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Infinite limit of a function at infinity and its phenomenology*

Abstract. In this paper we aim to characterise and define the phenomena of the infinite limit of a function at infinity. Based on the intuitive and formal approaches, we obtain as results five phenomena organised by a definition of this limit: intuitive unlimited growth of a function, for plus and minus infinity, and intuitive unlimited decrease of a function, for plus and minus infinity (intuitive approach), and the one way and returned phenomenon of infinite limit functions (formal approach). All this is intended to help overcome the difficulties that pre-university students have with the concept of limit, contributing from phenomenology, Advanced and Elementary Mathematical Thinking, and APOS theory.

Introduction

The notion of limit has been widely studied over the last 40 years in the field of Mathematics Education. This research has focused both on students (Jirotková, Littler, 2003; Jutter, 2006; Kidron, 2011; Valls et al., 2011; Morales et al., 2013; Douglas, 2018) and on active and trainee teachers (Movshovitz, Hadass, 1990; Kattou et al., 2009; Lestón, 2012; Arnal-Palacián et al., 2022; Arnal-Palacián, Claros-Mellado, 2022; Pérez-Montilla, Arnal-Palacián, 2023). All of them have made it possible to delimit the knowledge of their difficulties, obstacles and errors (Cornu, 1983; Sierpinska, 1985; Hitt, 2003; Vrancken et al., 2006; Irazoqui, Medina, 2013; Morales et al., 2013; among others).

Precisely around the difficulties of teaching and learning, some lines of research have been built in recent decades. However, in most cases studying the boundary in a general way (Tall and Schwarzenberger, 1978; Tall and Vinner, 1981; Cornu, 2002; Fernández et al., 2017; Marufi et al., 2018). In recent years, some

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studies point out that it is essential to investigate each boundary in a particular way (Morales et al., 2013). This perception is shared by Claros (2010), Sánchez (2012) and Arnal-Palacián (2019), who, using phenomenology in the sense given by Freudenthal (1983), have contributed to the teaching and learning of the finite limit of a sequence, finite limit of a function at a point and infinite limit of a sequence, respectively. This contribution has been able to occur precisely from the characterization of some phenomena, leaving behind the current trend of teaching the limit from an algorithmic and algebraic approach (Vrancken et al., 2006). The phenomena characterized for the aforementioned limits have made it possible to study the learning of the finite limit of a sequence of high school students, and the teaching of the finite limit of a function at a point with active teachers and the infinite limit of a sequence with teachers in training.

From these last three studies and the need to study each limit in a particular way, the reason why this research for the infinite limit of a function at infinity arises, giving rise to the following research question: Can the definition of the infinite limit of a function at infinity contribute to teaching and learning by highlighting relevant phenomena, as suggested by Freudenthal (1983)?

To address this question, the objective is to characterise and define some phenomena organised by the definition of the infinite limit of a function at infinity.

Achieving this objective has the potential to alleviate the challenges associated with the to the infinite limit of a function at infinity in the teaching-learning process, as was the case with the three previous phenomenological studies. In other words, the research could provide a solution to some didactic problems generated by the use of notions, ideas and definitions related to the infinite limit of a function at infinity.

1. Theoretical Framework

The theoretical framework of the present study considers three fundamental pillars: Advanced and Elementary Mathematical Thinking, phenomenology, in the sense given by Freudenthal (1983), and APOS theory.

The choice of these three fundamental pillars is determined by the following reasons. First, following Tall (1991), it is possible to go deeper into the cognitive development present in the teaching and learning processes of concepts related to infinitesimal calculus, which due to their difficulty should be placed within Advanced Mathematical Thinking. Secondly, phenomenology allows the analysis of mathematical contents (Freudenthal, 1983). Finally, the APOS theory can provide an explanation of how individuals conceive infinity. This could be the first step towards the development of pedagogical strategies aimed at aiding students understanding and applying the types of transformations necessary for the solution of various problems involving the concept of infinite limit (Weller et al., 2004).

1.1. Elementary Mathematical Thinking versus Advanced Mathematical Thinking

Since the 1980s, the mathematical community and, in particular, the area of mathematics education has been concerned with how people who are professionally engaged in mathematics think, that is, with mathematical thinking. In this study we are concerned with both Elementary Mathematical Thinking (EMT) and Advanced Mathematical Thinking (AMT). Although for Dreyfus (1991) there is no clear distinction between many of the processes of EMT and AMT, he does consider AMT to be more focused on definitional abstraction and deduction. Garbin (2015) furthermore includes representation, translation and abstraction in AMT. On the other hand, EMT is more related to mathematical notions accompanied by examples or considered intuitively or at least not formally defined. Between EMT and AMT, there should be a transition stage, which helps to transfer learning from teachers to students, increasing the frequency and relevance of demonstration and definition. In addition, it should encourage changes in the way students perform routine tasks and how they deal with information and carry out mathematical processes (Garbin, 2015).

Tall (1991) and Dreyfus (1991) elaborated a cognitive theory regarding the development and growth of Advanced Mathematical Thinking. Tall (1991) highlighted, among several concepts that should be part of AMT because of their difficulty, limits and infinity. Despite this statement, which would include the infinite limit of a function at infinity as part of AMT, according to the studies of Edwards et al. (2005), the concept of infinite limit is part of elementary or advanced mathematical thinking depending on the work related to it.

1.2. Phenomenology

Phenomenology is understood as a philosophical discipline, which began to develop in the 20th century. In this paper, when we speak of phenomenology in the sense given by Freudenthal (1983), we do so as the component of his didactic analysis. Freudenthal gives the name *phenomenology* to his method of analysis of mathematical contents, in which he starts from the contraposition between the terms *nooumenon* and *phenomenon*. In other words, this philosophical reflection is based on the contrast between the objects constructed in concepts, which are called objects of thought, and which will be called *nooumenon*, and the situations that these mathematical objects organise, when one has acquired experience, which will be the *phenomena*. Other authors, such as Gravemeijer and Terwel (2000), determine that the situations must be selected in such a way that they can be organised by the mathematical objects that the students have had to construct. The object to be considered will be the *nooumenon*, and this describes and analyses the *phenomenon*.

In previous studies on the phenomenology of the limit, it was certain phenomena have already been characterized: intuitive simple approximation and one way and returned phenomenon in sequences (Claros, 2010), intuitive double approximation and one way and returned phenomenon in functions (Sánchez, 2012), and intuitive unlimited growth, intuitive unlimited decrease and one way and returned phenomena in sequences with an infinite limit (Arnal-Palacián, 2019). These phenomena have one characteristic in common, the use of two approaches: intuitive and formal. It is from the intuitive approach that it is possible to characterise the intuitive phenomena that make it possible to take the first limit candidate, and subsequently characterise the formal phenomena that determine the value of the limit. It is precisely these phenomena characterised for other types of limits that serve as the starting point for characterizing of the phenomena of the infinite limit of a function at infinity. From among the three previous studies of the phenomenology of the limit, for the specific case of the infinite limit of a sequence, after consulting experts, the following definition was selected to analyse the phenomenology of this type of limit.

"Let K be an ordered field and $\{a_n\}$ a sequence of elements of K. The sequence $\{a_n\}$ has a "plus infinite limit", if for each H element of K, there exists a natural number v, so that $a_n > H$ for all $n \geq v$" (Linés, 1983, p.29).

Based on this definition, from an intuitive and formal approach, the following phenomena were characterised (Arnal-Palacián, 2019; Arnal-Palacián et al., 2020):

- intuitive unlimited growth: an increasing sequence fulfils the idea that the values of the sequence become larger and larger. If $n > m$, then $a_n > a_m$ (a_n general term of the sequence). By checking this for several values, we intuitively deduce that the sequence is increasing.

- intuitive unlimited decrease: we observe that a decreasing sequence fulfils the idea that the values of the sequence become smaller and smaller, small being understood as those negative numbers whose absolute value is greater and greater. If $n > m$, then $a_n < a_m$.

- one way and returned phenomenon in sequences with an infinite limit: For the limit plus infinity we can observe two processes that determine this phenomenon in sequences of limit plus infinity:

  - The first process, called "one way", corresponds to the fragment: "if for every element H of K, there exists a natural number v".
  - The second process, called "return" corresponds to the fragment "so that $a_n > H$, every time $n \geq v$".

In the latter phenomenon, the idea and return is manifested by observing these two processes together. In particular, by interpreting and applying the processes included in the very definition of the infinite limit of a sequence, requiring the construction of a function $H \to n(H)$ (See Figure 1). This decomposition, "one way" and "return", could help students to understand the whole process, without neglecting each of the mathematical notions involved.

The importance of taking into account both intuitive and formal phenomena is justified for epistemological, didactic and cognitive reasons.

The importance of considering intuitive phenomena is motivated by the fact that they allow us to rule out a limit candidate or to hypothesise about it. However, the intuitive phenomena do not guarantee that the limit candidate is the

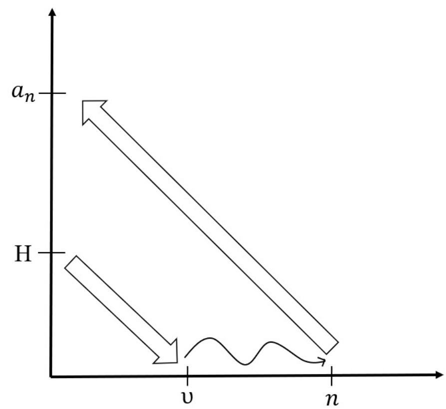

Figure 1: One way and returned phenomenon for infinite limit

true one, so it is necessary for the teacher to use the formal phenomenon. Moreover, considering the notion intuitively allows us to develop the infinite limit of a sequence from Elementary Mathematical Thinking. This type of thinking makes it possible to generate an image concept, in the sense given by Tall and Vinner (1981).

For this reason, we consider of vital importance the use of the formal phenomenon, because only through it we can be certain about the actual nature of that limit candidate. This certainty will be determined when the teacher presents the one way and returned processes that allow us to determine whether the limit candidate has been well chosen (Claros et al., 2013). In this case, the phenomenon will be linked to Advanced Mathematical Thinking, in which the processes of abstraction, demonstration and generalisation can take place.

In order to address the phenomenon of one way and returned in the classroom, a formal definition needs to be considered by the teacher and to analyse each of the notions involved. In the case of infinite limits, the notions to be taken into account are: dependence, both of the dependent and independent variable; infinite processes, discrete or continuous; types of infinity, actual and potential; and annotation, in both variables (Arnal-Palacián, 2019). With all this, the teacher, knowing both the intuitive and formal phenomena, will be able to provide answers, at different times, for learning the notion of the infinite limit of a sequence. The following are three examples of phenomena that could be observed in previous studies on the infinite limit of a sequence (Arnal-Palacián, 2019; Arnal-Palacián et al., 2020), mainly considered as the starting point of the present research.

- Example 1. Let the following sequence be provided in the tabular representation system (see Figure 2).

Following the intuitive unlimited growth phenomenon, we can observe that the terms of the sequence become larger and larger. The sequence is not subject to an upper limit and seems to grow in an unlimited way. This allows us to deduce that the sequence has a limit $+\infty$ (Arnal-Palacián et al., 2020).

n	1	10	100	1 000	...	→	+ ∞
a_n	2	101	10 001	1 000 001	...	→	+ ∞

Figure 2: Example of intuitive unlimited growth phenomena with tabular representation (Vizmanos, Anzola, 1996, p.160)

- Example 2. Let the following sequence be presented in the graphical representation system (see Figure 3)

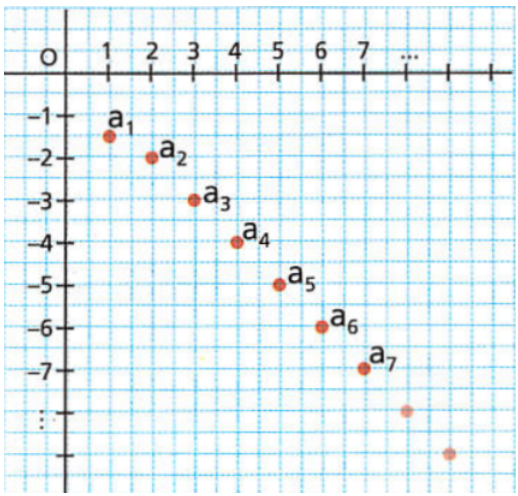

Figure 3: Example of intuitive unlimited decrease phenomenon with graphical representation system (Vizmanos, Anzola, 1996, p.160)

Observing the graph, it becomes evident that the terms of the sequence decrease as it progresses and there is no real number greater than all the other values in the sequence. This observation indicates the presence of the intuitive phenomenon of unlimited decrease. The sequence appears to be decreasing and we can therefore deduce that its limit will be $-\infty$ (Arnal-Palacián et al., 2020).

- Example 3. Let the sequence $s(n) = n^2$ be represented graphically, accompanied by its algebraic expression (see Figure 4).

We start at a real number, for example $H = 49$ (Figure 3), situated at the Axis Y and go “one way” to a natural number, in our example $v = 7$, situated at the Axis X, and “return” from $n = 8 (n \geq v)$ to a real number of the sequence, $a_8 = 64 (a_n > H)$ (Arnal-Palacián et al., 2020).

One way: Given $H = 49$, there is a v natural number, for example $v = 7$.

Return: with $n \geq v$, for example $n = 8$, we have $a_8 = 64 > 49 = H$.

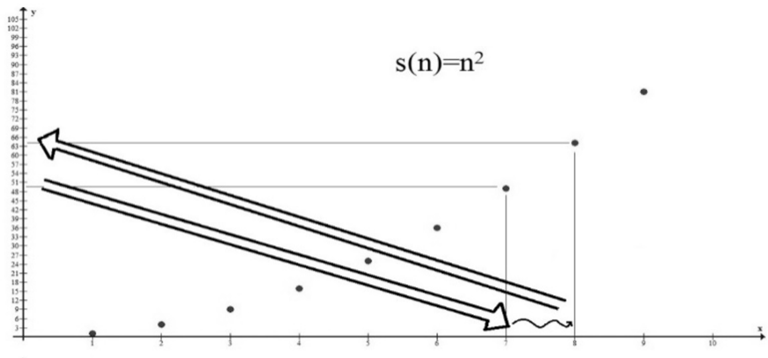

Figure 4: Example of one way and return phenomenon with graphical representation system (Arnal-Palacián et al., 2020)

Other examples could have been taken of some values taking H, where n is the smallest number giving the inequality $a_n > H$.

$$H = 10,\ n = 4 \to a_4 = 16 > 10$$

$$H = 100,\ n = 11 \to a_{11} = 121 > 100$$

$$H = 10\ 000,\ n = 101 \to a_{101} = 10\ 201 > 10\ 000$$

1.3. APOS Theory

In APOS theory, the understanding of a mathematical concept is achieved by an individual's reflection on mathematical problems and the solution given in a given social context, through the construction and reconstruction of certain mental structures and their organisation into schemas (Dubinsky, 2014).

APOS consists of the following mental structures: Actions (A), Processes (P), Objects (O) and Schemas (S). In addition, it needs different mechanisms, such as internalisation, encapsulation, coordination, inversion, de-encapsulation, thematisation and generalisation. Dubinsky (2014) described how to move from one mental structure to another through each of the mechanisms outlined above. See Figure 5.

We take APOS theory as a fundamental pillar because, according to Blázquez et al. (2008), students do not manage to interpret a formal definition of a limit easily and this definition is soon forgotten. The intuitive conceptions that students have prevail after the formal definition of limit is presented, and where the need arises to relate the formality with the intuitive ideas that students previously had (Artigue, 1998).

To facilitate the teaching and learning of the notion of infinity, Roa-Fuentes and Oktaç (2014) defined a genetic decomposition. The objects resulting from the application of an infinite process can be a context in which to analyse the limit. Performing a small number of iterations constitutes an action. Through the

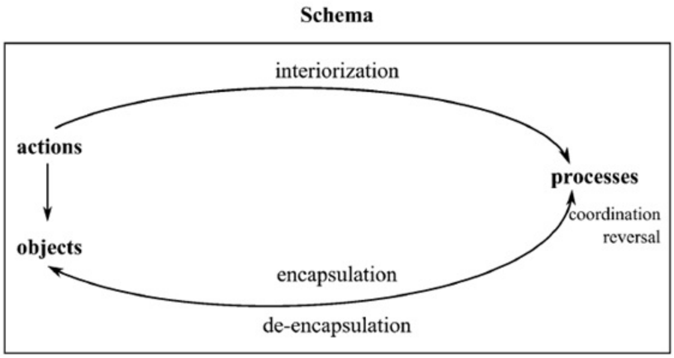

Figure 5: Mental structures and mechanisms of APOS theory (Arnon et al., 2014)

internalisation of these actions, an individual can use the structure of the resulting process to imagine repeating the actions indefinitely. Given an infinite process, internalisation and encapsulation allow one to think about what happens after the process has finished. An infinite iteration process conception develops when the individual is able to coordinate multiple instances for a finite process.

2. Method

The present study has a qualitative approach with an exploratory character (Elliot, Timulak, 2005) in order to describe the phenomena organised by the infinite limit of a function.

In the previously described studies dealing with the phenomenology of the limit, two types of phenomena were identified: intuitive and formal. Intuitive phenomena are manifested when contemplating, in a non-rigorous way, the limits in their dynamic facet. When a rigorous analysis of the notions is carried out, formal definitions are arrived at.

The definitions used in previous studies on the phenomenology of the limit (Claros, 2010; Sánchez, 2012; Arnal-Palacián, 2019), were selected through consultations with experts, including secondary school mathematics teachers and university professors of mathematics didactics. In this study we took the one used by Linés (1983) in the manual *Principles of Mathematical Analysis*. This manual was referenced in the research conducted by Arnal-Palacián (2019) and Arnal-Palacián et al. (2020). Since the present research is based on the previous study, it is precisely the definition given in this manual that is considered for the present study.

This definition for the limit plus infinity of a function at plus infinity is the following:

"Be $f : X \to \mathbb{R}$, with $X \subset \mathbb{R}$ not bounded at the top. f is said to have a limit $+\infty$, when it tends to $+\infty$, if for each real number H there exists a real number K so that $f(x) > H$ for all $x \in X$ satisfying $x > K$. It is written $\lim\limits_{x \to +\infty} f = +\infty$" (Linés, 1983, p.201-202).

The author considers the following limits:

$$\lim_{x\to+\infty} f = -\infty,\ \lim_{x\to-\infty} f = +\infty,\ \lim_{x\to-\infty} f = -\infty \ y\ \lim_{x\to\infty} f = \infty.$$

However, the definition of each of these three limits is not presented in a particular way. Therefore, when it is necessary to resort to them, an analogy will be established by adapting each of the signs.

3. Results

Intuitive and formal phenomena characterised from a definition of the infinite limit of a function at infinity, how these phenomena can contribute to elementary and advanced mathematical thinking, and how these phenomena could be presented in the classroom following APOS theory are presented below.

3.1. Intuitive phenomena

From an intuitive approach, if we analyse the definition presented, and its analogue for the limit plus infinity when x tends to minus infinity, we can observe that the values of the function become larger and larger as we 'advance' in $\mathbb{R}$. Consequently, it can be intuited that the function is unboundedly increasing, i.e. it grows unboundedly. Therefore, we can state that the intuitive unlimited growth is observed in these definitions. Therefore, we have the most infinite limit of a function when x tends to most infinity, $\lim_{x\to+\infty} f = +\infty$, and also when we have the most infinite limit of a function when x tends to minus infinity, $\lim_{x\to-\infty} f = +\infty$. By adaptation of the notation of the intuitive unlimited growth phenomenon (u.i-g.), in this case we will call this phenomenon intuitive unlimited growth for functions (u.i-g.f).

From an intuitive approach, if we analyse the definitions for the limit minus infinity when x tends to plus infinity and also when x tends to minus infinity, we can observe that the values of the function become smaller and smaller as we 'advance' in $\mathbb{R}$. Consequently, it can be intuited that the function is non-inferiorly bounded decreasing, i.e. it decreases unboundedly. Therefore, we can state that the intuitive unlimited decrease is observed in these definitions, i.e., for $\lim_{x\to+\infty} f = -\infty$, and for $\lim_{x\to-\infty} f = -\infty$. By adaptation of the notation of the intuitive unlimited decrease phenomenon (u.i-d.), in this case we will call this phenomenon intuitive unlimited decrease for functions (u.i-d.f).

Given that the terminology could give rise to confusion, because there is no single equivalent intuitive phenomenon for those defined for the infinite limit of a sequence, given that for the function we can take positive and negative real values, the phenomena described above are specified with a "+" or "-" sign. This difference will be determined when we have $x \to +\infty$ or $x \to -\infty$. Thus, +u.i-g.f. ($\lim_{x\to+\infty} f = +\infty$), -u.i-g.f. ($\lim_{x\to-\infty} f = +\infty$), + u.i-d.f. ($\lim_{x\to+\infty} f = -\infty$), − u.i-d.f. ($\lim_{x\to-\infty} f = -\infty$)

3.2. Formal phenomenon

From a formal approach, two processes can be observed that determine the one way and returned phenomenon in infinite limit functions (o.w.r.i.f.). Let us analyse this phenomenon in the definition presented.

- The first process, called "one way", corresponds to the fragment: "if for each real number H there is a real number K".

- The second process, called "return", corresponds to the fragment "so that it is $f(x) > H$ for all $x \in X$, that complies $x > K$".

The feedback is manifested by observing both processes together. In particular, by interpreting and applying the processes included in the definition of the infinite limit of a function. This requires the construction of a new function $H \to x(H)$.

Established an H on the $Y - axis$, we "go" from it to a K belonging to the real numbers (not unique) and "return" considering $x > K$ for which we will have $f(x) > H$. In this way a real function is constructed which takes real values and which we denote in a simplified way as $(H, x(H))$. This function that has been constructed is univocally linked to the function we are working with. The particularity of this function is that it starts from the $Y - axis$ and goes to the X-axis.

This process, as was already the case for the notion of the infinite limit of a sequence, does not show any fundamental differences if we modify the sign of the limit.

3.3. Comparison with phenomena of the infinite limit of a sequence

Despite the similarities between the infinite limit of a sequence and the infinite limit of a function at infinity, and giving value to studying each limit independently, we obtain the following comparison (Table 1):

These differences found from a phenomenological point of view, motivated by their mathematical characteristics, could influence the teaching and learning of both notions.

3.4. Phenomena as a contribution to AMT and EMT

Taking into account the previous sections, we affirm:

- The phenomena intuitive unlimited growth for functions (+u.i-g.f. and -u.i-g.f) and intuitive unlimited decrease for functions (+u.i-d.f. and -u.i-d.f) are characterised from an intuitive point of view. They are related to the concept image, created to handle the concept of the infinite limit of a function at infinity and the mental images associated with it.

- The one way and returned phenomenon in infinite limit functions (o.w.r.i.f.), is characterised from a formal point of view. It is intrinsically related to the concept of the definition of the infinite limit of a function at infinity, since its characterisation arises from the very definition of this concept.

Table 1: Comparison between phenomena with infinite limit

		Sequence		Function	
Phenomena	**Intuitive approach**	$a_n \to +\infty$, $n \to +\infty$ intuitive unlimited growth	$a_n \to -\infty$, $n \to +\infty$ intuitive unlimited decrease	$f \to +\infty$, $x \to +\infty$ + intuitive unlimited growth for functions	$f \to -\infty$, $x \to +\infty$ + intuitive unlimited decrease for functions
		$n \to -\infty$ It does not exist		$f \to +\infty$, $x \to -\infty$ - intuitive unlimited growth for functions	$f \to -\infty$, $x \to -\infty$ - intuitive unlimited decrease for functions
	Formal approach	one way and returned phenomenon in sequences with an infinite limit		one way and returned phenomenon in infinite limit functions	

Following the study of Edwards et al. (2005) we place the infinite limit of functions at infinity, considering the four intuitive phenomena (+u.i-g.f., -u.i-g.f., +u.i-d.f. and -u.i-d.f.) at EMT. With these phenomena we can only perform the calculation of the limit, but not a proper abstraction of the concept of infinite limit.

On the other hand, the one way and returned phenomenon we do place it within the AMT, since it requires deductive thinking and rigorous reasoning. The relationships between these phenomena and elementary (EMT) and advanced mathematical thinking (AMT) are reflected in the following diagram. See Figure 6.

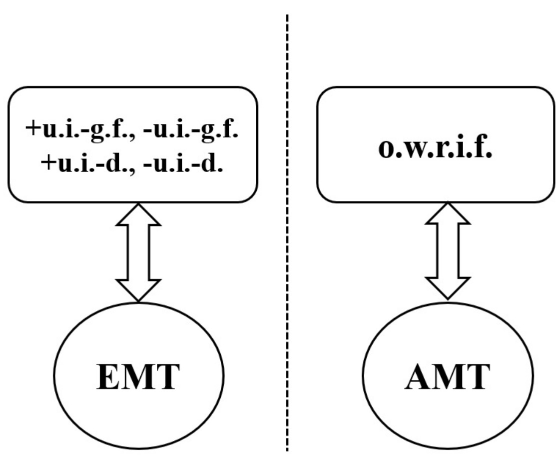

Figure 6: Infinite limit phenomena of functions at infinity taken independently

On the other hand, the joint use of these five phenomena will also imply the use of elements specific to the AMT (see Figure 7).

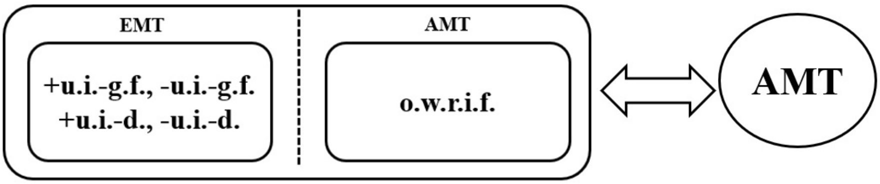

Figure 7: Infinite limit phenomena of functions at infinity taken jointly

3.5. Phenomena as a contribution to APOS theory

In the case in question, the infinite limit of a function at infinity, actions are proposed for learning this concept. These actions are based on the four intuitive phenomena, depending on whether the limit is $+\infty$ or $-\infty$, which will offer a first candidate for the limit. By reiterating this action, the student reflects on it and internalises it in a process. The process used in our particular case will be the construction of the one way and returned phenomenon in infinite limit functions. If the individual is able to conceive all the processes and actions carried out when calculating the infinite limit of a function, we can truly say that the student has constructed the cognitive objective infinite limit of a function for the particular case we are working on. See Figure 8.

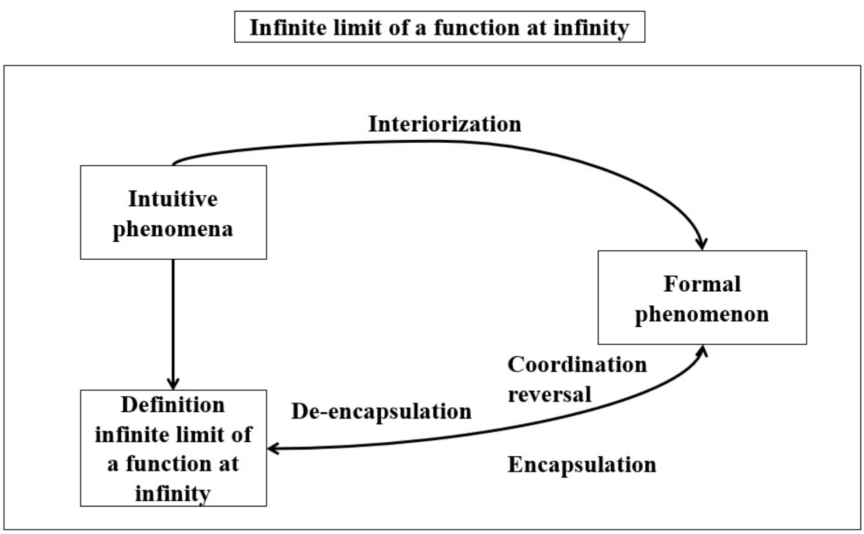

Figure 8: APOS theory on the infinite limit of functions at infinity

4. Conclusions

As a result of this research, we can conclude that the four intuitive phenomena show a first candidate for a limit, and this can be confirmed through the one way and returned phenomenon, when we are able to construct a function x(H) that satisfies both processes: one way and returned. The five characterised phenomena are intended as an aid to overcome the difficulties that students have when they have to tackle and understand tasks related to the concept of limit.

Intuitive phenomena are the first to allow us to obtain whether infinity is a candidate for a limit. However, these phenomena do not guarantee that this candidate is the true one, so the intervention of formal phenomena is necessary. For this reason, it will be precisely with the combination of the five phenomena and, especially, taking into account the intuitive and formal approaches that the teacher will be able to respond to some of these difficulties. From this research we are aware that their acquisition is not immediate and new obstacles may arise.

As has been observed during this work, it has been important to study the limit in a particular way, following the guidelines of Morales et al. (2013). There are phenomenological differences between the infinite limits of a function at infinity themselves, plus and minus infinity. This could influence the teaching and learning of the notion, and thus the students' understanding of it.

Likewise, with these five phenomena characterised from a definition of the infinite limit of a function at infinity, some of the difficulties pointed out by the literature (Tall and Vinner, 1981; Cornu, 2002; Fernández et al., 2017; Marufi et al., 2018) could be avoided, and not only show the limit from an algorithmic and algebraic approach (Vrancken et al., 2006). Moreover, given that students do not manage to interpret a formal definition of a limit easily (Blázquez et al., 2008), the study of the infinite limit of a function based on its phenomena allows for an exercise of the limit with a reflection on it.

The definitions of the infinite limit of a function at infinity that a teacher may consider in the classroom will be situated in Advanced Mathematical Thinking or Elementary Mathematical Thinking, depending on the approach. It will be on the basis of the phenomena characterised how to approach it in the classroom. Taking into account the phenomena, you will be able to establish a criterion of how to present to your students the notion of the infinite limit of a function at infinity.

Although the infinite limit of a function at infinity was initially placed within Advanced Mathematical Thinking, following the indications of Edwards et al. (2005), depending on the phenomenon used in the classroom, aspects of the EMT or the AMT will have to be mobilised. In fact, in our case, the four intuitive phenomena will belong to the EMT because they show the first candidate of limit, and not an abstraction of the concept itself. To move from EMT to AMT, the mental structure and mechanisms generated from APOS theory could be used.

From the characterisation of the phenomena presented above, we can make the following hypothesis: if we work with students on the phenomena of intuitive unlimited growth for functions, intuitive unlimited decrease for functions, and one way and returned phenomenon in infinite limit functions, we can ensure that these constitute good mental objects of the infinite limit of a function at infinity.

Although we have made progress in the phenomenological study of the infinite limit, continuing the line initiated by Arnal-Palacián (2019) and Arnal-Palacián et al. (2020), it should be noted that it is important to note that there is still work to be done. New lines of research emerge from this study, such as, for example: the phenomena organised by a definition of the infinite limit of a function at a point and the finite limit of a function at infinity, completing phenomenological studies, the limits that are most commonly encountered in the classroom.

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