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The identity $7^{3n} + 7^{3n+1} = (2 \cdot 7^n)^3$ and its generalizations*

Abstract. Starting with the identity $7^{3n}+7^{3n+1} = (2\cdot 7^n)^3$ and its sibling, we prove that for any positive integer m, the Diophantine equation $x^n + x^{n+k} = z^m$ has infinitely many solutions in nonzero integers x, z, n and k. We show that in case $k > 1$ the solutions come from Catalan's Conjecture. We also solve three similar Diophantine equations.

1. Introduction

The role of generalization in mathematics is immense - start with simple problem and try to find a greater picture of it. During the course conducted by the author, a student presented part of the book by Nowicki (Nowicki, 2012). On one of the slides the following identities were presented (they are labeled as Fact 1.7.4 and Fact 1.7.5 in the book):

$$7^{3n} + 7^{3n+1} = (2 \cdot 7^n)^3, \tag{1.1}$$
$$26^{3n} + 26^{3n+1} = (3 \cdot 26^n)^3. \tag{1.2}$$

These identities appear as the solution to the following problem: find the number whose two consecutive powers add up to the cube. One of the other challenges given to the student was to think of other possible identities of this kind.

The student could not find further solutions, but the author noticed that identities (1.1) and (1.2) follow a specific pattern, where the base numbers are of the form $t^3 - 1$ (with integer t). This lead to more solutions of the general equation

$$x^n + x^{n+1} = z^m, \tag{1.3}$$

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and eventually, to a complete solution of that equation. Equation (1.3) is a special case of far more general Diophantine problem

$$x^a + y^b = z^c. \tag{1.4}$$

Special cases of equation (1.4) involve many famous problems in number theory, including Fermat's Last Theorem or Catalan's Conjecture.

The famous Catalan's Conjecture posed in 1844 stated that there are no two consecutive perfect powers other than 8 and 9. In the language of Diophantine equations the conjecture can be formulated as follows: the equation

$$x^a - y^b = 1$$

with unknown x, z, a and b, all at least 2, has only one solution in positive integers $(x, y, a, b) = (3, 2, 2, 3)$. The conjecture has been proven positive by Mihăilescu with the help of cyclotomic units (Mihăilescu, 2004).

Identities (1.1) and (1.2) solve the equation (1.4) with $c = 3$ and $a = 3n, b = 3n+1$ and inspire us to search for the solutions of (1.4) under reasonable constrains. More specifically, we consider the following form of the equation:

$$x^n + x^{n+k} = z^m, \tag{1.5}$$

where x, z are integers, $n \geq 0$ and $k \geq 0$. We also consider a few variations of (1.5), such as

$$x^{n+k} - x^n = z^m, \tag{1.6}$$

$$x^n + 2x^{n+1} = z^m, \tag{1.7}$$

$$x^n + x^{n+1} + x^{n+2} = z^3. \tag{1.8}$$

Note that a variety of similar equations are considered in the literature, see (Powell, et al, 1978) for the solution of $x^n + 1 = y^{n+1}$ or (Nowicki, 2012) and the references therein.

We will use the following immediate observation throughout the article.

Lemma 1.1
If a is an integer and n, k are positive integers, then we have

$$\gcd(a^n, a^k + 1) = 1.$$

Proof. Let p be a prime number dividing a^n. Then $a^k + 1 \equiv 1 \pmod{p}$ and thus $\gcd(a^n, a^k + 1) = 1$.

2. The equation $x^n + x^{n+k} = z^m$

To demonstrate our approach, we start by solving the equation (1.5).

Let us briefly describe some of the *simple* cases. We start with $k = 0$ and $n = 0$.

If $k = 0$, then the equation (1.5) becomes

$$2x^n = z^m,$$

which implies $z = \sqrt[m]{2}x^{\frac{n}{m}}$. The remaining consideration depends on m.

In the first case we take $m = 1$, then it is easy to see that $(x, z, n) = (t, 2t^\ell, \ell)$ with $t \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ is the solution to our simplified problem.

Let $m > 1$, then $z = \sqrt[m]{2} \cdot x^{\frac{n}{m}}$ has to be an integer, which implies $x = 2^{\frac{p}{q}}$ for some integers p and q with $\gcd(p, q) = 1$. This implies

$$z = 2^{\frac{q+pn}{mq}}$$

is an integer, so $mq|q + pn$ and therefore, $q|n$. Thus, we can write

$$z = 2^{\frac{1+pn'}{m}}$$

for some positive integer n'. The fraction $\frac{1+pn'}{m}$, for fixed m, is an integer for infinitely many pairs (p, n'). To find any such pair, fix $\ell \geq 1$ and write $m\ell - 1 = p \cdot n'$. Therefore, for any $m > 1$ we can write the complete set of solutions:

$$(x, z, n) = \left(2^p, 2^\ell, n'\right), \qquad \text{with } m\ell - 1 = p \cdot n'.$$

For example, if we take $m = 11$ and $\ell = 3$, then

$$(x, z, n) = \left(2^{2^c}, 2^3, 2^{5-c}\right), \qquad c = 0, \ldots, 5.$$

We now move to the case $n = 0$. Here, the equation becomes

$$1 + x^k = y^m$$

which is, for $k, m > 1$, equivalent to Catalan's problem. In the case $k = 1$ and fixed m we have

$$1 + x = z^m$$

and the solution is $(x, z) = (t^m - 1, t)$ for $t \in \mathbb{Z}$. Similarly we solve the case $m = 1$. These are the simple solutions and in our further consideration are skipped.

To solve the equation (1.5) in general we go back to identities (1.1) and (1.2) again and at first we solve the case $k = 1$, that is, we solve equation (1.3). A careful consideration of these leads to the following.

Theorem 2.1
The solution of (1.3) *with fixed* m, $n > 0$ *and* $k = 1$ *is*

$$(x, z, n) = (t^m - 1, t(t^m - 1)^\ell, m\ell), \tag{2.1}$$

where $t \in \mathbb{Z}$, $\ell \in \mathbb{N}$.

Proof. We check that (2.1) satisfies (1.3). Indeed, we have

$$\begin{aligned} x^n + x^{n+1} &= (t^m - 1)^{m\ell} + (t^m - 1)^{m\ell+1} \\ &= (t^m - 1)^{m\ell} t^m \\ &= (t(t^m - 1)^{\ell})^m. \end{aligned}$$

Suppose now that (x, z, n) is a solution to (1.3). We have

$$z^m = x^n + x^{n+1} = x^n(1 + x)$$

and since $\gcd(x, x + 1) = 1$, we find $s, t \in \mathbb{Z}$ with $\gcd(s, t) = 1$ and

$$\begin{cases} z^m &= s^m t^m, \\ s^m &= x^n, \\ t^m &= 1 + x. \end{cases}$$

It follows from the third equation that $x = t^m - 1$, implying $s^m = (t^m - 1)^n$. Excluding trivial case $x = 1$ (which leads to $z = 2$ only if $m = 1$, so one of the simple solutions) we see that $s \geq t$ and so $s^m > t^m - 1$, which gives

$$s = (t^m - 1)^{\frac{n}{m}}.$$

If $\frac{n}{m} \notin \mathbb{Z}$, then $t^m - 1 = u^q$ for some $u \in \mathbb{Z}$ and $q \in \mathbb{N}$ so that $q \cdot \frac{n}{m}$ is an integer. But, using Catalan's conjecture, this is possible only when $u = m = 2$ and $t = q = 3$, implying n is even. So, let $n = 2\ell$ and thus,

$$t = 3, \quad s = 3^{\ell}, \quad x = 3, \quad z = 4 \cdot 3^{2\ell}.$$

Consider the case $\frac{n}{m} \in \mathbb{Z}$, so $n = m \cdot \ell$ for some integer $\ell \in \mathbb{N}$. It follows that $s = (t^m - 1)^{\ell}$ and thus

$$x = t^m - 1, \quad z = t \cdot (t^m - 1)^{\ell},$$

which by the initial reasoning gives a valid solution for any t and ℓ, concluding the theorem.

We note that the solution $(x, z, n) = (3, 4 \cdot 3^{2\ell}, 2\ell)$ (which comes from the case $m \nmid n$) is in fact the solution described by $t = m = 2$ in equation (2.1), which clarifies why such a solution is not included in the statement of the theorem.

Remark 2.2 *Notice that the solutions* (2.1) *are generated from the solution corresponding to* $\ell = 1$ *in the following sense: if* (x, z, n) *solves* (1.3)*, then so does* $(x, z \cdot x, n+m)$*. This motivates us to call the solutions for* $\ell = 1$ *primitive solutions.*

Example 2.3 *We present the primitive solutions for selected* m *in Table 1. Notice the appearance of identities* (1.1) *again and* (1.2) *in the third row of the table.*

Take for example the primitive solution

$$7^3 + 7^4 = 14^3.$$

Table 1: General form of solution for $2 \le m \le 5$ and the primitive solutions.

m	solution	example solutions ($\ell = 1$)		
2	$(t^2-1, t(t^2-1)^\ell, 2\ell)$	$3^2+3^3=6^2$	$8^2+8^3=24^2$	$15^2+15^3=60^2$
3	$(t^3-1, t(t^3-1)^\ell, 3\ell)$	$7^3+7^4=14^3$	$26^3+26^4=78^3$	$63^3+63^4=252^3$
4	$(t^4-1, t(t^4-1)^\ell, 4\ell)$	$15^4+15^5=30^4$	$80^4+80^5=240^4$	$255^4+255^5=1020^4$
5	$(t^5-1, t(t^5-1)^\ell, 5\ell)$	$31^5+31^6=62^5$	$242^5+242^6=726^5$	$1023^5+1023^6=4092^5$

Multiplying both sides by 7^3, 7^6, 7^9 and so on we obtain all solutions corresponding to the case $m = 3$ and $t = 2$. Notice also that if we didive the identity by 7^3, we obtain $7^0 + 7^1 = 2^3$ - a solution corresponding to the simple case $n = 0$ described in the beginning of this section.

Theorem 2.4
The solutions of (1.5) *with $k > 1$, $m > 1$ and $x, z \neq 0$ are*

$$(x, z, n, k, m) = (2, 3 \cdot 2^\ell, 2\ell, 3, 2). \tag{2.2}$$

Proof. Suppose (x, z, n, k, m) solves (1.5). Then

$$z^m = x^n(1 + x^k)$$

and since $\gcd(x^n, 1+x^k) = 1$, we have $1+x^k = t^m$ for some $t|z$. The only solution to this equation is $(x, t, k, m) = (2, 3, 3, 2)$ and thus (1.5) becomes

$$2^n \cdot 9 = 9 \cdot y'^2$$

for some $y' \in \mathbb{Z}$. It follows that $y' = 2^{\frac{\ell'}{2}}$ for some $\ell' \in \mathbb{N}$. Taking $\ell' = 2\ell$ for $\ell \in \mathbb{Z}$ we see the general solution to (1.5) is of the desired form.

Substituting (2.2) to (1.5) yields the identity

$$2^{2\ell} + 2^{2\ell+3} = (3 \cdot 2^\ell)^2$$

and completes the proof.

2.1. The equation $x^{n+k} - x^n = z^m$

We consider the equation (1.6). The following two results are derived in the same way Theorem 2.1 and Theorem 2.4 and thus the details are omitted.

Theorem 2.5
The solution of (1.6) *with fixed m, $n > 0$ and $k = 1$ is*

$$(x, z, n) = (t^m + 1, t(t^m + 1)^\ell, m\ell),$$

where $t \in \mathbb{Z}$, $\ell \in \mathbb{N}$.

Theorem 2.6
The solutions of (1.6) *with $k > 1$, $m > 1$ and $x, z \neq 0$ are*

$$(x, z, n, k, m) = (3, 2 \cdot 3^\ell, 3\ell, 2, 3).$$

3. Companion equations

In this section we partially solve equation (1.7) and completely solve equation (1.8). The partial solution comes from the coefficient 2 appearing in the equation, which increases the complexity of the equation significantly. In fact, the complete solution of such an equation leads to yet another very general Diophantine equation. We refer the reader to the details below.

3.1. The equation $x^n + 2x^{n+1} = z^m$

The equation (1.7), surprisingly, is much more difficult to solve and not all solutions can be explicitly written down.

We begin the solution by noting that

$$z^m = x^n(1+2x)$$

and since $\gcd(1+2x, x^n) = 1$, we obtain that there are co-prime integers s, t such that

$$\begin{cases} z^m &= s^m t^m, \\ s^m &= x^n, \\ t^m &= 1+2x. \end{cases}$$

It follows from the third one that

$$x = \frac{t^m - 1}{2}$$

and thus t is odd, so $t = 2u+1$ for some integer u.

We now consider several cases. First, assume that $n > 1$ and $m|n$. Then $n = m \cdot \ell$ and it is easy to give the solution

$$(x, z, n) = \left(\frac{(2u+1)^m - 1}{2}, (2u+1)\left(\frac{(2u+1)^m - 1}{2}\right)^{\ell}, m\ell \right),$$

valid for each integer u and $\ell \geq 1$.

Suppose now that $n = 1$, which translates the initial equation to

$$x(1+2x) = z^m.$$

Consider substitution $t^m = 1 + 2x$ again and notice that

$$x = \frac{t^m - 1}{2} = s^m.$$

This is equivalent to

$$t^m - 2s^m = 1, \tag{3.1}$$

which in case $m = 2$ is Pell equation. Using standard algorithm of solving such equations we find that

$$|t| = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2}, \qquad |s| = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{2\sqrt{2}}$$

and $k \in \mathbb{N}$ defines all integer solutions. Indeed, it is easy to check that if

$$t_k = \frac{(3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k}{2},$$

then t_k is odd for all k and thus $\frac{t_k^m - 1}{2}$ is an integer. (for example, taking $k = 6$ we have $t_6 = 19601$. Then we get $x = 192\ 099\ 600$ and $z = 271\ 669\ 860$).

In case $m \geq 3$ equation (3.1) is a Thue equation. It follows from the result of Bennett (Bennett, 2001, Theorem 1.1) that in such case we have at most one solution in positive integers for each m. Since the proof is not constructive, it is not viable to provide a direct set of solutions.

Suppose that m does not divide n. Then, following similar part of the proof of Theorem 2.1 we get

$$s = \left(\frac{t^m - 1}{2}\right)^{\frac{n}{m}}$$

and thus $\frac{t^m - 1}{2} = u^p$ for some $u \in \mathbb{Z}$ and positive integer p. This translates to a very general equation

$$2u^p + 1 = t^m. \tag{3.2}$$

As far as Author is concerned, this equation has not been studied in the whole generality. There are of course trivial cases, such as $m = p = 2$ which reduce to Pell equation. On the other side, we were able to find many numerical solutions to that equation, here we showcase a few of them with varying m. For $m \geq 5$ we present the solution with lowest possible pair (p, u) in the lexicographic order.

$$\begin{aligned}
(u, p, m, t) &= (2, 2, 2, 3), \\
(u, p, m, t) &= (12, 2, 2, 17), \\
(u, p, m, t) &= (70, 2, 2, 99), \\
(u, p, m, t) &= (82, 15, 2, 319241167726353), \\
(u, p, m, t) &= (576, 15, 3, 79883254312781), \\
(u, p, m, t) &= (11, 2, 5, 3), \\
(u, p, m, t) &= (9062, 23, 7, 11099716939425), \\
(u, p, m, t) &= (7997, 27, 9, 552368705167), \\
(u, p, m, t) &= (9720, 35, 11, 5192725770939), \\
(u, p, m, t) &= (7761, 42, 13, 3895673179533), \\
(u, p, m, t) &= (4381, 51, 17, 143429207143).
\end{aligned}$$

We also note that there are many solutions such that $\frac{p}{m}$ is an integer, for example

$$\begin{aligned}
(u, p, m, t) &= (1989, 12, 3, 19718890002291), \\
(u, p, m, t) &= (341, 20, 4, 5483140746069), \\
(u, p, m, t) &= (640, 35, 7, 118550708827139), \\
(u, p, m, t) &= (544, 55, 11, 50741214944823),
\end{aligned}$$

and many more. In order to keep the difficulty of the paper on equal level, we do not delve into the problem of solving equation (3.2) (even in special cases) and thus we finish our consideration here.

3.2. The equation $x^n + x^{n+1} + x^{n+2} = z^3$

Let us consider the equation (1.8). Then,

$$x^n(1 + x + x^2) = z^3$$

and since $\gcd(1 + x + x^2, x) = \gcd(1 + x^2, x) = 1$, we know that

$$1 + x + x^2 = y^3 \tag{3.3}$$

for some integer y. The equation (3.3) can be seen as

$$(2x + 1)^2 + 3 = 4y^3$$

and substituting $X = 2x + 1, Y = y$ yields

$$X^2 + 3 = 4Y^3.$$

This equation has only two pairs of solutions (see (Tzanakis, 1984)):

$$(X, Y) = (\pm 1, 1), \quad (X, Y) = (\pm 37, 7).$$

Thus there are four solutions to (3.3):

$$(x, y) = (-1, 1), \quad (x, y) = (0, 1), \quad (x, y) = (-19, 7), \quad (x, y) = (18, 7).$$

Let us consider all four cases.

- If $(x, y) = (-1, 1)$, then $(-1)^n = z^3$. Thus if n is odd, $z = 1$ and if n is even, $z = -1$.
- If $(x, y) = (0, 1)$, then we obtain only trivial solution $x = z = 0$.
- If $(x, y) = (-19, 7)$, then the equation becomes

  $$(-19)^n \cdot 343 = 343 \cdot z'^3.$$

  Thus $z' = (-19)^{\frac{n}{3}}$ and $3|n$. This implies that

  $$(x, z, n) = (-19, 7 \cdot (-19)^\ell, 3\ell)$$

  solves (1.8) provided $\ell \in \mathbb{N}$, that is, we have the following identity

  $$(-19)^{3\ell} + (-19)^{3\ell+1} + (-19)^{3\ell+2} = \left(7 \cdot (-19)^\ell\right)^3.$$

- The case $(x, y) = (18, 7)$ is similar and gives

  $$(x, z, n) = (18, 7 \cdot 18^\ell, 3\ell)$$

  for some $\ell \in \mathbb{N}$, giving the identity

  $$18^{3\ell} + 18^{3\ell+1} + 18^{3\ell+2} = \left(7 \cdot 18^\ell\right)^3.$$

We can summarize the above with the following result.

Theorem 3.1
The complete solution to the equation (1.8) *is described by the following cases (in each one we consider $\ell \in \mathbb{N}$):*

1. $(x, z, n) = (-1, (-1)^\ell, \ell)$,
2. $(x, z, n) = (0, 0, \ell)$,
3. $(x, z, n) = (-19, 7 \cdot (-19)^\ell, 3\ell)$,
4. $(x, z, n) = (18, 7 \cdot 18^\ell, 3\ell)$.

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