does not ask to prove/demonstrate/show some property but uses reasoning as a support to reach the solution explicitly (see Fig. 4 and 5). Not every solved task is of this kind, as many are solved only algorithmically without providing any reasons why the given steps are valid (see Fig. 1).

After this phase, national teams re-coded their data with the code categories. After that, there was a cross-examination phase. As the languages of instruction in both countries are very similar and Slovaks and Czechs understand each other when speaking both in their mother tongues, no translations of the content of textbooks were needed. The data were then merged in the software and analysed using methods of descriptive statistics.

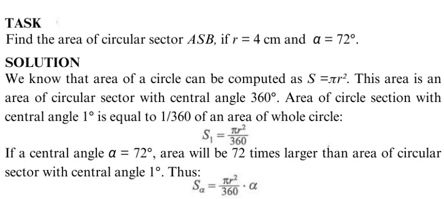

TASK
Find the area of circular sector ASB, if $r = 4$ cm and $\alpha = 72°$.
SOLUTION
We know that area of a circle can be computed as $S = \pi r^2$. This area is an area of circular sector with central angle 360°. Area of circle section with central angle 1° is equal to 1/360 of an area of whole circle:
$$S_1 = \frac{\pi r^2}{360}$$
If a central angle $\alpha = 72°$, area will be 72 times larger than area of circular sector with central angle 1°. Thus:
$$S_\alpha = \frac{\pi r^2}{360} \cdot \alpha$$

Figure 4: Solved task involving R&P (K2) from Slovak textbook (Šedivý et al., 2000b, p. 122)

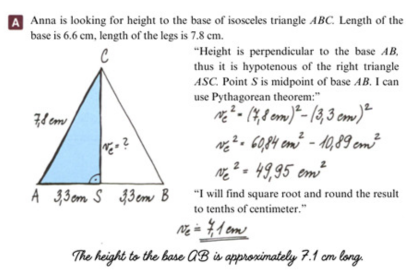


Figure 5: Solved task involving R&P (K2) from Czech textbook (Odvárko & Kadleček, 2012b, p. 30)

Results

Since the order and distribution of the thematic units are different in both countries, we present the results by grades. Tables 2 and 3 show the total number of solved geometric tasks in the examined Slovak and Czech textbooks by grade. The tables also show how many geometric R&P tasks are solved, and their number is also expressed as a percentage. The tables indicate whether the solved geometric R&P tasks belong to the K1 or K2 categories.

Tables 2 and 3 show that Slovak textbooks have more solved geometrical tasks than Czech textbooks. However, the percentage of solved geometric tasks focused on R&P is higher in Czech textbooks. Thus, in Czech textbooks, solved geometric tasks are mainly focused on R&P. In absolute terms, the number of solved geometric tasks focused on R&P is higher in Slovak textbooks in grades 6, 7, and 8. In grade 9, there is the same amount. The lowest number of R&P tasks in Slovakia's 9th grade might be caused by the fact that the curriculum is mainly focused on refreshing or practising thematic units from lower grades; therefore, fewer tasks are focused on R&P.

Table 2: Number of solved geometric (R&P) tasks in SK and CZ by grades

Slovakia								
Grade	**6^{th}**		**7^{th}**		**8^{th}**		**9^{th}**	
Number of solved geometric tasks	**70**		**43**		**55**		**65**	
Number of solved geometric RP tasks	**23 (32.9%)**		**31 (72.1%)**		**30 (54.6%)**		**11 (16.9%)**	
	K1	**K2**	**K1**	**K2**	**K1**	**K2**	**K1**	**K2**
	9	**14**	**15**	**16**	**10**	**20**	**7**	**4**
Czech Republic								
Grade	**6^{th}**		**7^{th}**		**8^{th}**		**9^{th}**	
Number of solved geometric tasks	**20**		**10**		**30**		**12**	
Number of solved geometric RP tasks	**11 (55.0%)**		**9 (90.0%)**		**26 (86.7%)**		**11 (91.7 %)**	
	K1	**K2**	**K1**	**K2**	**K1**	**K2**	**K1**	**K2**
	10	**1**	**3**	**6**	**11**	**15**	**8**	**3**

The solved geometric tasks focusing on R&P into K1 and K2 were divided. Figure 6 shows the distribution of these categories in Slovakia and the Czech Republic. The relative distribution is similar across the two countries, with K2 slightly predominating in Slovakia and K1 in the Czech Republic. From this point of view, textbooks are similar.

Table 3 shows which *ways of reasoning* (Sevinç et al., 2022) were present and to what extent. In addition to the categories from the framework, the results are divided into grades, countries, and categories K1 and K2. The original framework contains the category *Other* as well. The category is omitted here as every task could be classified in some of previous six categories.

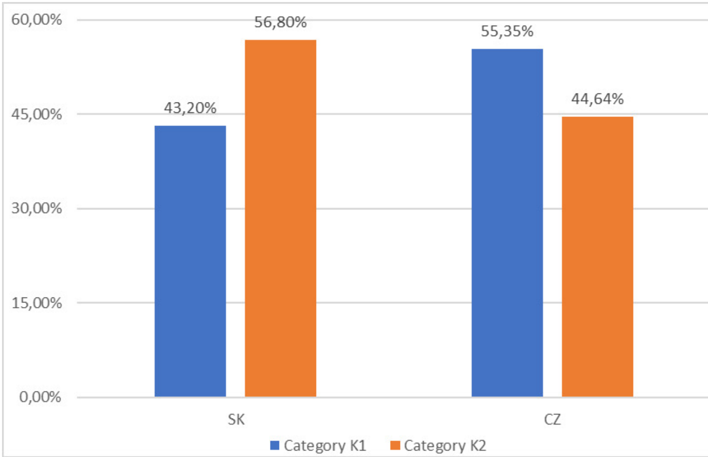


Figure 6: Percentual distribution of tasks in categories K1 and K2 by countries

Table 3 shows that:

- The sixth-grade Czech textbooks include only R&P geometrical tasks from the K1 category except for one task. This is related to the fact that most R&P tasks belong to categories 5, 5a, and 5b, as this way of reasoning is associated with assignments where the need for proof is directly expressed more than with K2, which is closer to category 6.
- Category 5b was only represented in Slovak and Czech textbooks in the sixth grade. This may be due to the pupils' age; for younger pupils, the empirical form of reasoning might be more accessible than deductive forms (categories 2, 6, 6a, 6b, 6c).
- In higher grades, the tendency to use empirical ways of reasoning is smaller; for example, only deductive forms can be found in Czech textbooks in the seventh grade.
- In the eighth and ninth grades, deductive forms of proof are predominant. The other forms also occasionally occur but in smaller numbers.
- *Reasoning by analogy* is found within the solved geometric tasks focused on R&P in textbooks only once – in the ninth grade of the Czech textbook. The reason might be the difficulty of choosing the fitting analogy, which must be easy for the teacher to use and easy for pupils to understand yet not lead to creating misconceptions. Since the authors of the analysed textbooks are probably aware of these issues, they avoid reasoning by analogy in geometry.
- Tasks that use *Appeal to authority* as a way of reasoning have a specific status. These tasks all fell into the K1 category.

Table 3: Summary of our analysis' results by grades and countries

	6th grade						7th grade						8th grade						9th grade					
	SK			CZ			SK			CZ			SK			CZ			SK			CZ		
		K1	K2		K1	K2		K1	K2		K1	K2		K1	K2		K1	K2		K1	K2		K1	K2
1. Appeal to authority	3	3	0	2	2	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0
2. Simple (1-step) deduction	0	0	0	1	0	1	7	6	1	0	0	0	8	2	6	2	0	2	2	1	1	0	0	0
3. Mathematising	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4. Reasoning by analogy	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
5. Reasoning with empirical arguments/ specific cases	1	1	0	1	1	0	3	2	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
a) making claims and generalizing	1	0	1	1	1	0	0	0	0	0	0	0	1	0	1	2	2	0	0	0	0	2	2	0
b) justification of a claim	1	1	0	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6. Developing conclusions/ justifying/ refuting through deductive reasoning	16	3	13	2	2	0	14	2	12	6	0	6	18	5	13	17	4	13	8	6	2	7	5	2
a) generic example	1	1	0	0	0	0	7	5	2	2	2	0	2	2	0	0	0	0	1	0	1	1	0	1
b) counterexample	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0
c) systematic enumeration	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0

- In our analysis, no task belonged to the *Mathematising* as there were no solved geometrical R&P tasks requiring transcribing word problems from real life into mathematical form.

From the Table 3 we can see that within category K1, several ways of reasoning appeared. *Empirical ways of reasoning* appeared only in this category (except for 3 Slovak tasks). Tasks categorised as *Appeal to authority* or *Reasoning by analogy* occur only in this category.

Tasks whose solution was categorised as *deductive ways of reasoning* (2, 6, 6a, 6b, 6c) occurred in both K1 and K2 categories. Apart from the three Slovak tasks, category K2 consists exclusively of tasks focused on deductive reasoning. These three Slovak tasks included *empirical argumentation*, even if the task did not specifically ask for any argumentation.

In K1 tasks, we also analysed if the reasoning involved is explanatory or only provides conviction. As expected, explanatory R&P could only be found in a deductive way of reasoning (2, 6a, 6b, 6c). The results are to be compared in Table 5.

Given that the *Appeal to authority* way of reasoning cannot be understood as proper argumentation, we did not assign roles to these tasks. These tasks directly required R&P, but the explanation in the true sense was not present. (The framework's authors consider appeal to authority a so-called null explanation.) Even though there is no explanation, some pupils might still find these convincing (Harel & Sowder, 1998).

As a secondary characteristic, we also observed the form of representation within the argumentation in the analysed tasks (manipulatives, real-world situations, verbal, symbolic, graphical). Since we analysed the solved tasks from the area of geometry, verbal, symbolic, and graphical appeared to a large extent. These forms often occurred in pairs or triplets. In Czech textbooks, manipulatives were present in two cases; such tasks did not occur in Slovak textbooks. Real-world situations were not found in either Slovak or Czech textbooks.

Discussion

This article compared selected Slovak and Czech mathematics textbooks for grades 6 to 9. We focused on geometry while we compared different ways of reasoning (Sevinç et al., 2022) in solved tasks with a focus on R&P. The main questions were:

1. Are there any substantial differences in the ways of reasoning in the area of geometry between Slovak and Czech textbooks?

In the analysed Slovak textbooks, there are more solved geometric tasks as well as more solved geometric R&P tasks. However, the percentage ratio of geometric R&P tasks to solved geometric tasks is higher in the analysed Czech textbooks. Therefore, the solved geometric tasks in Czech textbooks are primarily focused on R&P.

When categorising the tasks according to the framework, we found that in the sixth grade, several tasks involved *empirical ways of reasoning*. In higher grades, the number of such solved tasks is lower, or they do not occur. Deductive ways of reasoning prevail there (and overall, similarly to the results of Stacey & Vincent (2009)). This development can be observed in both Slovak and Czech textbooks.

The paper introduced categories K1 and K2 to differentiate solved geometric R&P tasks. From the point of view of the distribution of these categories, the textbooks in both countries are similar. Within category K1, several *ways of reasoning* were present. Most tasks involving empirical ways of reasoning belong to this category (all tasks in Czech textbooks and all but three Slovak tasks). Tasks

Table 4: The role of reasoning by categories and countries

	SK		**CZ**	
	Explanation	**Verification/conviction**	**Explanation**	**Verification/conviction**
1. Appeal to authority	0	0	0	0
2. Simple (1-step) deduction	6	3	0	0
3. Mathematising	0	0	0	0
4. Reasoning by analogy	0	0	0	1
5. Reasoning with empirical arguments/specific cases	0	4	0	1
a) making claims and generalizing	0	0	0	5
b) justification of a claim	0	1	0	3
6. Developing conclusions/justifying/refuting through deductive reasoning	10	6	10	1
a) generic example	7	1	2	0
b) counterexample	0	0	2	1
c) systematic enumeration	0	0	2	0
Total	**23**	**15**	**16**	**12**

using *appeal to authority* and *reasoning by analogy* occur only in this category and only in a small amount (similarly as in the case of Silverman and Even (2016)). Tasks solved with *deductive ways* occurred in both K1 and K2 categories. Apart from the three Slovak tasks, category K2 consists exclusively of tasks focused on the *deductive way of reasoning.*

For tasks from the K1 category, we also distinguished whether the reasoning provides an explanation or only convinces/verifies (De Villiers, 1990; Hanna, 1990). Among analysed tasks, deductive ways of reasoning mostly have an explanatory role, while *empirical* only convince or verify. These findings align with the authors' expectations, as deductive procedures can provide insight into the inner workings of what is to be justified. It is interesting, however, that in Slovak textbooks, three empirical tasks could also be considered explanatory as the empirical method led to a sufficient generalisation offering an explanation.

2. What is our experience with the used framework?

The following specifics were noticed while using the *ways of reasoning* framework (Sevinç et al., 2022). *Reasoning by analogy* is found within the solved geometric tasks focused on R&P in textbooks only once – in the ninth grade in the Czech textbook. As already discussed, we believe this is deliberate.

Nevertheless, the occurrence of such tasks justifies the inclusion of this category within the framework. In our analysis, no task belonged to the *mathematising* category. This might be due to the nature of the analysed textbooks. This way of reasoning could be used more with other types of textbooks or in another area than geometry. The original framework also has a seventh category called o*ther.* We did not use this category in the results section because we could classify all the solved geometric tasks focused on R&P into the six mentioned categories. Overall, we can evaluate the use of the framework as satisfactory, yet more detailed description of different framework categories with examples would be beneficial for future use.

Discussion and Conclusions

As part of our research, solved geometric R&P tasks in selected Slovak and Czech mathematics textbooks for grades 6 to 9 were analysed. Among these textbooks, there were similarities in the presence of different ways of reasoning (Sevinç et al., 2022) and in the representation of categories K1 and K2. We attribute this result to the countries' cultural proximity, the long-term history of joint education, and the selection of textbooks. It would be interesting for future research to find out how valid these results are for other areas of mathematics or other textbooks used in these countries. An extension of the research could also be an analysis of Slovak and Czech upper secondary school textbooks.

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