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The specialised knowledge of middle school teachers concerning the concept of function*

Abstract. This study presents research focusing on in-service mathematics teachers and their approach to tasks aimed at developing of functional thinking. The analysis is based on the model of The Mathematics Teachers' Specialised Knowledge – model MTSK by Carrillo et al. (2018). The research question is the following: What "specialised knowledge", in the sense of the MTSK framework, is revealed when a middle school teachers solve and discuss the selected tasks? The research was conducted with 9 teachers, who voluntarily participated in the professional development. The teachers solved a set of 30 tasks that were selected with a focus on 4 aspects of the concept of function, different representations and transitions between them and different actions that learners engage in the task solving. In this study, we analysed the teachers' written solutions and the discussion with the teachers about a particular task. Reflecting on their work as learners was very important in moving from dealing with the mathematics itself to shifting to developing their knowledge of mathematics teaching (KMT) and knowledge of features of learning mathematics (KFLM) in terms of the MTSK framework. The results helped us to identify gaps in teachers' specialised knowledge and consequently will help us to improve professional development courses for both in-service and pre-service teachers.

Introduction

At all levels of mathematics education, the development of functional thinking has been seen as a core area of mathematics since the beginning of the twentieth century (Vollrath, 1986). This is important because the relevance of functional thinking in private, academic and professional contexts, together with various empirically documented difficulties of students, implies that support specifically for

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functional thinking (and not just for mathematics as a whole) is essential (e.g., Leinhardt et al., 1990; Thompson, Carlson, 2017). Reasons for students' difficulties with functional thinking may be the abstract nature of 'functions', accessible only through specific representations (e.g., graph, equation, table, situation description), or the need to move between mathematics and real-world contexts (e.g., Ronda, 2015; Ostermann et al., 2018; McCulloch et al., 2022). Therefore, researchers continue to ask the fundamental question: "What do teachers need to know and be able to do to teach functions effectively?" In order to conceptualise the different types of knowledge that middle school teachers should have in order to develop their students' functional thinking, the Mathematics Teachers' Specialised Knowledge (MTSK) model by Carrillo et al. (2018) was chosen as a theoretical framework.

This paper is part of a larger study in which we have developed a tool to explore the specialised knowledge of middle school teachers in particular with respect to topic concerning the concept of the function. The complexity of teachers' mathematical knowledge makes it challenging to determine exactly which knowledge is important for a good mathematics teacher (Zakaryan, Ribeiro, 2019).

Therefore, we investigate the specialised knowledge of in-service mathematics teachers in teaching important aspects of functional thinking in Slovak middle schools (12–15 years old). As a tool for the research we decided to use the specific mathematical tasks. The reason for this is that mathematical challenge play a very important role in the professional development of mathematics teachers. Teachers need to have a deep and broad understanding of school mathematics in order to be able to offer challenging mathematics to students (e.g., Zaslavsky, Leikin, 2004). In-service teachers' solutions to selected tasks were the starting point for the research described in this paper. On the basis of the in-service teachers' solutions to selected tasks and the audio recordings of the discussion during the professional development, we tried to better understand, interpret and characterise the knowledge in the sub-domains of MTSK in the context of the topic of function, and possibly to identify critical aspects of their knowledge.

In this paper, we will analyse the teachers' solutions to one of the selected tasks and the joint discussion about it. When analysing them, we also drew on research already carried out on characterisation of the specialised knowledge of secondary school mathematics teachers about the concept of function (Espinoza, 2020). The research question addressed in this paper is framed as follows: What 'specialised knowledge', in the sense of the MTSK framework, is revealed when a group of middle school teachers solve and discuss a task on function?

The following two sections provide an overview of the MTSK model as a theoretical framework for different types of teachers' specialised knowledge, and the theoretical background to the development of functional thinking.

The Mathematics Teacher's Specialised Knowledge model

Many authors have investigated how mathematics teachers' knowledge of mathematics content influences their choice of teaching methods and their management of the teaching process. They have found that a key aspect in the development of mathematics teachers' pedagogical reasoning is the relationship between mathematical content knowledge and pedagogical content knowledge. The need to go deeper into the knowledge that can be used for teaching, and consequently the need for appropriate tools or models that facilitate analysis and possibly allow recommendations to be made for teacher training, led Carrillo et al. (2018) to develop a model (based on Shulman's model, 1986) that allows for a deeper analysis of knowledge (understanding and interpreting rather than evaluating). This Mathematics Teacher's Specialised Knowledge (MTSK) model takes a holistic view of the professional nature of teachers' knowledge, ensuring that the definitions for each sub-domain of MTSK are constructed in terms of what the teacher uses/needs, without reference to other professions (non-teachers).

Following Shulman (1986), the MTSK model (see Figure 1) considers two main areas of knowledge:

- the knowledge that mathematics teachers have from the point of view of the scientific discipline in the context of education - the Mathematical Knowledge (MK) domain,
- the knowledge of mathematical content from the point of view of teaching and learning – the Pedagogical Content Knowledge (PCK) domain.

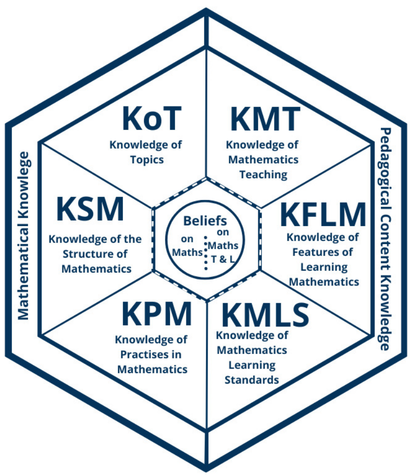


Figure 1: Model MTSK (adapted from Carrillo et. al, 2018, p. 241)

The MTSK model also includes beliefs about mathematics and about teaching and learning mathematics in the centre of the model to emphasize the reciprocity between beliefs and knowledge domains.

Based on the rules and characteristics of the process of creating mathematical knowledge that enables teacher to teach content in a coherent way and to validate their own and their students' mathematical conjectures, Mathematical Knowledge is divided into three sub-domains: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM).

On the right side of the MTSK model is Pedagogical Content Knowledge (PCK), which is "a specific type of knowledge of pedagogy which derives chiefly from mathematics. It is the area of teachers' knowledge that is most closely related to classroom practice." (Carrillo et al., 2018, p. 246). Two sub-domains of PCK, referred to as Knowledge of Mathematics Teaching (KMT) and Knowledge of Features of Learning Mathematics (KFLM), are related to teaching and learning and the third sub-domain is Knowledge of Mathematics Learning Standards (KMLS).

In our work, we focused mainly on the right side of the model, namely on the sub-domains KMT and KFLM, and on the KoT sub-domain from the left side of the model.

Knowledge of Features of Learning Mathematics

"The sub-domain KFLM refers to the need for the teacher to be aware of how students think and construct knowledge when tackling mathematical activities and tasks. This sub-domain encompasses knowledge associated with features inherent to learning mathematics, placing the focus on mathematical content (as the object of learning) rather than on the learner." (Carrillo et al., 2018, p. 246). Categories of the KFLM sub-domain are listed in the Table 1. (In the right part of the table we introduce the abbreviation for each category that we will use later.)

Table 1: Categories of Knowledge of features of learning mathematics

1.	Theories (personal and institutionalized) of mathematical learning	KFLM1_TL
2.	Strengths and weaknesses in learning mathematics (in general/ with respect to specific content)	KFLM2_SW
3.	Ways students interact with mathematical content	KFLM3_MC
4.	Emotional aspects of learning mathematics (motivation, student's interests and expectations, ...)	KFLM4_EA

Knowledge of Mathematics Teaching

KMT refers to knowledge about how to present mathematical content during the lesson in the classroom. This sub-domain concerns knowledge that is intrinsically linked to content, to the exclusion of aspects of general pedagogical knowledge. "In KMT we locate knowledge of resources from the point of view of their mathematical content or the knowledge of approaching a structured series of examples to help students understand the meaning of a mathematical item." (Carrillo et al., 2013, p. 2991). Categories of the KMT sub-domain categories are listed in the Table 2.

Table 2: Categories of Knowledge of mathematics teaching

1.	Theories of mathematical teaching (both personal and institutional)	KMT1_TT
2.	Teaching resources (textbooks, manipulatives, technological resources)	KMT2_TR
3.	Strategies, techniques, tasks, and examples (with any potential limitations and obstacles)	KMT3_ST
4.	Knowledge of different ways of representing specific content	KMT4_RC

Knowledge of Topics

The sub-domain KoT involves an in-depth knowledge of mathematical content and its relevance to the topics taught by the teacher. It emphasises the knowledge that students are expected to acquire, with a deeper, more conceptual and more formal understanding. Four categories are included in this sub-domain of knowledge, which are listed in Table 3. "The term topic refers to content items within the definable knowledge areas making up the mathematics syllabus. It is important to note that the topics are specific components within these areas and can vary according to each country's curriculum." (Carrillo et al., 2018, p. 242)

Table 3: Categories of Knowledge of topics

1.	Knowledge of procedures (including connections to items within the same topic)	KoT1_KP
2.	Properties and their basic principles, definitions, and foundations	KoT2_PR
3.	Knowledge of different ways of representing content (graphically, algebraically, ...)	KoT3_DR
4.	Phenomenology and applications	KoT4_PA

Teacher's Specialised Knowledge related to developing functional thinking

This research deals with the development of teachers' specialised knowledge in the context of functional thinking. It is part of a larger body of research within the FunThink project. Based on the literature and the research within the project, functional thinking can be characterized as "the process of building the concept of function and reasoning with and about functions" (Blanton et al., 2015; Pittalis et al., 2020; Vision document, 2021). It is considered key to mathematical thinking because it is connected to different areas of mathematics and has applications in a wide range of problem situations.

Research in mathematics education has consistently focused on how to describe, understand (e.g., Vollrath, 1986), and support students' functional thinking (e.g., Lichti, Roth, 2018). The concept of function, which is central to functional thinking, has had a long and difficult developmental history, with different views of function emerging. This has been reflected in different views of the concept of function in mathematics education. According to Doorman et al. (2012) and Pittalis et al. (2020) we can distinguish four main aspects of views on the concept of function in functional thinking.

<u>The function as an input-output assignment:</u> This view on function as an input-output machine stresses the operational and computational character of the concept of function (Vision document, 2021). It also comes into play when patterns and structures are investigated (recursive patterning).

<u>The function as a dynamic process of co-variation:</u> This aspect concerns the notion that "two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other." (Thompson, Carlson, 2017, p. 444)

The function as a correspondence relation: A correspondence relation includes identifying a correlation between variables, using the function rule to predict far-function values, and finding the value of one variable given the value of the other. (Confrey, Smith, 1995)

The function as a mathematical object: "A function is a mathematical object that can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view on the same object." (Doorman et al., 2012, p. 1246)

Teachers who are aware of these different views of function should be able to better understand and differentiate what students' views of function are (KFLM), and therefore could choose appropriate types of tasks that could be used in the classroom to develop students' functional thinking (KMT).

Functions can be represented in a variety of ways, including symbolic equations (a formula); a table; a graph; language – a verbal description (a situation with some context); or a nomogram. Hart (1981) investigated which representations students choose when solving multiple-choice tasks. In her work, she concluded that students show certain trends in their use of representations. The factors she identified as influencing students' choice of representation were: students' experience with the given representation, the context in which the given representations are worked with, students' confidence in their own ability to understand symbolic notation, and formal mathematical language. Ronda (2015) argues that the main goal of teaching functions should be to see the invariant properties of a function (view of the function as a mathematical object) in its different representations, and to understand the advantages and disadvantages of each representation in relation to the context in which it is worked with. "Each of the representations can emphasize a particular property of the represented function. However, simply knowing how to work with the different representations does not necessarily lead to an understanding of the function." (Ronda, 2015). An important aspect of the KoT sub-domain in the MTSK model is the ability to use and switch between multiple representations. "The ability to identify and represent the same thing in different representations, and the flexibility to move from one representation to another, allows one to see rich relationships, develop better conceptual understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems." (Even, 1998). According to Yerushalmy and Schwartz (1999), using different representations of the concept of function develops a richer and deeper understanding of mathematical concepts and encourages students to use a richer set of symbolic and graphical representations. Similarly, Cox et al. (2003) and Dikovic (2009) argue that using different representations and linking them positively influences students' understanding of mathematical concepts and thus their attitudes. Dufour-Janvier et al. (1987), in research investigating the accessibility of representations, concluded that the use of different representations makes mathematics more attractive and interesting and reduces difficulties in learning mathematics. Teachers who consider of this could be helpful to their students' learning, which is an important part of the KFLM sub-domain.

In terms of tasks selected with the goal of developing functional thinking, Leinhardt et al. (1990) identify two actions that learners engage in with such tasks –

interpretation and construction. These categories are neither exhaustive nor mutually exclusive. Interpretative tasks are focused on issues of pattern, continuation, or rate; or determining when specific events or conditions are met. Construction tasks related to the act of generating something new - constructing a graph, constructing an algebraic function for a graph, plotting points from data. "Whereas interpretation does not require any construction, construction often builds on some kind of interpretation." (Leinhardt et al., 1990, p. 13). Construction and interpretation tasks are typical for the domain of graphs and functions, and they are divided into 4 groups:

- Prediction – the action of using given part of a graph to infer where other points on the graph should be (not explicitly given or plotted) or how other parts of the graph should look.
- Classification – actions involving (a) deciding whether a particular relation is a function; (b) identifying a function among other relations; or (c) identifying a particular type of function among other functions.
- Transition – involves (a) the act of recognizing the same function in different forms of representation; (b) identifying for a specific transformation of a function in one representation its corresponding transformation in another representation (c) constructing one representation of a function given another one.
- Scaling – tasks that focus attention on the axes and their scales and on the units being measured.

Considering different types of tasks that require the use of different strategies, techniques, methods with potential limitations and barriers is part of KMT. In addition, it can potentially help teachers identify strengths and weaknesses of students' learning within the development of functional thinking (KFLM).

Methodology

Participants and context

The study involved nine middle school teachers who volunteered to take part in the study as part of the professional development program within national project IT Academy. Each of them obtained a degree in Mathematics in combination with another science subject (3 teachers – Mathematics and Physics, 2 teachers – Mathematics and Informatics, 1 teacher – Mathematics and Biology, 2 teachers – Mathematics and Geography, 1 teacher – Mathematics). They were teachers from the eastern part of the Slovak Republic (Košice region), two men and seven women aged between 28 and 52. Informed consent was obtained from all in-service teachers involved in the study.

In Slovakia, the concept of function is explicitly mentioned only in the last year of the middle school. Teachers in this type of schools usually deal only with the linear function and its properties. They usually introduce it as a function with the formula $y = ax+b$, whose graph is a straight line, examine its properties depending on the coefficients a and b, solve typical application problems leading to a linear function. In Slovak middle school textbooks, the input-output and correspondence aspects predominate. The concept of slope is not explicitly introduced.

The professional development program within the national project IT Academy was designed to enable teachers to develop their mathematical knowledge and pedagogical content knowledge in a way that promotes a constructivist perspective of teaching. The goal of the program included the need to prepare teachers for innovative and reform approaches to their teaching. It also aimed to support teachers' ability to reflect on their teaching and to promote teachers' sensitivity to students and their ability to assess students' mathematical understanding. The professional development program was divided into 5 one-day sessions over a 2-months period. The last two sessions were devoted to the topic of Functions and to the development of functional thinking of their students. The second author was one of the lectors of this professional development program.

Data Instrument and Collection

The data collection procedures used in this study included teachers' written solutions to 30 tasks (designed research tool) and audio recordings of discussions (including discussions between teachers in groups of three and discussions between teachers and lector) focused on the research tool.

As mentioned above, the research tool consisted of a set of 30 tasks that varied along three dimensions:

i) the mode of representation in which the function concept was presented (language, formula, graph and table);

ii) the four aspects of the function concept according to Doorman (2012) and Pittalis (2020);

iii) the action of a learner – whether the task is interpretation or construction (according to Leinhardt et al., 1990) – each category (both interpretation and construction) can be further divided into prediction, classification, translation and scaling tasks.

Fifteen tasks of the set were selected from Slovak and Czech mathematics textbooks, ten tasks from research articles devoted to the development of functional thinking, and five tasks from popular websites for mathematics teachers from the Czech and Slovak Republic. The tasks were chosen to cover equally all aspects of the concept of function, different representations of a function (graph, table, language, formula) and transitions between representations (see Tables 4, 5, 6). This division is not disjunctive, and some tasks can be assigned to several aspects and several transitions between representations, depending on the student's level of knowledge, the solution method used, etc.

In addition, following Leinhardt et al. (1990), we selected tasks to cover all action of a learner when solving them (see Table 7). Again, the division of tasks

Table 4: The numbers of representations of the function used in the assignments of the tasks

Graph	Table	Language	Formula
3	11	22	2

Table 5: The number of tasks for which the corresponding aspect of the function can be used in the solution process

Input-output	Covariation	Correspondence	Object
17	17	22	8

Table 6: The number of tasks for which the corresponding transition between representations of the function can be used in the solution process

G→L	G→T	G→F	L→G	L→T	L→F	F→G	F→L	F→T	T→L	T→F	T→G
10	3	3	10	10	7	9	3	1	3	8	9

is not disjunctive, as some tasks consist of several parts. The least covered category is the Interpretation – Prediction task, which is in line with Leinhardt et al. who wrote: "At the heart of most prediction tasks is an action of construction, which can be done either physically or mentally." (1990, p.13) As for the category Construction – Classification, these tasks are typical for Slovak textbooks, so we prefer tasks from other categories.

Table 7: The number of tasks according to an action of a learner

Construction				Interpretation			
Translation	Classification	Prediction	Scaling	Translation	Classification	Prediction	Scaling
12	**3**	**7**	**5**	**8**	**7**	**1**	**6**

The research tool has been developed with the understanding that teachers need to have a deep and broad knowledge of school mathematics in order to provide challenging mathematics for their students. Solving similar problems is the main learning activity of their students, but for teachers it can serve as a vehicle for professional growth beyond mathematical knowledge (Zaslavsky, Leikin, 2004). The complete research tool is available as an appendix in Slabý (2022).

The teachers were asked to solve the set of 30 tasks in advance. 7 teachers solved each of the tasks, 2 teachers did not submit a complete set of tasks. All 9 teachers participated in the two sessions devoted to the topic Function and therefore in the whole discussion related to this topic.

At the beginning of the first session, the teachers were asked to work in groups of three to discuss the tasks from the set. To guide their discussion, we asked them to focus on the following questions:

- Do you think the task is important for teaching mathematics? If yes, why? If not, why not?

- What are the advantages and disadvantages of the task?
- Which task would be challenging for students and why?

After the discussions in the groups, the teachers chose tasks for a joint discussion. According to their selection, we discussed the eight tasks from the set of 30 tasks in more detail.

In this study we will analyse the teachers' solutions to one task (task 25) from the task set (see Figure 2) and the common discussion about the selected task. We chose this task for the common discussion because it is a non-standard task for Slovak middle school teachers in the functional context. Moreover, the teachers had a rich discussion about this task in groups. The task is realistic and its text is easy to understand for middle school students. It is formulated as word problem (representation using language) and it is possible to solve it using different aspects of the function concept and different transitions from the language representation of the function to another representation. According to Leinhardt et al. (1990), the task belongs to the Construction - Prediction task action because it requires the prediction of the most advantageous offer.

Helen wants to organize a birthday party in the children's playground. She decides between the following offers:

- Playground A. The price for each guest is 15 €. No additional fees are payable.
- Playground B: The price for each guest is 12 €. In addition, a fixed cost of 50 € is payable.
- Playground C: The price for each guest is 18 €. A discount of 30 € will be given on the final total price.

Which offer is the most advantageous for her?

Figure 2: Task 25 from set of tasks

Data analysis

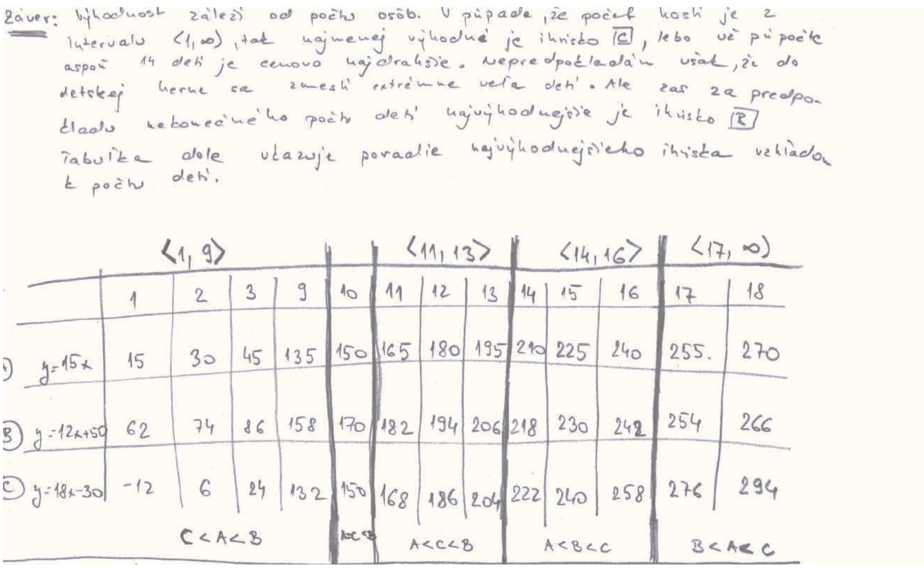
From teachers' solutions we can gain partial access to their KoT. Three teachers (Greta, Vera and Lara) used all aspects of the concept of function and all the common representations of the function in their solutions and showed all categories of KoT (described in the Table 3) in their solution. However, when they sketched the graphs of the functions they used straight lines without hesitation even though the domain of the function was discrete. This problem is known as continuous versus discrete graphs (e.g., Leinhardt et al., 1990).

Other three teachers didn't use graph in their solution (Michael, Patrick and Jodie). Patrick and Jodie used table. Jodie read the result of the task directly from the table and interpreted it within the task context. Patrick first wrote and solved equations and then interpreted the result within the task context. Michael only used the equations to solve the problem. These teachers showed their KoT from the category Phenomenology and application. Jodie and Patrick used the table and the language representation to solve the problem. Patrick also used the equations. Therefore, they both also demonstrated the category Knowledge of different ways of representing content from their KoT.

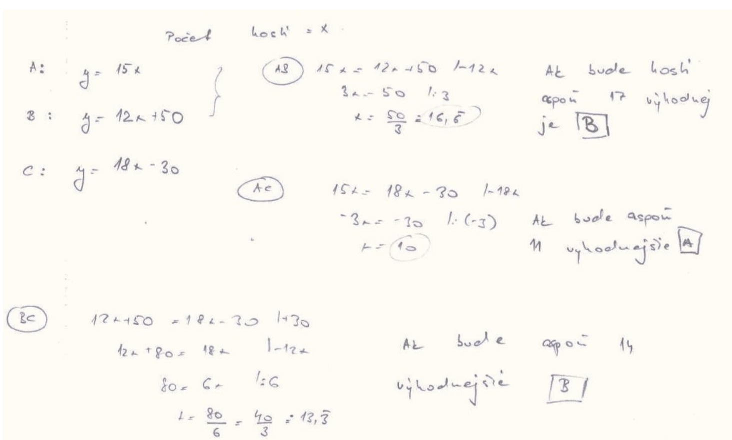
Sabina didn't try to solve the problem. She wrote: "It's hard to say, depending on how many children will be at the party." It seems that she has problems with modelling the task using functions. The other two teachers (Emma and Kate) did not submit a solution to the task. Examples of three teachers' solution (Patrick, Greta and Michael) to the task are shown in Table 8.

Table 8: Teachers' solution of the task 25 and their analysis

Patrick's solution:



	⟨1, 9⟩					⟨11, 13⟩			⟨14, 16⟩			⟨17, ∞)	
	1	2	3	9	10	11	12	13	14	15	16	17	18
A) $y = 15x$	15	30	45	135	150	165	180	195	210	225	240	255.	270
B) $y = 12x+50$	62	74	86	158	170	182	194	206	218	230	242	254	266
C) $y = 18x-30$	−12	6	24	132	150	168	186	204	222	240	258	276	294
	C<A<B				A=C<B	A<C<B			A<B<C			B<A<C	



If there will be at least 17 guests, B is OK

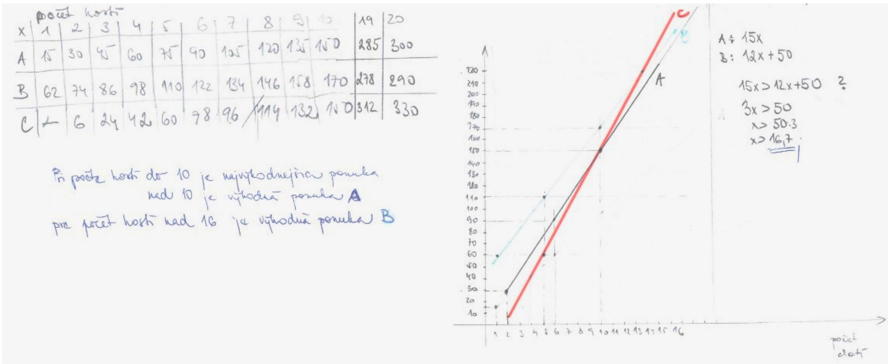
If there will be at least 11 guests, A is OK

If there will be at least 14 guests, B is OK

Advantageous depends on the number of people. If the number of guests is in the interval $\langle 1, \infty)$, the C playground is the least advantageous because it is the most expensive with at least 14 children. I don't expect to fit an extremely large number of children into the playground. But again, assuming an infinite number of children, playground B is the cheapest. The table below shows the ranking of the most preferred playground with respect to the number of children.

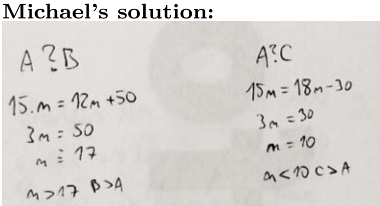
Table 8: Teachers' solution of the task 25 and their analysis (continued)

Greta's solution:



For up to 10 guests, offer C is the most advantageous. For more than 10 guests, offer A is preferred. Offer B is preferred for over 16 guests.

Michael's solution:



To found out what kind of knowledge from PCK domain was demonstrated by the teachers we analysed the discussion between teachers and lector concerning the task 25. Below we provide a transcript of this discussion.

Discussion:

1	Lector:	Try to say something about this task (the discussion about the task 25 began).
2	Patrick:	How many children were there?
3	Greta:	We said exactly that, that the children would ask and how many children would be there?
4	Sabina:	That's exactly what we said that the kids will ask: and how many kids will be there?
5	Jodie:	And the point here is that they're supposed to find out that at a given number this is ideal, at a given number this is ideal. So, I would do this task cross-curricular and I would use excel to do it. It would be quite efficient.
6	Lector:	Are you missing the number of how many children are supposed to be there?
7	Sabina:	Or at least the maximum number that can participate.
8	Lector:	Some of you have sorted it out. So, try to say how you approached it.
9	Kate:	I set the number, for example, that there would be ten children at the party.
10	Lector:	You've pitched the number, you've put that if there are ten, then after the A-point it will be 15x10, etc. Did anybody approach it differently?

11	Greta:	I put one guest, two, three, four, five, six, up to 20 and then I pulled up the graphs and there was an overflow somewhere in there. So, I came up with that for up to 10 guests, offer C is the best deal, over 10 is offer A, over 16 is offer B (see Greta's solution, Table 8).
12	Lector:	And if there were two? Or one guest?
13	Patrick:	One guest is good for C.
14	Vera:	So, from 2 to 9 is C, for 10 both A and C, from 11 to 15 A, 16 both A and B, 17 is then already B.
15	Lector:	Is there something in this task that's good for the sake of teaching maths (it's a linear function)? How is it different, from the previous ones?
16	Jodie:	The graph starts in the fourth quadrant, below zero.
17	Greta:	Scale problem.
18	Vera:	This task is very interesting.
19	Greta:	Just keep looking for where it starts. Where do the advantages, disadvantages start?
20	Vera:	Which offer is most advantageous to her (party organiser). Whether those kids would know... But also, this whole question of what that means most advantageous, in the context of whether they can still break it down into intervals in some way, that when which offer is more advantageous.
21	Michael:	I compared A-B, A-C (see Michael's solution, Table 8). That A sort of standard, and what I came up with, I've already consolidated that. I can imagine that I would address this with them as a discussion at such a time, that to show them how to solve such harder problems, to think together, prescription we learn, graph.
22	Greta:	At least that, that's from reality. We choose which restaurant we're going to go to have a party or a wedding or whatever. It's not always convenient when I only have a menu for 12€.
23	Michael:	Is it a nice task that they say - wow - that's the way to go?
24	Greta:	It's nice that it's not just A, just B or just C, but that it's related from the number of guests.
25	Michael:	I could imagine including this task.
26	Greta:	But only towards the end for practice, after the takeover.
27	Michael:	Or so that I would do the introduction, and then by discussion, that you try yours.
28	Vera:	I don't think it's so obvious through the equations. That's why I drew the graphs, because once they have the graphs, they can just see it.
29	Greta:	Only when the graph though is not accurate.
30	Vera:	Then they mislead themselves...

The authors independently coded the data according to the categories of the KFLM, KMT and KoT sub-domains of the MTSK model and then compared their coding. The differences in the codes were discussed and finally the coding was unified. The result is shown in Table 9. The code in the second column of the table contains the abbreviation of the sub-domain; a number that tells us the order of the category from Tables 1, 2 and 3; and two letters that help us to navigate through the categories (e.g., Knowledge of Features of Learning mathematics – 1. Theories of mathematical learning is coded as KFLM1-TL).

Table 9: Teachers' knowledge in action

Sub-domain	Code of category	Category specification	Replicas
KFLM	KFLM1-TL	–	–
	KFLM2-SW-1	Scaling/graph misconceptions	[16–17]
	KFLM2-SW-2	Difficulties with graphing the function when it is not appropriate to have the same scale on both axes (in connection with the previous group discussion)	[17]
	KFLM2-SW-3	Difficulties with translating the task to formal language (how to interpret advantageous)	[20]
	KFLM2-SW-4	Difficulties identifying function to its graph	[30]
	KFLM3-MC-1	Awareness about students' thinking – they may think that the number of children is missing. Students will not recognize a function situation	[3–4]
	KFLM3-MC-2	It's difficult for students to compare functions using equations	[28]
	KFLM4-EA	-	
KMT	KMT1-TT-1	Organization of task presentation – constructivist view	[21]
	KMT1-TT-2	Organization of the presentation – traditional procedure	[26]
	KMT1-TT-3	Organization of the task presentation – constructivist view (according to group discussion Michael prefer to start the lesson with the task to explain properties of linear functions)	[27]
	KMT2-TR-1	Using a MS Excel to solve problems on functions	[5]
	KMT3-ST-1	Potential of the task – realistic task, understandable context for pupils	[5]
	KMT3-ST-2	Potential of the task – realistic context leads to unusual graph (graph starts below x-axis)	[16]
	KMT3-ST-3	Potential of the task – realistic task in which students have to adjust scale on axes to draw a graph (in connection with the previous group discussion)	[17]
	KMT3-ST-4	Potential of the tasks – linking presentation – formula and graph	[21]
	KMT4-RC	–	
KoT	KoT1-KP-1	The domain of the function is not determined	[7]
	KoT1-KP-2	A comparison between functions can be solved through equations as well as by graph	[28]
	KoT1-KP-2	Determining outputs for concrete inputs	[9], [11]
	KoT1-KP-3	Solving a problem of comparison of some functions can be done by intervals	[20]
	KoT2-PR	–	
	KoT3-DR-1	A function can be expressed by an algebraic expression as well as by a graph	[28]
	KoT3-DR-2	Using graph to solve the task is problematic because it is not accurate	[29]
	KoT4-PA-1	A situation that can be model by a function	[5], [24]
	KoT4-PA-2	Function linked to situation of optimisation	[20]
	KoT4-PA-3	Real situations than can be model by a function	[22]

As can be seen in the Table 9, teachers touched on eight out of twelve categories of the KMT, KFLM and KoT sub-domains of the MTSK model in discussion. Seven categories were touched upon more than once. Thus, the discussion of the task and its solution led teachers to reflect on different aspects of teachers' specialised knowledge.

Results and discussion

Model MTSK

This section presents and discusses the evidence of 'specialised knowledge' in the sense of the MTSK framework, as revealed by in-service middle school mathematics teachers when solving and discussing the particular task. The results are presented and discussed in relation to the main sub-domain to which they correspond.

Knowledge of Features of Learning Mathematics

Strengths and weaknesses in learning mathematics – ***KFLM2-SW***

From the dialogue to the task 25 it can be seen that the teachers are aware that the task is suitable for building a correct understanding of the graph of a linear function. They are aware that it allows to avoid several misconceptions mentioned in the literature (e.g., Leinhardt et al., 1990) regarding the notion of linear function and its representations. In particular, they point out the problem of the need to put different scales on the coordinate axes and also the problems related to the transition from the informal language of the text of the task to the language of mathematics, namely the need to recognise that the task can be modelled using linear functions and, consequently, to identify an interpretation of the word most advantageous in the language of mathematics.

Ways students interact with mathematical content – ***KFLM3-MC***

In two of the replicas, the teachers were concerned with how the students would respond to the task presented. In replica 4 (KFLM3-MC-1) Sabina shows an awareness of how students think about and communicate with each other about certain content. The teachers classified the task as an atypical task that the students would eventually struggle with. (Sabina found the task atypical and challenging, suggesting that some teachers try to avoid such tasks and do not use them in mathematics lessons. Her opinion was probably also influenced by the fact that she herself struggled with the task). In replica 28 (KFLM3-MC-2), Vera explained that she thought it was easier for students to see comparisons between different functions using graphs than using equations.

Knowledge of Mathematics Teaching

Theories of mathematical teaching (both personal and institutional) – ***KMT1-TT***

Teachers with more than 15 years of teaching experience were introduced to Bloom's Taxonomy, which was very popular at the time of their undergraduate training, as part of their teacher training. Apparently also influenced by this theory, which appeared in both general and subject pedagogical preparation, they preferred to teach the new concept in the way from the easier to the more difficult. They consider it more appropriate to set the task only after introducing the concept of linear function and exploring it in simpler tasks. On the other hand, teacher Michael, who had the least experience of the group of teachers (4 years), might already have been influenced by the constructivist theories that have dominated teacher education in recent years. According to him, the task is realistic and would motivate students to explore linear functions using different representations and thus motivate them to study linear functions. Of course, these different preferences of teachers may also be influenced by their beliefs.

*Teaching resources – **KMT2-TR***

One teacher (Jodie in replica 5) also mentioned that to solve this problem it would be appropriate to use a spreadsheet calculator, namely MS Excel, which is available in all middle schools in Slovakia and its use is part of the curriculum.

*Strategies, techniques, tasks, and examples – **KMT3-ST***

Three teachers state that the task has great potential for teaching. Their reasons are: the task makes it possible to link students' knowledge from different areas, students have to deal with a scale problem when solving it and with a problem with the intersection with the y-axis, which is lying below the x-axis. The task can also be used to link different representations of a linear function (language, table, graph, formula). On the other hand, none of the teachers mentioned the possible limitation of the task, which is that the domain of the functions is only natural numbers. Moreover, as mentioned above, three teachers sketched the graphs of the functions and used straight lines instead of discrete points lying on a straight line. The additional limitation is that the context of the task only offers modelling with increasing linear functions. Therefore, the teacher needs other tasks to create the complex view of the concept being studied.

Knowledge of Topic

*Knowledge of procedures – **KoT1-KP***

In five replicas, the four teachers discuss the methods they used to solve the problem. The discussion mentions the input-output aspect (replicas 9 and 11), the covariance aspect (replica 20) and the correspondence aspect (replica 28).

*Knowledge of different ways of representing content – **KoT3-DR***

We observed from both the teachers' solutions and the dialogue that the teachers were aware that the task could be represented in a few ways, while being aware of some of their limitations (replica 29). They also showed ways of linking representations in their solutions.

*Phenomenology and applications – **KoT4-PA***

As the task is related to real context teachers in four replicas touched the category Phenomenology and applications. They mentioned modelling a real situation by a function (replicas 22, 24) and linked the task to the situation of optimisation (replica 20).

Final comments

Our work was aimed at revealing the specialised knowledge of in-service teachers, which was manifested during a guided discussion focused on a set of tasks. Such a practice in teacher professional development has been recommended by several researchers (e.g., Zaslavsky & Leikin, 2004; McCulloch et al., 2022). During the shared discussion the teachers chose the tasks they would like to discuss together. They preferred tasks that are not usually included in Slovak middle school textbooks. The discussion usually started with the presentation of the solution to the problem (KoT) and then moved on to the subdomains belonging to the PCK. As the opinions on these tasks were initially quite different, the teachers were forced to refine their argumentation in order to convince their opponent.

There were tasks in the set where teachers' mathematical knowledge dominated the discussion; for example, the task 30 (see Figure 3) led teachers to reflect on the definition of the concept of a linear function, how they introduce this concept in the classroom and how this affects their students' problem solving (the category of the KPM subdomain of MTSK). Other tasks that were discussed (such as task 25 presented earlier) led more to the development of PCK.

Decide, whether the functions below are linear. Justify your reasoning.

(a) $y = 14x$

(b) $y = 14$

(c) $x = 14$

(d) $y = \frac{2}{x} + 1$

(e) $y = \frac{x}{2} + 1$

(f) $y = \frac{x^2}{x} + 1$

Figure 3: Task 30 from set of tasks

The set of tasks provided a rich opportunity for teachers to experience a different kind of learning (compared to their previous experiences), which required their preparation in advance and, consequently, their cooperation and activity in groups and shared discussion. The tasks chosen by the teachers for shared discussion led to rich dialogue in which the teachers not only demonstrated their specialised knowledge but also enable us to identify some gaps in their knowledge. Three teachers had problem with modelling task 25 using functions (KoT4-PA). In relation to the discussion of task 30 (see Figure 3), we identified that teachers were uncertain about the definition of the term linear function (KoT2-PR). Task 25 (see Figure 2) and also the other two tasks showed that teachers were unaware of the problem of continuous versus discrete graphs (a common misconception among students when deciding whether a graph is or should be represented in a continuous or a discrete manner – see Leinhardt et al., 1990) (KFLM2-SW and KoT2-PR). A non-standard problem on linking the representation of a function using a graph and a language with an ambiguous solution showed that teachers have difficulty in solving such tasks (KoT2-DR and KoT4-PA).

In our research, we focused on the teachers' knowledge related to the development of functional thinking in the period when students are introduced to the concept of function, start to use formal function notation and work with the letter as a variable (students around 14-15 years old in Slovakia). Research that has addressed teachers' knowledge in the development of functional thinking has mostly focused on the earliest stages and is often linked to working with patterns (e.g., McAuliffe, Vermeulen, 2018; Wilkie, 2014). Other studies have addressed issues with specific concepts or properties from the topic of functions such as Zaslavsky et al. (2002), Yoon, Thompson (2020), or with specific aspects of teaching, such as Ostermann et al. (2018). They have typically used tests to determine levels of understanding, usually in combination with an interview. Instead of testing teachers, we seek to understand teachers' knowledge by having them discuss tasks designed for students as they spontaneously formulate and discuss their ideas, while we observe their knowledge in action. To analyse this knowledge, we use the MTSK model, which we believe better reflects the specificities of mathematics, rather than the Model of Mathematical Knowledge for Teaching (Hill et al. 2008) used in the context of function, e.g., in Steele et al. (2013) or Wilkie (2014). The presented categorisation according to sub-domains of the MTSK model provides us a better understanding of teachers' knowledge and allows us to examine not only what knowledge the teacher exhibits, but also the depth of this knowledge. We also believe that the selected tasks could be a medium for developing different areas of teachers' knowledge.

A limitation of the study is the relatively small sample size. The results may be different in a different group of in-service teachers. Therefore, future research should include a larger sample in order to obtain more generalised data. We also plan to analyse solutions and dialogues related to other tasks from the set to get a more comprehensive picture of the state of teachers' knowledge. We believe that such an analysis is an important starting point for the designing of professional development courses for in-service and pre-service teachers to meet the needs for improvement in the development of functional thinking.

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